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Described is a general college curriculum in mathematics that can be taught by a college staff with as few as four teachers. This report considers four related problems in mathematical education, particularly in colleges of arts and sciences. First, there is the problem of uncertainty about curriculum content and sequence. Second, the "revolution" in school mathematics has enabled the student of superior ability to move far ahead by the time he enters college. Third, not only is there a widening spread of capabilities among incoming students but there are now many kinds of mathematical knowledge which students seek. Finally, there is a problem of economy of staff and the more important economy of intellectual content. The program described is a basic one designed for the student who will use college mathematics and who brings to college a substantial high school mathematics preparation. (RP)

# A GENERAL CURRICULUM IN MATHEMATICS FOR COLLEGES

A REPORT TO

THE MATHEMATICAL ASSOCIATION OF AMERICA

1965

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

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January 1965

The Committee on the Undergraduate Program in Mathematics is a committee of the Mathematical Association of America charged with making recommendations for the improvement of college and university mathematics curricula at all levels and in all educational areas. Financial support for CUPM has been provided by the National Science Foundation.

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Others who assisted in the preparation of course outlines or other parts of this report, some members of CUPM and some not, include: Louis Auslander, R. Creighton Buck, George Carrier, Charles DePrima, Bernard Friedman, Leonard Gillman, Frederick Mosteller, Barrett O'Neill, A. H. Taub, and A. B. Willcox.

Every college, large or small, needs a basic mathematics program simple enough to staff and operate, yet substantial and flexible enough to accommodate to today's diversity of students and their objectives. The general college curriculum in mathematics which CUPM proposes in this report consists of an economical list of semester-course offerings for a college department, from which a multiplicity of individual student programs can be constructed. CUPM believes that college teachers will find considerations of feasibility dominant in this report, for this program can be taught by a staff of only four teachers and is a simple basis from which the more specialized and advanced programs recommended by the Panels of CUPM can be extended.

Because of the importance of this subject, CUPM chooses not to issue the results of its study of the problem as a set of recommendations made on its own authority. Instead, we hereby present our findings as a report to the Mathematical Association of America and seek its acceptance by the Association. We request the Board of Governors to receive it and transmit it to the Sections of the Association for their evaluations and such resolutions of endorsement or dissent as they may wish to adopt on the system of modular units for mathematical curricula that we propose here.

Outline of this Report

I. THE PROBLEM.

1. Mathematics in a college of arts and sciences. This report considers four related problems current in mathematical education, particularly in colleges of arts and sciences.

First, there is uncertainty about the curriculum. Many of us can well remember when a firm tradition decreed that college mathematics should consist of the sequence: college algebra, trigonometry, analytic geometry, differential calculus, integral calculus, differential equations, theory of equations, advanced calculus, .... That old structure which comfortably regulated college mathematics has fallen apart, and in the wake of the break-up there is much confusion about what college departments should now be doing.

Second, the generally welcome revolution in school mathematics has enabled a student who has superior ability and up-to-date high school training to move far ahead of the normal student by the time he enters college. This process has already created a greater diversity of entering students than we ever experienced before - and it has only begun! The spread in the mathematical capability of entering students will become much greater still. The department which cannot offer a mathematics program suitable for the superior student either does not attract him in the first place, or, if he comes, may kill his interest by its standard mathematics curriculum and will, in any case, not use his time to the best advantage.

Third, not only is there a widening spread of capabilities among incoming students but there are now many more kinds of mathematical knowledge which they seek; thus there is a multiple output as well as a multiple input to the mathematics department "black box." This is brought about by the computer, the increasing mathematization of the biological, management, and social sciences, by the growth of space science, and by the modern emphasis on such subjects as probability, combinatorics, logic, operator theory, and functional analysis in the fields of engineering and the physical sciences. Then there is the explosive increase in the number of students who wish to major in mathematics and expect to earn a living from it after completing their bachelor's degrees.[1] Moreover, colleges are expected to train many more young people to teach the "new mathematics" in the schools.

Finally there is a problem of economy; economy of staff and the more important economy of intellectual content. A small mathematics staff cannot offer a limitless diversity of courses for entering college students. Indeed, even in a large university where a diversity of courses can be staffed, one cannot assume that the student entering at eighteen knows what his professional interests will be, or that he will receive proper advice on the course he should take, or that the university can re-sort and repair the students who change their plans. In a large university the problem of economy in the mathematics program may be one of getting along with as few departments of mathematics as possible. The assumption underlying this report is that there is centralized planning of undergraduate mathematics in the institution.

The program outlined in this report begins with five basic semester courses in analysis, probability, and linear algebra. If the resources of the department permit, it may well be worthwhile to have separate honors courses covering the same material. But while students may be grouped in these courses according to their ability, we recommend as a matter of educational policy that the five basic courses should not be taught in different styles for students with differing major fields and professional objectives.

2. The relevant previous reports of CUPM. The Commission on the Undergraduate Program in Mathematics has issued a number of curricular recommendations which originated from Panels, each

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1. Soon after World War II the number of mathematics majors graduating from college in any one year was much less than those in either chemistry or geology. According to a study by C. B. Lindquist, Trends in Degrees 1954-55 to 1969-70, The Educational Record, Washington, D. C., Spring, 1964, the number of mathematics majors graduating in 1955 was 4,034 compared with a total of 11,202 in all physical sciences and 22,589 in engineering. In 1963-64 the number of mathematics majors passed all physical sciences combined, and, according to Lindquist's projections, should reach 39,000 by 1970, compared with 29,100 in all physical sciences and 46,100 in engineering. Thus the mathematics department nationally is approaching the size of the undergraduate engineering school. The percentage of mathematics majors in all bachelors degrees (including humanities) was 1.3 percent in 1955 and Lindquist predicts it will rise to 5.3 by 1970.

charged with the responsibility for some particular aspect of college mathematics.[2] A list of their reports containing curricular recommendations follows: "Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists," "Tentative Recommendations for the Undergraduate Mathematics Program for Students in the Biological, Management, and Social Sciences," "Pregraduate Preparation of Research Mathematicians," and a sequel showing how students can be prepared for graduate work in contemporary college departments entitled "Preparation for Graduate Study in Mathematics," "Recommendations for the Training of Teachers of Mathematics," "Recommendations on the Undergraduate Mathematics Program for Work in Computing," "Report of a Committee on Applied Mathematics in the Undergraduate Curriculum," and a forthcoming report on a five-year curriculum in the area of mathematical engineering. Other special reports will appear from time to time.

The program that we offer in this report is an economical one which can serve as a basis for richer and more advanced programs wherever and whenever these are feasible. The various panel recommendations of CUPM represent what we have to say about richer programs in special areas of college mathematics and are by no means superseded by this document, for considerations of staff limitation have played a major part in our recommendations, and seriously affected the upperclass program. Thus we have attempted a compromise among a number of objectives. It is perhaps surprising that such a variety of needs has been met with a set of only 14 courses. However, a school on the basis of its own special objectives may choose to design its program by leaning more heavily on the recommendations of one of the Panels.

3. Nature of this report. CUPM has now conducted an all-Panel study in order to describe a general college curriculum in mathematics economical enough so that it can be taught by a college staff with as few as four teachers, and so constructed that it will meet the specifications of our various Panel recommendations as well as any single, teachable program can. The program we present is a basic one designed for the student who will use college mathematics and brings to college a substantial high school preparation for it. It is not specially aimed at the honors student but, on the other hand, it does not provide for deep remedial instruction. It incorporates no mathematics appreciation course for liberal arts students of modest

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2. Obtainable free of charge by writing to CUPM Central Office, P. O. Box 1024, Berkeley, California 94701

mathematical attainment; it does not cover the special training needs of teachers, particularly of elementary school teachers. We offer some suggestions on these and other special services in Chapter V of this report, services which in many colleges the mathematics department is expected to perform but which require special staffing or other adaptation. Like any department chairman, CUPM is forced to provide first for the type of teaching service that has been the main business of college mathematics for many years, the teaching which college departments are best equipped to handle.

In offering the following solution we have deliberately left it as general and loosely structured as possible in order to avoid being overly prescriptive and to allow for natural local variations. This report is based not only on the Panel reports of CUPM but also on the studies of some of the leading departments of mathematics in the country. We have worked in close consultation with representatives of the School Mathematics Study Group to provide for a suitable articulation with the progressing secondary school program. In fact, we have tried to make a program for the immediate future of college mathematics teaching in so far as we can determine what that will be. We have tried to avoid the temptation to make a new, startling, or original proposal and have chosen instead a more conservative report which reflects current conditions in the colleges and can be taught with existing textbooks. Indeed, the program outlined in the following section is of necessity more conservative than some now being planned for superior students in the eleventh and twelfth school years. The great bulk of college students are not "superior" and have not had the benefit of modern accelerated high school curricula for superior students. Some may regard our approach to rigor as reactionary. Nevertheless some aspects of our proposed program do call for new textbooks, do invite novel approaches to the problem, and do call for an ultimate rearrangement of the college mathematics curriculum which may be regarded as radical at the present time.

In a number of ways, our program should be regarded as minimal. We discuss a total of only 14 semester courses; this total surely should be increased if staff resources permit. We have made no provision for special honors courses, or for tutorials. Moreover, it is likely that many of our recommended courses can be improved by various innovations. We urge that colleges which can exceed our minimum goals should do so. In fact, a number of college and university departments already do exceed them.

The take-off point of our proposal is our examination of the body of material called "elementary functions." It consists of a precalculus study of the idea of a function and of particular classes of functions: polynomials, rational functions, algebraic functions, logarithms, general powers, exponential and trigonometric functions. These are studied using algebra and coordinate geometry techniques. In its old fashioned form this material was called "algebra, trigonometry, and analytic geometry" and has long occupied a year in the curriculum in the first year of college or the last year of high school. Though there is a strong tendency to move this material to high school as early as the eleventh grade, much of it is still taught in colleges.

In the study of the elementary functions it is unprofitable to postpone too long the use of calculus. The second semester of "elementary functions" can become "the calculus of elementary functions" and can do so naturally without violation of teaching experience. We call this second semester of study of elementary functions "Introductory Calculus," and designate it as Mathematics 1, while the first semester of this year's material is precalculus and is designated as Mathematics 0. Thus we regard the year sequence Mathematics 0-1 as an adaptation of the standard time-honored material for this level; and this idea determines the nature of our elementary calculus more specifically than the notion of offering either a rigorous or intuitive introduction to calculus for its own sake.

We approach the problem of obtaining the flexibility to accommodate to the diversity of achievement and ability of entering college students by a combination of two devices. We propose a basic set of semester courses rather than year courses, for a student can take advantage of advanced placement in semesters much more easily than in whole years. Moreover these semester units combine into more different sequences than year units do, offering more entrances and more exits. Then, since advanced placement does not necessarily afford a higher intellectual level for a superior student, as honors courses do, we propose to spiral the approach to rigor, increasing the level of rigor as one advances through the sequence. In this way advanced placement does imply that the superior student who qualifies finds himself in a more challenging environment than he would without advanced placement.

## II. OUTLINE OF THE PROPOSED BASIC PROGRAM.

CUPM proposes that, in general, a college should offer every semester as needed the five semester-courses in mathematics for the lower divisions listed in Section 1 below. We suggest also that, where admission requirements still do not eliminate the teaching of college algebra and trigonometry before calculus, the course in Elementary Functions, Mathematics 0, described in Section 2 below should be offered in the first semester. These six offerings will enable students entering with different levels of achievement and different interests to move ahead as rapidly as possible without marking time for a semester or repeating high school courses while waiting for the next needed course to come up.

Although we recognize that the majority of colleges must now teach the material in Mathematics 0 to some of their students, this subject should be completed in high school and will be to an ever increasing extent. Hence, we suggest that this course be eliminated by admission requirements wherever possible, and, in any case, held to one semester when taught in a college where the high school prerequisites to Mathematics 0 can be met.

### 1. Lower division courses.

This list of courses is deliberately given with bare "college catalogue" descriptions, for we do not propose to specify the content of these courses in detail. A more detailed understanding of the Committee's intent can be gathered from sample outlines in Chapter VII of this report.

Mathematics 1. Introductory Calculus. (3 or 4 semester hours) [Prerequisite: Elementary functions. See Section 2 below.] Differential and integral calculus of the elementary functions with associated analytic geometry.

Mathematics 2, 4. Mathematical Analysis. (3 or 4 semester hours each) [Prerequisite for Mathematics 2: Mathematics 1. Prerequisites for Mathematics 4: Mathematics 2 and 3.] Techniques of one-variable calculus, limits, series, multivariable calculus, differential equations.

Mathematics 3. Linear Algebra. (3 semester hours) [Prerequisite: Mathematics 1.] Systems of linear equations, vector spaces, linear dependence, bases, dimension, linear mappings, matrices, determinants, quadratic forms, orthogonal reduction to diagonal form, eigenvalues, geometric applications.

Mathematics 2P. Probability. (3 semester hours) [Prerequisite: Mathematics 1.] An introduction to probability and statistical inference making use of the calculus developed in Mathematics 1.

We consider two versions of Mathematics 2, 4. Detailed sample outlines are given in Chapter VII. The Committee majority prefers an analysis sequence in which Mathematics 1 is an intuitive introduction to the differential and integral calculus of the elementary functions: polynomials, rational functions, logarithm, exponential function, algebraic functions, trigonometric functions. This would be followed by Mathematics 2 as multivariable calculus, still involving simple functions and keeping the technical demand at a reasonable load. The level of rigor is elevated only slightly. Mathematics 3 then provides algebraic technique for a stronger development of multivariable methods. The level of rigor and abstraction advances. Mathematics 4 in the preferred version contains a systematic attack on the theory of limits in single and multivariable calculus, both to remove logical difficulties and to provide computational technique. It includes linear differential equations, making strong use of linear algebra.

The preferred version, advancing multivariable calculus from the last third to the middle third of the course, would be difficult to teach from existing textbooks. Partly for this reason we include also a more conventional version of Mathematics 2, 4 in which Mathematics 2 conventionally continues the single variable calculus to more advanced techniques and applications, including limits and series and a brief introduction to differential equations. This material demands for honesty a somewhat more rigorous Mathematics 2 than does the multivariable calculus version. Then Mathematics 3, linear algebra, intervenes as before. Finally Mathematics 4 is multivariable calculus and linear differential equations making use of linear algebra. The main differences between this version and the standard current calculus courses are: the introductory calculus unit complete in Mathematics 1, the interpolation of an unusually substantial amount of linear algebra in Mathematics 3, the inclusion of elementary differential equations, and the strong use of linear algebra in multivariable calculus and linear differential equations.

2. Precalculus mathematics.

The prerequisites for Introductory Calculus, Mathematics 1, include the two components, A and B, below.

A. Three years of secondary school mathematics. The usual beginning courses in algebra (perhaps begun in eighth grade) and geometry account for two of these years. The remaining year should include: quadratic equations; systems of linear and quadratic equations and inequalities; algebra of complex numbers; exponents and logarithms; the rudiments of numerical trigonometry; the rudiments of plane analytic geometry, including locus problems, polar coordinates and geometry of complex numbers; and arithmetic and geometric progressions.

B. A study of elementary functions, their graphs and applications, including polynomials, rational and algebraic functions, exponential, logarithmic and trigonometric functions; an introduction to three dimensional analytic geometry.

The components, A and B, of the prerequisites to Mathematics 1 will normally require 3 1/2 or 4 years of high school mathematics. It is the component B and selected topics from A for which Mathematics 0 may be needed as a college course.

Mathematics 0. Elementary Functions and Coordinate Geometry. (3 or 4 semester hours) [Prerequisite: At least three years of high school algebra and geometry equivalent to A above.] An outline of the course is component B above.

3. Some critical considerations and local variations. We have used semester courses of three or four semester hours as the building blocks of our program. One obvious form of local variation might be to put these together in year-course units, for example, Mathematics 1 and Mathematics 2, Mathematics 3 and Mathematics 4, Mathematics 0 and Mathematics 1. We hope that where this is done it will not be done at the expense of flexibility or at the expense of the advanced placement provisions. Another local variation will concern the exchange of material between Mathematics 2 and Mathematics 4. In fact some teachers will prefer to let Mathematics 1, 2, 4, be taught in sequence, then Mathematics 3.

We would like to encourage colleges to experiment with the form of Mathematics 1, 2, 4, in which Mathematics 2 contains the multivariable calculus, postponing to Mathematics 4 the thorough treatment of limits and series and some of the more technical parts of single variable calculus. There are several reasons for believing that such an arrangement may be better than the conventional one in which single variable calculus is thoroughly exhausted before turning to multivariable calculus as a more advanced subject. Teachers are familiar with the fact that when this is done the essential multivariable concepts have been rushed or slighted to build up techniques of integration in closed form along with certain geometric applications; interesting subjects, but ones whose values diminish in a new era of numerical methods. By contrast, the multivariable calculus need not be as difficult as these ramifications of single variable calculus and offers more new ideas to contribute to the first year of calculus. Thus the proposed arrangement of Mathematics 1, 2 provides a more comprehensive introduction to calculus than the conventional form, especially for students stopping at Mathematics 2, without necessarily involving the student in elaborate technique or sophistications of rigor and provides a good basis for our suggested spiral to a higher level of rigor in the second year with the more technical material of limits, series, approximations, numerical methods, and linear differential equations.

Furthermore, the proposed arrangement is attractive for better service to engineering and physics for it sets up early many things that engineers and physicists want to do; for example it permits an early introduction of the moments of solid figures. It is also more useful for students in other areas, such as social science and business. They will have an early introduction to multivariable calculus in Mathematics 2 and, if time is limited, can bypass the more technical material of Mathematics 4.

Finally the proposed arrangement provides an early introduction to multivariable analysis in Mathematics 2 which is continued in the linear algebra, Mathematics 3, continued in Mathematics 4 and beyond that to Mathematics 5, the course in vector fields, Stokes' theorem, and partial differential equations. This sustained development of multivariable techniques with the substantial support of linear algebra at the right place should do much to improve the student's mastery of them. As a local variation the linear algebra could be placed in any one of the slots for Mathematics 2, 3, 4, or 5, or intermixed with the calculus topics. While no body of calculus material necessarily requires a prior formal study of linear algebra, this can be used with great effectiveness in both multivariable calculus and linear differential equations.

Our arguments for placing a formal course in linear algebra in the first semester of the second year are more concerned with the values of the subject itself and its usefulness in other sciences than with linear algebra as a prerequisite for later semesters of calculus. Let us first consider prospective mathematics majors. Their official commitment to major in mathematics is usually made before the junior year of college. It is desirable that this decision be based on mathematical experience which includes college courses other than analysis. For these students linear algebra is a useful subject which involves a different and more abstract style of reasoning and proof. The same contrasts could be obtained from other algebraic or geometric subjects but hardly with the same usefulness that linear algebra offers.

The usefulness of linear algebra at about the stage of Mathematics 3 is becoming more and more apparent in physics and engineering. In physics it is virtually essential for quantum mechanics which is now being studied as early as possible in the undergraduate curriculum, especially in crystal structures where matrix formulation is most appropriate. In engineering, matrix methods are increasingly wanted in the second year or earlier for computation, for network analysis, and for linear operator ideas. The basic ideas and techniques of linear algebra are also essential in the social sciences and in business management. Students in these specialities are best served by an early introduction to the material in Mathematics 3.

We think, however, that Mathematics 3 is about the earliest stage at which the subject can profitably be taught to undergraduates generally. It can be taught to selected students in high school, though the high school version of the subject tends to be somewhat lacking in substance. High school students do not have a sufficiently broad scientific or mathematical background to motivate it and have not yet reached the stage of their curriculum when they can use it outside the mathematics classroom.

It is our view that the best way to handle introductory differential equations is to embed the subject in the elementary analysis sequence as something less than a three-hour course. We have provided about 24 lessons for differential equations in Mathematics 2, 4. With some decline in the importance of integrating in closed form in favor of numerical methods, some of the time formerly spent on integration technique can profitably be allocated to finding the unknown function in simple differential equations. Technically this is a collection of tricks which subsumes integration in closed form as a special case, but the differential equations ideas are richer in applications and scientific interpretations. When this first exploratory phase of the study of differential equations is absorbed into the calculus it sets the stage for the first formal course in differential equations to have a better theoretical structure than it has as the conventional introductory differential equations designed to follow calculus.

When introductory differential equations is included, the elementary analysis sequence, Mathematics 1, 2, 4, with the usual emphasis and conventional texts may require 12 semester-hours. We believe that it is possible with economies to complete it in 9 semester-hours, but we concede that this is somewhat conjectural. Therefore in situations where courses have only three hours it may be necessary to extend the Mathematics 1, 2, 3, 4, sequence into the third year. Although this device is a good safety valve, in this report we will consider that our elementary sequence Mathematics 1, 2, 3, 4, has 12 to 15 semester-hours and <u>includes</u> introductory differential equations, whether the sequence is extended into the third year or not. Hence we make no provision for the introductory ordinary differential equations in our advanced undergraduate sequences, though our suggested outline for Advanced Multivariable Calculus, Mathematics 5, includes a first introduction to partial differential equations.

4. <u>Machine computation</u>. The prevalence of the high-speed automatic computer affects the teaching of mathematics in a very general way. Many mathematically trained students will work closely with computers, and even those who do not should be taught to appreciate the type of algorithmic approach that enables a problem to be handled by a machine. This point of view should therefore be presented, along with the more classical one, at appropriate places in calculus, differential equations, linear algebra, etc. For the benefit of those students who eventually do serious computing (perhaps 25 percent of mathematics majors fall in this category), a semester course in numerical analysis is very helpful.

If there is a computer on the campus, or if one is otherwise accessible, it is likely that elementary programming instruction will be available to the students early in their academic careers. This, in turn, makes it possible to take advantage of the computer throughout the mathematics program, and material should be presented to make use of this opportunity. If the courses have already been modified as indicated above, this material might consist merely of some additional problems to be done on the computer.

The introduction of the general student to machine computation, and the adaptation of mathematics courses to use it, is subject to continuing study in CUPM as it is in the mathematical community.[3] (The training of career specialists in computation is a separate matter, treated briefly in Chapter V, Section 3 of this report and in more detail in a separate CUPM report, loc. cit. p. 6.)

3. CUPM expects to have more to say about this in a future report.

## III. STRUCTURE OF THE ENTIRE PROGRAM.

1. Lower division courses.

CUPM proposes that the mathematics department offer the following list of semester courses every semester.

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|---|---|
| Mathematics 1: | Introductory Calculus |
| Mathematics 2P: | Probability |
| Mathematics 2, 4: | Mathematical Analysis |
| Mathematics 3: | Linear Algebra |

2. Upper division courses.

The following list of typical courses might be offered once a year, or in some cases, in alternate years, to meet the needs of mathematics majors of various types, advanced placement students who are still in their second year of college, or majors in other fields requiring advanced mathematical training. All have Mathematics 1, 2, 3, 4, but not 2P, as prerequisites except where indicated. Locally, these upper division courses will often be combined into year courses. The minimal mathematics major program should include at least six semester courses from the list of upper division courses. (See Chapter IV below.) The order is a rough indication of the level. (Sample outlines appear in Chapter VII.)

Mathematics 5. Advanced Multivariable Calculus. Vector fields, Stokes' and Green's theorems, introduction to partial differential equations.

This course in analysis, but of strongly geometric character, is necessary for mathematics majors whose interests are in pure or applied mathematics, and for physical science and engineering students. It is much less appropriate for social science or statistics students, and has no particular relevance for secondary school teachers.

Mathematics 6. Algebraic Structures. Groups, rings, fields, and more advanced linear algebra.

The material is essential for pregraduate mathematics students and useful for computer science students as well. Prospective high school teachers need it. Many physical science students now are finding it important.

Mathematics 7. Probability and Statistics. Moments of distributions and Stieltjes integrals; joint density functions; conditional means; moment generating functions; sequences of random variables; Markov chains; stochastic processes. Statistical inference; estimation; deciding between hypotheses; relationships in a set of random variables; regression; analysis of variance; design of experiments.

A number of different courses fall within this heading. They vary by the degree to which they emphasize data analysis and by the weights (some of which may be zero) given to the topics enumerated above. The particular course offered should meet the needs of students and reflect the training of the instructor. The course builds on the background obtained in the basic sequence and in Mathematics 2P. Minimum prerequisites are Mathematics 1, 2, 3, 2P. Depending upon the choice of topics and emphasis, Mathematics 4 may also be required.

This course is elective for mathematics majors, essential for pre-graduate statistics and mathematically oriented biological, management, or social science students. It is now considered important for engineering students, particularly in communication fields or industrial engineering. Theoretical physicists and chemists also find it desirable.

Mathematics 8. Numerical Analysis. Numerical integration and numerical solution of differential equations. Numerical methods in linear algebra, matrix inversion, estimation of characteristic roots. Error propagation and stability. Oriented towards machine computation.

This course is elective for mathematics majors and important for the large number of students who expect to enter the computation field. It is basic for computer science majors and useful for physical science and social science students.

Mathematics 9. Geometry. A single concentrated geometry theory from a modern axiomatic viewpoint, and not intended to be a descriptive or survey course in "college geometry." If the college undertakes the training of prospective secondary school teachers, the essential content of this course is Euclidean geometry. A more widely ranging full year course in the same spirit is desirable if it is possible. Other subjects which offer the opportunity of the appropriate depth include topology, convexity, projective geometry, and differential geometry. With the decline of analytic geometry in the first year a serious introduction to geometric ideas and geometric proof is valuable at about this stage for all undergraduates concentrating in mathematics. For the purposes of applied mathematics the differential geometry form of this course is most often the essential one.

Mathematics 10. Applied Mathematics. A course to illustrate the principles and basic styles of thought in solving physical problems by mathematical methods. Content depends upon the people involved.

The offering of this course depends upon the availability in the faculty of someone who understands this difficult subject thoroughly. Without such a person it is probably wise to omit the course. Such a course, when available, is recommended for all mathematics majors so that they can acquire a feeling for the nature of applied mathematics and, with less emphasis, for students in other fields. In some institutions a good course in theoretical mechanics taught in the physics or engineering department might be an acceptable substitute.

Mathematics 11, 12. Introductory Real Variable Theory. Preferably a year-course but if necessary may be offered in a one-semester version or combined with complex analysis in a one-year course. A course in which the student learns to prove the basic propositions of real variable theory: those concerned with sets of real and complex numbers, particularly Dedekind completeness in the reals, Archimedean property, set terminology, cardinality; topology and limits in metric spaces, Euclidean spaces, Borel-Lebesgue theorem and its consequences, Cauchy construction, continuity, uniform continuity, continuous images of compact sets, algebra of continuous functions, differentiation, Taylor's theorem with remainder, implicit function theorem. Second semester: Riemann-Stieltjes integration, series of numbers, series of functions, Weierstrass approximation theorem, series expansions in powers and in orthonormal systems, Fourier series.

This kind of course is highly desirable for pregraduate mathematics students, who, upon completion of this course should be ready to begin a graduate course in measure and integration or functional analysis. The topics and skills are basic in such fields of analysis as differential equations, calculus of variations, harmonic analysis, complex variables, probability theory, and many others. For undergraduates, an extensive coverage of subject matter, especially in the directions of abstract topologies and functional analysis, may be sacrificed in favor of active practice by the student in proving the theorems.

Mathematics 13. Complex Analysis. The complex numbers, elementary functions and their mappings. Complex limits and power series. Analytic functions. Conformal mapping and boundary value problems. Contour integrals, Cauchy's theorem and integral formula. Taylor and Laurent expansions, residues, analytic continuation, integral transforms and their inversion identities, maximum modulus theorem, Liouville's theorem.

Many prefer to have this course precede the real variables. It is important for mathematics majors, engineering students, applied mathematicians, theory-oriented physics and chemistry students.

Although we have described the upper division mathematics in semester courses, as part of the general plan for this report described in the introduction, we point out the possibility of handling these advanced subjects by independent study, directed study, tutorials or seminars. This is especially applicable in a small college where it may not be possible to organize classes in every needed subject. It is also applicable when suitable textbooks do not exist for the courses described. Experience with this form of teaching for talented young students has been quite good. As part of the curriculum work following this report CUPM hopes to recommend some reading materials by which these subjects can be learned in directed study.

IV. REASONS FOR CHOOSING THIS FORM OF SOLUTION.

1. Multiple track flexibility. In our opinion the best possibility for accommodating the diversity of incoming students, for providing them with suitable starting points and multiple track programs with outputs leading to graduate work in mathematics, or sciences, graduate or professional work in social sciences, business or medicine, careers as applied mathematicians of many kinds, or computer scientists, or secondary school teachers, or for just getting enough education in mathematics to be informed in the modern world, lies in offering an open stock assortment of the smallest building blocks of educational programs, semester-courses.

There are in general two ways to provide for the student who enters college with superior ability or advanced high school achievement or both. One can teach such a student the same material as the ordinary students but in a special way designed for him. This requires special honors courses which may be very good where they are possible. Another way to accommodate a student who comes with advanced capability is to offer him advanced placement. It is possible, with our flexible set of offerings and within our restricted economy, to place an entering student in Mathematics 2, 2P, 3, or 4, whichever is appropriate.

Exit points are also frequent. Thus the program provides a more suitable compromise between the whole of calculus and no calculus than does the conventional course structure.

Here are some typical course sequences it provides.

(a) Two six semester-hour programs for liberal arts students, majors in the social and behavioral sciences, and business administration students: Mathematics 1, 2 or Mathematics 1, 2P.

(b) An advanced placement two-year sequence for mathematics or physical science students: Mathematics 2, 3, 4, and 5 (or 2P, or 7) as appropriate.

(c) A mathematics major program for students bound toward graduate mathematics: Mathematics 1, 2, 2P, 3, 4, 5, 6, 10, 11, 12, 13. A stronger major would be desirable but this is adequate to enter good graduate schools at the present time. With two semesters advanced placement the student still has Mathematics 7, 8, 9, from which to complete a better major.

(d) A mathematics major program for applied mathematicians:[4] Mathematics 1, 2, 2P, 3, 4, 5, 6, 8, 10, 11, and 12 or 13. This exhausts the maximum total number of courses we have budgeted for a major in these sequences. The CUPM ad hoc committee on applied mathematics desires a much heavier concentration including Mathematics 9 (as Differential Geometry), a year of Applied Mathematics 10 including some topic in depth, Mathematics 7 or a course in ordinary differential equations, and <u>Introduction to Computer Science</u> (described in Chapter V, Section 3, below).

(e) A program for prospective graduate students in the biological sciences, the social sciences, or business administration: Mathematics 1, 2P, 2, 3, 4, 7. (Should also include <u>Introduction to Computer Science</u>. See Chapter V, Section 3.) Substantially meets the BMSS Panel recommendations.

(f) The mathematics content of a program for program for prospective computer scientists (not mathematics majors) is: Mathematics 1, 2, 2P, 3, 4, 6, 7, 8.

(g) A physics major bound for graduate work in physics would find good support in: Mathematics 1, 2, 3, 4, 5, 2P, 10, 13. Mathematics 6, 7, 8, 11, 12, would also be appropriate for him.

(h) The minimal program recommended for mathematics teachers of grades 9-12 by the Teacher Training Panel of CUPM can be met by: Mathematics 1, 2, 3, 4, 6, 2P, 7, 9, and two semester electives, provided that Mathematics 9 is a year course.

These sequences serve to illustrate the flexibility of the proposed offerings.

2. <u>Advanced placement.</u> The number of students now entering college with a calculus course completed in high school is of the order of 1%, about three or four per year in an entering class of 300, if the college gets its share. This is not enough to make up a special honors program on any but the costly tutorial basis.

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4. The ad hoc committee on applied mathematics will soon report in detail on undergraduate programs in applied mathematics. As was the case in pregraduate training, their report will necessarily be more specialized and advanced than this report.

Comparatively few of these high school graduates with calculus can qualify for a whole year's advanced placement, so that advanced placement recognition of these able and ambitious students and their teachers will be slow in traditional college programs. We may assume that high schools which teach calculus will teach a year course and an appreciable proportion of the graduates will qualify for two semester's advanced placement in college, but several times as many will qualify for one semester's advanced placement. Thus it will promote the early study of calculus if colleges will offer advanced placement in one-semester as well as two-semester units.

In this way, while the high school calculus movement grows rapidly, as it is expected to do, the one-semester at a time advanced placement afforded by our proposed system of course units will permit an orderly transition of the first unit of calculus to the high schools and then two semester units, and so on. Thus, as the high school program advances, the sequence we propose will provide smooth and flexible progression of lower college subjects to high school while it takes care of individual students by advanced placement, all without the necessity of periodically reconstructing the college program. The first step in this progression is, of course, the completion of the transfer of Mathematics 0 to high school.

3. Economy. Let us examine how this general college curriculum might work out in a department whose staff is large enough to teach 12 to 14 sections at one time. A typical first semester might look like the following:

| | |
|---|---|
| Mathematics 1 | 3 sections |
| Mathematics 2 | 1 section (small) |
| Mathematics 2P | 1 section (small) |
| Mathematics 3 | 2 sections |
| Mathematics 4 | 1 section (small) |
| Lower division mathematics | 8 sections |
| Upper division and service mathematics, without teacher training. | 4 to 6 sections |
| Total | 12 to 14 sections |

Relatively few colleges in the United States that have a reasonably broad spectrum of students have departments so small that they could not handle this number of sections; but they exist and, for them, we concede that our proposed program would have to be modified. Moreover, the allocations above do not take adequate account of the service load of Mathematics 0, and of special courses, when the department is committed to the training of teachers while offering a general curriculum in mathematics at the same time. Such extra instruction would require staff members in addition to those already engaged in the general curriculum in mathematics.

4. Rigor. Even if it were possible to make a precise statement about the level of rigor we expect in this program, that would be undesirable, for the appropriate level of rigor varies from classroom to classroom and we should not attempt to dictate what it should be. Let us agree that it is the level of rigor in the student's understanding which counts and not only the rigor of the text or lecture presented to him. Our definite proposal is that the advancement of a student from the first calculus course, Mathematics 1, to the second shall not depend upon his being able himself to make rigorous epsilon-delta proofs. The problem, as we see it, is to devise a presentation which, while not strictly logical at all stages, will nevertheless be intelligible, and which will convey the ideas of the calculus in forms which are intuitively valid, can later be made exact, and are made rigorous as the student advances.

We shall make no attempt to give a definition of what we mean by a valid first exposition. It seems better to give some illustrations.

(1) It was known to the ancients that the area of a circle is given by the formula

$$A = \pi r^2.$$

We realize today that this formula does not have a meaning until area is defined; and Jordan measure, to which the ancients tacitly appealed, was not defined until the nineteenth century. Nevertheless we cannot deny that meaningful mathematics was based on the formula well before that time. Thus mathematical understanding is a matter of degree, and some kinds of informal understanding can be adequate for a long time.

(2) In a similar way, a student may get along, at least for a while, without the formal definition of a limit. But limits, and all other concepts of the calculus, should be taught as concepts in some form at every stage. For example, the fundamental theorem

of the integral calculus involves two concepts: the "limit" of a sum and the antiderivative. Supposing that f is continuous and

$$\int_a^b f(x)dx$$

has been defined by approximating sums, the theorem states that

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F' = f$. There is, to begin with, no obvious relation between the two sides of this equation, and only a proof can make it credible. One natural proof depends on proving, starting from the definition of an integral, that if

$$G(x) = \int_a^x f(t)dt$$

then $G' = F' = f$, whence G and F differ by a constant, which can only be F(a). Thus a simple test to determine whether a student understands the fundamental theorem is to ask him to differentiate

$$G(x) = \int_0^x \sqrt{1+t^8}\, dt.$$

If he does not know how, he does not understand the theorem. It is dishonest to conceal the connection between the two concepts by conditioning the student to accept the formalism without his being aware that the concepts are there. On the other hand, to give the student only the concepts without making him fully aware of the formalism is to lose sight of the aspect of calculus that makes it such a powerful tool in applications as well as in pure mathematics.

(3) A "cookbook" would teach students to solve maximum problems by setting the derivative equal to zero and accepting the result, perhaps after "testing" to see that the second derivative is negative. A thoroughly rigorous book would demand careful proofs of the existence of a maximum of a continuous function, Rolle's theorem, and so on. What we suggest for the first calculus course is a clear statement of the problem of maximizing a function on its domain, a precise statement of such pertinent properties as the existence of the maximum, and the examples to indicate that the maximum, if it exists, is to be sought among endpoints, critical points, and points where the derivative does not exist.

(4) These remarks are meant as descriptions of the style in which the study of calculus should begin, not of the style in which it should end. One of the ultimate objectives is an exact mathematical understanding; and the question as we see it, is not whether we should

achieve this, but rather when, and how, and after what preliminaries. This is a delicate question, and we have no simple answer to offer. One possibility is a "spiral" technique, in which every concept is attacked several times. Or we might proceed informally up to a certain point, and then formalize. But in any case, we should start where the student really is, and proceed to where he should be. Mathematical sophistication should be introduced little by little.

Specifically in the analysis sequence of the first two years which we are recommending we suggest that a student who stops with Mathematics 2 will at least know something useful, have enough technique to use it, and will have a grasp of the basic concepts of calculus in a form as precise as he can appreciate and precise enough so that it can be made rigorous.

One who goes through Mathematics 4 will have considerably developed his capacity to understand mathematical ideas exactly and to reason deductively himself; one who has worked his way through our Introductory Real Variables, Mathematics 11, 12, will have a still deeper insight, but he will probably still be naive by the logician's standards. In particular we consider it important that before a student declares a major in mathematics at the end of his second year he should have some real understanding of what it means to think as a mathematician.

Finally, there is a still further reason for this intuitive start growing more rigorous as the two-year sequence unfolds. We are interested in providing for the beginning to be made in high school. High school teachers are likely to be better than the college teachers at developing intuitive and manipulative skills with their small classes meeting five times a week, while the college teacher is more likely to be oriented towards the theory, lecturing as he does with fewer class meetings and less drill. Hence the best calculus teaching team may well be one in which the high school teacher provides the first introduction and the technical drill, while the college teacher provides the second and deeper look at the subject. With these statements on the constraints governing the selection of the approach to rigor we leave it to teachers to decide the matter for themselves, knowing that they will vary widely.

V. PROGRAMS FOR SPECIAL CONDITIONS.

1. One-year mathematics to satisfy a B.A. degree requirement. We have not described a special year-course in mathematics appreciation for students in liberal arts colleges. We think it is better for the student to take Mathematics 1, 2 or Mathematics 1, 2P. These ways of satisfying a liberal arts requirement open more doors for the student than any form of terminal appreciation course. We observe that the student may choose Mathematics 2 or Mathematics 2P after his performance in Mathematics 1 is established. In particular many students who are oriented towards social science will choose Mathematics 2P. A student who is required to take mathematics and is not prepared to enter Mathematics 1 might take Mathematics 0, 1. The one-semester calculus, Mathematics 1, as described makes this a satisfactorily complete sequence and still leaves the way open for the student to continue with Mathematics 2 or 2P if he so desires.

For those students for whom even the Mathematics 0. 1 sequence is not possible or appropriate there are two general possibilities. In many colleges students have been taking and will continue to take a year-course of material like that in Mathematics 0. This can be done in an intellectually satisfactory form, and still leave the door open to the analysis sequence, though it is our view that such a year-course is largely remedial in character, repeating in college mathematical subjects which should have been mastered in an accredited high school program. This type of precalculus analysis may be the best solution for students who have not learned mathematics as rapidly as the advanced college preparatory group of students but have continued the study of college preparatory mathematics through four years of high school in a "sequential program." These students appear in technical junior colleges in considerable numbers. These remarks do not have the force of a recommendation, since CUPM has not yet considered in detail this important curricular problem.

Among the remaining college students for whom neither precalculus analysis nor calculus is appropriate we recognize a sizable number who are preparing for elementary school teaching. Their needs should be met by special courses discussed in paragraph 2 below.

The remaining rather large category of students whose preparation is inadequate forces the college mathematics department to choose between undesirable alternatives. The students we are considering include many who were admitted to college under state law after graduating from high school with only two units of "General

Mathematics." Generally speaking, one cannot isolate their needs, if any exist, for college mathematics. Their motivation to study it is usually poor or negative. When the college mathematics department has to decide what to do with such students it may consider three general approaches:

a) Provide deep remedial instruction of level below Mathematics 0;

b) Use a humanistic approach to present a cultural course about mathematics;

c) Do not teach mathematics to these students.

CUPM cannot make a recommendation of its own except to suggest that for many colleges at the present time the last alternative should be considered seriously. That is, do not try to teach mathematics to these students in college, assuming instead that the mathematical component of their general education was completed in high school. One reason for preferring this to a) above is that manpower shortages in college mathematics teaching are so serious that there is doubt that we can afford to undertake remedial instruction in the face of the small yields to be expected from it. Moreover, colleges should not relieve high schools of their responsibility to do their share of the mathematics teaching by doing it for them.

Others have considered the cultural mathematics course for liberal arts students who have a distribution requirement for the bachelor's degree to satisfy in mathematics.[5] Again CUPM has not studied this problem, not only because of the previously mentioned teaching manpower shortages, but also because we are not convinced that most mathematics teachers are equipped to do good teaching in a humanistic style, even if it is desirable, or that students can be motivated to profit from it under any but a most exceptional teacher.

2. Elementary teacher training. Many colleges have an obligation to train elementary school teachers, and the general college curriculum in mathematics here described will not accomplish this

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5. Alder, Henry L. An address to the Association at Amherst, Massachusetts. MAA Monthly, January 1965, pp. 60-66.

Jones, B. W. Elementary Concepts of Mathematics, 2nd ed., Macmillan Co., New York, 1963.

Kline, Morris. Mathematics in Western Culture, Oxford University Press, 1953.

Richardson, Moses. Fundamentals of Mathematics, Macmillan Co., New York, 1958.

Stein, Sherman. Mathematics, The Man-made Universe, Freeman and Co., 1963.

purpose. We urge that such colleges offer the full four semesters of course work especially designed for elementary teachers ("Level I") on the structure of the real number system, basic concepts of algebra, and informal geometry.[6]

3. Training of career specialists in computer science. The training of career specialists in computer science is not a service which a mathematics department should be expected to perform by itself. Computer science is not mathematics, though the two subjects are closely linked. If a college wishes to make a start in training students who are skillful in the use of the computer, and can staff such a course effectively, it should offer:

Introduction to Computer Science. (3 semester hours) First year level. Description of the computer and its logical structure, functions of the parts of a computer. Algorithms, programming languages, problem solving in numerical and non-numerical situations.

This course and the courses Mathematics 1, 2, 3, 4, 6, 2P, 8, 11, 13 will meet the mathematics major requirements suggested in the CUPM "Recommendations on the Undergraduate Mathematics Program for Work in Computing," May, 1964.

6. American Math. Monthly, Vol. 67, No. 10, December, 1960. See also CUPM "Course Guides for the Training of Teachers of Elementary School Mathematics."

VI. IMPLEMENTATION.

1. Textbooks. The pivotal introductory calculus of the elementary functions, Mathematics 1, complete in one semester, departs somewhat from the conventional calculus textbook organization and will be hard to teach from big, complicated, conventional texts. There are some simpler and more compact texts now available which give a fair approximation to it but we need new texts with suitable following text materials for Mathematics 2, 3, 4, to get the best results. These have long been needed anyway, for calculus courses have become big packages of eight to twelve semester hour length which are accessible only to a rather limited clientele of students who have the required technical ability, the prerequisite training, the time and the interest to undertake such a large commitment. We hope that authors will take note. The population wishing to study calculus has increased several times over but the demand is for shorter, simpler, less technical, and not so all-or-nothing versions of the subject.

At some sacrifice of flexibility, advanced placement, and other advantages, the elementary analysis unit as a whole, that is, Mathematics 1, 2, 4, can be covered with conventional texts much more easily than it can in the special packaging into semester units that we have designed here. In this approximate sense the proposed program need not wait on new texts for implementation.

2. Adoption of the program. CUPM believes that the wide adoption in colleges of course programs conforming to this general pattern will go a long way towards the elimination of the confusion which now exists, will speed up the process of orderly transfer to high school of mathematical subjects which can be appropriately taught there, will leave the colleges at all stages of this transfer with a workable program, and will take care of both the individual students from schools where the new improved mathematics programs have taken hold, as well as the students from secondary schools where only the old-style preparation is yet available.

Instead of trying to secure national adoption of our program specifically as we conceive it, we invite colleges to adapt this program outline to their own needs and to publish their own detailed course outlines with comments on text materials used and classroom experience, particularly with reference to the handling of students who come with superior high school backgrounds. Such an exchange of information and such a broad distribution of the work will do much more than CUPM can accomplish with conferences, writing sessions, consultants, and recommendations, to reap the harvest of the pioneer experimental work which has already been done.

## VII. SAMPLE OUTLINES OF THE COURSES.

The following course outlines are intended in part as extended expositions of the ideas that we have in mind, in part as feasibility studies, and in part as proposals for the design of courses and textbooks. The intent is not prescriptive. For this reason we have given more than one outline for some of the courses. Some outlines, which we would have liked to include, are missing. Since these are not recommendations, we did not feel that it was necessary to present outlines of every course. Some of the courses, but not all, are accompanied by bibliographies. In these cases, the books are not being proposed as texts, but merely as indications of the intellectual content and spirit that we have in mind. In general, we gave citations which would be helpful for the design of courses implementing the outlines.

Some of the outlines may seem conservative. In any case, it should be understood that nothing in these outlines is intended to discourage original course design or further experimentation.

Mathematics 0.

a. Definition of function and algebra of functions. (5 lessons) Various ways of describing functions; examples from previous mathematics and from outside mathematics; graphs of functions; algebraic operations on functions; composition; inverse functions.

b. Polynomial and rational functions. (10 lessons) Definitions; graphs of quadratic and power functions; zeros of polynomial functions; remainder and factor theorems; complex roots; rational functions and their graphs.

c. Exponential functions. (6 lessons) Review of integral and rational exponents; real exponents; graphs; applications; exponential growth.

d. Logarithmic functions. (4 lessons) Logarithmic function as inverse of exponential; graphs; applications.

e. Trigonometric functions. (10 lessons) Review of numerical trigonometry and trigonometric functions of angles; trigonometric functions defined on the unit circle; trigonometric functions defined on the real line; graphs; periodicity; periodic motion; inverse trigonometric functions; graphs.

f. Functions of two variables. (4 lessons) Three-dimensional rectangular coordinate system; sketching graphs of $z = f(x, y)$ by plane slices.

Mathematics 1. Introductory Calculus. ( 3 or 4 semester hours) (First version) [Prerequisite: Mathematics 0.]

The purpose of this course is to introduce the ideas of derivatives and integrals with their principal interpretations and interrelations and to develop the simpler techniques of differentiation and integration for the elementary functions studied in Mathematics 0.

a. The integral and derivative. (6 lessons) Step functions and their integrals, the exact area under a monotonic curve, the definite integral. Limit of a function, slope of a curve, tangents, instantaneous rates, numerical derivatives. Continuity of addition, multiplication, and inverse; algebra of limits, existence of limits, continuous functions.

b. Differential calculus of polynomials and rational functions. (9 lessons) Derived functions, D(u+v), Dcu, Duv, $Du^n$, differentiation of polynomials. Interpretation and applications of derivatives: slopes, rates, sign of the derivative, acceleration, scale factor in mapping, marginal quantities in economics. Mean value theorem, increasing functions, the differential equation DF(x) = 0. Taylor's theorem, higher order derivatives, cut-off Taylor's series as approximations, the second derivative, convexity, constant acceleration, maxima and minima. Differentiation of rational functions, maxima and minima with side conditions.

c. Antiderivatives and integrals of polynomials. (8 lessons) The antiderivative, differential equation $DF(x) = f(x)$, antiderivative formulas, fundamental theorem of calculus. Integration by parts, algebraic properties of integrals, applications, area, mean value of a function, falling bodies, work, volumes, moments.

d. Logarithms and exponentials. (6 lessons) Antiderivatives of negative powers, applications, introduction of log x as the missing antiderivative of $x^{-1}$. Theory of logarithms and general powers. Exponential functions. Applications.

e. Calculus in Euclidean geometry. (4 lessons) Tangent and normal, orthogonal curves, angle between two curves, circle and parabola, focal property. Arc length, polar coordinates.

f. Trigonometric functions. (6 lessons) Derivatives and integrals of trigonometric functions. Periodic motion, simple harmonic motion. Inverse trigonometric functions, $e^{i\theta}$.

This completes 39 lessons for a 3 semester-hour course.

g. Applications and extensions. (Optional) (1-14 lessons) Differences and approximations, differentials, related rates, calculus in economics, least squares, damped motion, chemical kinetics, empirical methods, numerical methods: trapezoidal rule, Simpson's rule, Newton's method.

Mathematics 1. Introductory Calculus. (3 semester hours) (A second version)

Differential Calculus.

a. Introduction. (7 lessons) The ideas of the differential calculus are introduced by the intuitive solution of an extreme value problem. The slopes of the graph of $x^2$ and of a general quadratic function are calculated. Velocity and rate of change. Calculation of derivatives of $1/x$, $\sqrt{x}$; derivative at $x = 0$ of $f(x) = x \sin (1/x)$ for $x > 0$, $f(0) = 0$, attempted inconclusively.

Limit and approximation. Limits of sums, products, quotients. Statements of extreme and intermediate value theorems. (Pictorial motivation; no epsilonics.)

b. Technique and applications of differentiation. (14 lessons) Differentiation of linear combinations, products, $x^n$, polynomials, quotients, rational functions. Concept of inverse function; derivatives of inverses, $\sqrt{x}$, $\sqrt[n]{x}$, rational powers. Vanishing of the derivative at an interior extremum, idea of global and local extrema. Theorem that $f(x)$ has an interior extremum on $(a,b)$ for continuous $f$ such that $f(a) = f(b)$ and theorem that between successive extrema a continuous function is strongly monotone (pictorial demonstrations using extreme and intermediate value theorems). Location of extrema at endpoints of the domain or points where the derivative fails to exist, or at the zeros of the derivative. Solution of extreme value problems. Curve sketching use of sign of 1st derivative, 2nd derivative. Sign tests for interior extrema. Derivatives of circular functions. Inverse circular functions. Chain rule. Derivatives of implicitly defined functions. Tangent and normal. Law of the mean (pictorial approach), tangent as best linear approximation.

Integral Calculus.

c. Area and integral. (8 lessons) Area as limit of Riemann sums, idea of integral, integrable function. Squeeze by upper and lower sums, error estimates for monotone and piecewise monotone functions. Integrals of af, f + g. Interpretation of integral as signed area. $\int_a^b + \int_b^c = \int_a^c$. Integral as function of upper endpoint, derivative of "indefinite" integral. Fundamental theorem.

d. Applications. (6 lessons) Volume of solid of revolution; falling body problem. Definition of log x as integral; exponential function. Differential equation for exponential function; growth and decay. Differential equation for sine and cosine; uniqueness of solution to initial value problem; simple periodic phenomena.

e. Techniques of integration and applications. (4 lessons) Simple substitutions. Integration by parts.

Mathematics 2P. Probability. (3 semester hours) [Prerequisite: Mathematics 1.]

a. Probability as a mathematical system. (9 lessons) Sample spaces, events as subsets, probability axioms, simple theorems, finite sample spaces and equiprobable measure as special case, binomial coefficients and counting techniques applied to probability problems, conditional probability, independent events, Bayes' formula.

b. Random variables and their distributions. (13 lessons) Random variables (discrete and continuous), probability functions, density and distribution functions, special distributions (binomial, hypergeometric, Poisson, uniform, exponential, normal...), mean and variance, Chebychev inequality, independent random variables, functions of random variables and their distributions.

c. Limit theorems. (4 lessons) Poisson and normal approximation to the bionomial, Central Limit Theorem, law of large numbers, some statistical applications.

d. Topics in statistical inference. (7-13 lessons) Estimation and sampling, point and interval estimates, hypothesis-testing, power of a test, regression, a few examples of nonparametric methods.

Remarks:

For students with only the minimum prerequisite training in calculus (Mathematics 1), about 6 lessons will have to be devoted to additional calculus topics needed in Mathematics 2P: improper integrals, integration by substitution, infinite series, power series, Taylor's expansion. For such students there will remain only about 7 lessons in statistical inference. Students electing Mathematics 2P after Mathematics 4 will be able to complete the entire course as outlined above.

Mathematics 2.4. Version completing single variable calculus before multivariable.

Mathematics 2. (3 or 4 semester hours) [Prerequisite: Mathematics 1.]

a. Differentiable functions. (9 lessons) Summary outline of introductory calculus, limits of functions, proof of general chain rule, implicit functions, inverse functions, methods of integration (substitution and partial fractions), inverse trigonometric functions, applications.

b. Limits. (15 lessons) Limits of sequences, existence of limit, indeterminate forms, Newton's method, the integral as a limit, numerical integration, applications of integration (volume--shell and slice), improper integral, series of positive terms, series of real and complex numbers, power series, Taylor series.

c. Theory of curves. (6 lessons) Parametric representations of curves, tangent, arc length, curvature, curvilinear motion, curves in polar coordinates, tangent and normal, area of a sector.

d. $y' = f(x,y)$. (9 lessons) The tangent field representation, numerical solution, power series solution, applications to mechanics.

This completes 39 lessons for three semester-hours. This material may well require four semester-hours.

Mathematics 2, 4. Version completing single variable calculus before multivariable.

Mathematics 4. Multivariable Calculus. (3 or 4 semester hours)

[Prerequisites: Mathematics 2 and Mathematics 3.]

a. Real valued functions of several variables. (12 lessons) Limits and continuity in $E_n$, differential of functions of several variables, chain rule, partial derivatives, gradient, law of mean, maxima and minima, implicit functions, Lagrange multipliers, Taylor's series, approximations, applications.

b. Multiple integration. (12 lessons) Definition of the integral $\int f(x)dm$, existence, interpretation as area, volume, mass, mean. Numerical evaluation. Centroids, moments. Evaluation by repeated simple integrals. Cylindrical and spherical coordinates.

c. Linear differential equations. (15 lessons) Equations of the form $Mdx + Ndy = 0$, exact differentials, integrating factors. Linear differential equations with constant coefficients, nth order equations, systems of differential equations, linear differential equations with variable coefficients. Special problems, laws of motion, conservation of energy, two-body problem.

This completes 39 lessons for three semester-hours. This material may well require four semester-hours.

Mathematics 2, 4. Version introducing multivariable calculus in Mathematics 2.

Mathematics 2. Multivariable Calculus. (3 or 4 semester hours)

[Prerequisite: Mathematics 1.]

a. Vectors in three dimensions. (6 lessons) Scalar and vector products. Equations of lines and planes. Applications to geometry and physics. Vector-valued functions of a real variable, curves, derivative, tangent and velocity, arc length.

b. Real-valued functions of several variables. (12 lessons) Graphical representations, level curves and surfaces. Surfaces of the form $F(x, y, z) = \text{constant}$, quadratic surfaces. Limits, continuity, partial derivatives. Differentiable functions, gradient. Differential. Chain rule, tangent plane to a surface, normal. Related rates. Directional derivative, gradient as maximal increase vector of the function and as normal to level surfaces. Repeated partial derivatives, Taylor's theorem with remainder. Approximations, estimates of the remainder.

c. Integration. (9 lessons) Definition of the integral $\int f(x)dm$, existence, interpretation as area, volume, mass, mean. Numerical evaluation of the integral. Centroids, moments. Evaluation by repeated simple integrals. Applications. Cylindrical and spherical coordinates. Curve integrals. Applications, potential.

d. Differential equations. (12 lessons) Tangent fields given by $y' = f(x, y)$. Meaning of solution of curves. Picard's method for establishing existence. Numerical step-by-step solution. Systems $y_i' = f_i(x, y_1, \cdots y_n)$, numerical solution. Special case $y' = f(x)$. Special numerical methods; trapezoidal rule, Simpson's rule. Equations of the form $Mdx + Ndy = 0$, variables separable, exact differentials, integrating factors. Linear differential equations of first order.

Mathematics 2, 4. Version introducing multivariable calculus in Mathematics 2.

Mathematics 4. Theory and Techniques of Calculus. (3 or 4 semester hours) [Prerequisites: Mathematics 2 in multivariable calculus version and Mathematics 3.]

a. Limits and continuity in $E_n$. (4 lessons) Law of the mean. More careful treatment of chain rule, implicit functions and differentials.

b. Techniques of differentiation and integration. (12 lessons) Complicated differentiation problems involving implicit functions, etc. Integration by substitution, partial fractions, and use of tables. Inverse trigonometric functions. Improper integrals. Maxima and minima for functions of one and several variables, Lagrange multipliers.

c. Limits of sequences. (12 lessons) Computational iteration processes, Newton's method. Convergence of monotone sequences. Infinite sequences of real and complex numbers, absolute convergence. Power series of real and complex numbers, expansions of functions. Series solutions of $y' = f(x,y)$. Indeterminate forms, L'Hospital's rule.

d. Differential equations. (11 lessons) Linear equations. nth order linear equations with constant coefficients, first order equations in a vector variable. Special problems, laws of motion, conservation of energy, two-body problem.

Preface to Linear Algebra

In this program linear algebra is designed to follow Mathematics 2 before Mathematics 4. With suitable modifications of Mathematics 4, the linear algebra could be taught in the fourth semester, after three semesters of calculus, but CUPM favors the earlier position.

When linear algebra is taught as early as the first term of the second year of college one should examine carefully the choice of topics. We need something intermediate between the simple matrix algebra which has been suggested for high schools and the more sophisticated Finite-Dimensional Vector Spaces of Halmos. The pieces of linear algebra which have demonstrated survival value at this curriculum level for many years are: Systems of linear equations and determinants from college algebra, uses of vectors in analytic geometry, and the calculus of inner products and vector cross products. More modern ideas call for the introduction of matrices as rectangular arrays with elementary row operations, Gaussian elimination, and matrix products; then abstract vector spaces, linear dependence, dimension, and linear transformations with matrices reappearing as their representations. Finally, of all subjects in undergraduate mathematics after elementary differential equations the one which has the widest usefulness in both science and mathematics is the circle of ideas in unitary geometry: orthogonality, orthogonal bases, orthogonal expansions, characteristic numbers and characteristic vectors.

The following course descriptions are made up of this material with certain other prejudgments: 1) The course should be as geometric as possible to offset its natural abstractness; 2) Determinants should be treated with all possible brevity; 3) The next topic to abbreviate under pressure of limited time are the abstract vector spaces and linear transformations, topics which in a senior course for mathematics majors one would treat in detail. With these remarks we proceed with the course outlines.

We present two outlines. The first version has an algebraic and computational beginning exploiting the linear spaces of n-tuples matrices, and Gaussian elementary row operations.[7] This beginnig has the virtue that it makes accessible elementary applications not covered by the strictly geometric view of vector spaces which suppresses the meaning of the individual components. The second version of Mathematics 3, more familiar to most mathematicians, emphasizes the geometric view from the beginning and comes to matrices only for the representation of linear mappings. Computations are less accessible in this version, as are certain types of applications.

Mathematics 3. Linear Algebra. (3 or 4 semester hours) (Version with algebraic beginning) [Prerequisites: Mathematics 1 and 2.] (The multivariable form of Mathematics 2 is best for this purpose.)

a. Linear equations and matrices. (5 lessons) Systems of linear equations, equivalence under elementary row operations, Gaussian elimination, matrix of coefficients, row reduced echelon matrix, computations, solutions, matrix multiplication, invertible matrices, calculation of inverse by elementary row operations.

b. Vector spaces. (6 lessons) Vector spaces abstractly defined. Examples. Linear dependence and independence. Linear bases and subspaces, dimension. Inner product, length, angle, direction cosines, applications to line and plane geometry.

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7. A good presentation of this introduction appears in Hoffman and Kunze, Linear Algebra, Prentice Hall, 1961, Chapter 1. This book as a whole is more advanced than the course intended here as Mathematics 3. Applications accessible at this early stage may be found in Block, Cranch, Hilton, Walker, Engineering Mathematics, Vol. I,II, Cornell University, 1964, also in Kemeny, Snell, Thompson, Finite Mathematics, Prentice Hall, 1956, and in Johnston, Price, Van Vleck, An Introduction to Mathematics, Vol. I, The University of Kansas, 1963. The last named also treats machine computations in matrix calculations in an elementary way.

c. Linear mappings. (10 lessons) Mappings, linear mappings, kernel and image of a map. Rank of a map. The linear map associated with a matrix. Representation of linear maps by matrices, composition of maps and multiplication of matrices. Algebra of linear mappings and matrices. Change of basis in a linear mapping, similar matrices.

d. Determinants. (4 lessons) Definition of a determinant of a square array, properties of determinants. Cramer's rule. Inverse of a matrix. Determinant of transposed matrix and of the product of two matrices.

e. Quadratic forms. (14 lessons) Symmetric matrices and quadratic forms. Quadric surfaces. Effect of linear transformation. Rational reduction to diagonal. Invariance of index, positive definiteness. Inner product, orthogonal bases, Gram-Schmidt orthogonalization. Orthogonal expansions and Fourier rule. Orthogonal reduction of 2 x 2 quadratic form, application to plane conics. The general case, orthogonal reduction, characteristic roots and vectors, invariance. Calculation of characteristic roots and vectors. Cayley-Hamilton theorem, trace, discriminant and other scalar invariant functions. Applications to analytic geometry.

This completes 39 lessons for three semester hours. For further applications or in a four semester hour course one may include the following topics.

f. Vector cross product. (4 lessons) Definition and geometric interpretation, algebraic properties. Uses in line and plane geometry.

g. Differential calculus of inner product and cross product. (5 lessons) Differentiation of vector products, applications to curves, arc length, curvature, normal and binormal, torsion. Angular velocity, angular momentum.

h. Groups of symmetries. (6 lessons)
or
Small vibrations of mechanical systems. (6 lessons)

Mathematics 3. Linear Algebra. (3 or 4 semester hours) (Alternate geometric version) [Prerequisites: Mathematics 1 and 2.]

a. Vector spaces. (3 lessons) Vector spaces abstractly defined. Examples. Linear dependence and independence. Bases and subspaces. Dimension of linear space.

b. Linear mappings. (8 lessons) Linear mappings, kernel and image of a map. Rank of a map. Linear maps as a vector space. Choice of basis in vector space and how it determines base in vector space of linear maps, representation of linear maps by matrices, similar matrices. Composition of mappings and multiplication of matrices.

c. Linear equalities and matrices. (5 lessons) Relation between linear mappings and systems of linear equations. Existence of solution of linear equations in terms of associated linear mapping. Equivalence under elementary row operations of equations and matrix, row reduced echelon matrix, explicit method for calculating solutions, invertible matrices, calculation of inverse by elementary row operations.

d. Linear inequalities and convex sets. (4 lessons) Linear inequalities and half spaces. Simplices and convex linear combinations. Convex polyhedra and convex sets. Separating hyperplanes. The simplex method of linear programming.[8]

8. This is an optional topic.

e. Inner product and norms. (4 lessons) Inner product, length, angle, direction cosines, applications to line and plane geometry. Norm as determined by convex symmetric body. Introduction of orthogonal bases and Gram-Schmidt orthogonalization process, orthogonal expansion and Fourier rule.

f. Determinants. (4 lessons) Definition and elementary properties of determinants. Criterion for invertibility of square matrix.

g. Quadratic forms. (11 lessons) Symmetric matrices and quadratic forms. Quadric surfaces. Effect of linear transformations. Rational reduction to diagonal. Invariance of index, positive definiteness. Orthogonal reduction of 2 x 2 quadratic form, application to plane conics. The general case, orthogonal reduction, characteristic vectors and roots, invariance. Calculation of characteristic roots and vectors. Cayley-Hamilton theorem, trace, discriminant. Applications to analytic geometry.

Mathematics 5. Advanced Multivariable Calculus. (3 semester hours) (Conventional version)

The differential and integral calculus of Euclidean 3-space, using vector notation, leading up to the formulation and solution (in simple cases) of the partial differential equations of mathematical physics. Considerable use can and should be made of the students' preparation in linear algebra.

a. Vector algebra. (4 lessons) Dot and cross products, identities. Geometric interpretation and applications. Invariance under change of orthogonal bases.

b. Differential vector calculus. (8 lessons) Functions from $V_m$ to $V_n$, continuity. Functions from $V_1$ to $V_3$, differential geometry of curves. Functions from $V_3$ to $V_1$, scalar fields, directional derivative, gradient. Functions from $V_3$ to $V_3$, vector fields, divergence, curl. The differential operator $\nabla$, identities. Expression in general orthogonal coordinates.

c. Integral vector calculus. (15 lessons) Line, surface, and volume integrals. Change of variables. Green's, divergence, and Stokes' theorems. Invariant definitions of gradient, divergence, and curl. Integrals independent of path, potentials. Derivation of Laplace's, heat, and wave equations.

d. Fourier series. (6 lessons) The vector space of square integrable functions, orthogonal sets, approximation by finite sums, notion of complete orthogonal set, general Fourier series. Trigonometric functions as a special case, proof of completeness.

e. Boundary value problems. (6 lessons) Separation of variables. Use of Fourier series to satisfy boundary conditions. Numerical methods.

References:

Apostol, T.M. Calculus: Volume II. New York, Blaisdell Publishing Company, 1964.

Buck, R. C. Advanced Calculus, 2nd ed. New York, McGraw-Hill Book Company, 1964.

Crowell, R.H. and Williamson, R.E. Calculus of Vector Functions, Englewood Cliffs, New Jersey, Prentice-Hall, 1953.

Churchill, R.V. Fourier Series and Boundary Value Problems, New York, McGraw-Hill Book Company, 1941.

Kaplan, W. Advanced Calculus, Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1952.

Protter, M.H. and Morrey, C.B. Modern Mathematical Analysis, Reading, Massachusetts, Addison Wesley Publishing Company, Inc., 1964.

Mathematics 5. Advanced Multivariable Calculus. (3 semester hours) (Alternate employing differential forms.)

A study of the properties of continuous mappings from $E_n$ to $E_m$, making use of the linear algebra in Mathematics 3, and an introduction to differential forms and vector calculus, based upon line integrals, surface integrals, and the general Stokes' theorem. Application should be made to field theory, or to elementary hydrodynamics, or other similar topics, so that some intuitive understanding can be gained.

a. Transformations. (15 lessons) Functions (mappings) from $E_n$ to $E_m$, for n, m = 1, 2, 3, 4. Continuity and implications of continuity; differentiation, and the differential of a mapping as a matrix valued function. The role of the Jacobian as the determinant of the differential; local and global inverses of mappings, and the implicit function theorem. Review of the chain rule for differentiation, and reduction to matrix multiplication. Application to change of variable in multiple integrals, and to the area of surfaces.

b. Differential forms. (6 lessons) Integrals along curves. Introduction of differential forms; algebraic operations; differentiation rules. Application to the change of variable in multiple integrals. Surface integrals; the meaning of a general k-form.

c. Vector analysis. (4 lessons) Reinterpretation in terms of vectors; vector function as mapping into $E_3$; vector field as mapping from $E_3$ into $E_3$. Formulation of line and surface integrals (1-forms and 2-forms) in terms of vectors. The operations Div, Grad, Curl, and their corresponding translations into differential forms.

d. Vector calculus. (8 lessons) The theorems of Gauss, Green, Stokes, stated for differential forms, and translated into vector equivalents. Invariant definitions of Div and Curl. Exact differential forms, and independence of path for line integrals. Application to a topic in hydrodynamics, or to Maxwell's equations, or to the derivation of Green's identities and their specializations for harmonic functions.

e. Fourier methods. (6 lessons) The continuous functions as a vector (linear) space; inner products and orthogonality; geometric concepts and analogy with $E_n$. Best $L^2$ approximation; notion of an orthogonal basis, and of completeness. The Schwarz and Bessel inequalities. General Fourier series with respect to an orthonormal basis. Treatment of the case $\{e^{inx}\}$ and the standard trigonometric case. Application to the solution of one standard boundary value problem.

References:

Buck, R. C. Advanced Calculus, 2nd ed. New York, McGraw-Hill Book Company, 1964.

Flanders, Harley. Differential Forms, with Applications to the Physical Sciences. New York, Academic Press, 1963.

Fleming, Wendell H. Functions of Several Variables. Reading, Massachusetts, Addison-Wesley Publishing Co., Inc., 1965.

Goffman, Casper. Calculus of Several Variables. New York, Harper and Row, Publishers, 1965.

Maak, Wilhelm. An Introduction to Modern Calculus, trans. by G. Strike. New York, Holt, Rinehart and Winston, Inc., 1963.

Munroe, M.E. Modern Multidimensional Calculus. Reading, Massachusetts, Addison-Wesley Publishing Co., Inc., 1963.

Purcell, E.M. Electricity and Magnetism, Berkeley Physics Course, Vol. II. New York, McGraw-Hill Book Company, 1965.

Mathematics 6. <u>Algebraic Structures.</u> (3 semester hours)

This course is intended to introduce the basic algebraic properties of groups, rings, and fields, culminating in the fundamental theorem of the Galois theory and some indication of its uses. The material from Mathematics 3 on linear transformations and matrices is used mainly to provide examples of groups and rings.

a. <u>Introduction.</u> (2 lessons) Sets, relations, functions, operations, etc. Algebraic systems: integers, rationals, matrices, etc. Isomorphism and examples. Equivalence classes.

b. <u>Groups.</u> (10 lessons) Definitions and examples: algebraic examples, geometric transformations, permutations, matrices. Subgroups, cyclic groups, basic theorems, Lagrange's theorem. Homomorphism, normal subgroup, quotient group. The isomorphism theorems. Composition series, the Jordon-Hölder theorem.

c. <u>Rings.</u> (7 lessons) Definition and examples: integers, matrices, polynomials, etc. Integral domains, fields, quotient field. Homomorphism, ideals, residue class rings. Isomorphism theorems.

d. <u>Unique factorization domains.</u> (7 lessons) Examples of failure of unique factorization. Euclidean domains, integers, Gaussian integers, and polynomials over a field as examples. Division algorithm, highest common factor, unique factorization in Euclidean domain. If R has unique factorization so has R[x].

e. Fields. (6 lessons) Prime fields and characteristics. Extension fields. Simple algebraic extensions, minimal polynomial, structure of simple extension. Finite extensions, degree, structure.

f. Galois Theory. (7 lessons) Galois group. Splitting field. Separable extensions. Normal extensions. Subfield and subgroup. The fundamental theorem. Finite fields. Solvability in radicals (mainly discursive).

References:


Birkhoff, G. and MacLane, S. A Survey of Modern Algebra, rev. ed. New York, Macmillan Company, 1965.

Barnes, W.E. An Introduction to Abstract Algebra. Boston, D. C. Heath and Co., 1963.

Mostow, G.D., Sampson, J.H., and Meyer, J.P. Fundamental Structures of Algebra. New York, McGraw-Hill Book Company, 1963.

van der Waerden, B. L. Modern Algebra, Vol. I, rev. ed. New York, Frederick Ungar Publishing Co., 1953.

Mathematics 7. Probability and Statistics. (3 semester hours) (Statistics emphasis) [Mathematics 2P is a prerequisite.]

This version emphasizes statistical theory and inference from data. Although only theoretical topics are listed, students should perform some sampling experiments and should see examples and do exercises using real data.

a. Estimation. (10 lessons) Point estimators: unbiasedness, consistency, efficiency, sufficiency, minimum variance. Maximum likelihood, Bayes' estimates. Sampling theory and interval estimation. Loss function and estimators to minimize expected loss. Prior and posterior distributions.

b. Decision theory and testing hypotheses. (12 lessons) Loss function and two decision problems. Tests of hypotheses, Neyman-Pearson lemma, power of a test. Likelihood-ratio method with example of contingency tables. Consideration of prior knowledge and consequence of the decision taken. Sequential analysis. Some non-parametric procedures: sign test, Mann-Whitney.

c. Relationships in a set of random variables. (5 lessons) Matrix methods used for n-variate normal. Independence, correlation, regression, and prediction in general and for n-variate normal.

d. Linear models and design. (12 lessons) Regression and components of variance models. Analysis of variance. Treatment, block, interaction effects. Design of experiments so that estimators of effects exist which are uncorrelated and have suitable variances.

Block and factorial designs.

References:


Brownlee, K.A. Statistical Theory and Methodology in Science and Engineering. New York, J. Wiley and Sons, 1961.

Brunk, H.D. An Introduction to Mathematical Statistics, 2nd ed. New York, Blaisdell Publishing Company, 1964.

Cramer, H. Mathematical Methods of Statistics. Princeton, New Jersey, Princeton University Press, 1946.

Fraser, D.A.S. Statistics: An Introduction. New York, J. Wiley and Sons, 1958.

Freeman, H.A. Introduction to Statistical Inference. Reading, Massachusetts, Addison-Wesley Publishing Company, 1963.

Graybill, F.A. An Introduction to Linear Statistical Models, Vol. I. New York, McGraw-Hill Book Company, 1961.

Hogg, R.V. and Craig, A.T. Introduction to Mathematical Statistics. New York, Macmillan Company, 1959.

Kempthorne, O. The Design and Analysis of Experiments. New York, J. Wiley and Sons, 1952.

Lehmann, E.L. Testing Statistical Hypotheses. New York, J. Wiley and Sons, 1959.

Lehmann, E.L. Notes on Theory of Estimation. Associated Students Store, University of California, Berkeley, September, 1950.

Scheffé, H. The Analysis of Variance. New York, J. Wiley and Sons, 1959.

Weiss, L. Statistical Decision Theory. New York, McGraw-Hill Book Company, 1961.

Wilks, S.S. Mathematical Statistics, 2nd ed. New York, J. Wiley and Sons, 1962.

Mathematics 7. Probability and Statistics. (3 semester hours) (Probability emphasis) [Mathematics 2P is a prerequisite.]

This version emphasizes analytic probability theory and stochastic processes.

a. Random variables and distributions. (14 lessons) Stieltjes integral and average values over distributions. Characteristic or generating functions with examples: binomial, Poisson, normal, gamma, t and F distributions. Conditional distributions and conditional mean. Distribution of functions of random variables, convolution. Particular case of bivariate and multivariate normal density.

b. Sequences of random variables. (5 lessons) Mean and variance of sum of random variables. Law of large numbers. Central limit theorem. Statistical applications.

c. Markov chains. (10 lessons) Transition probabilities and matrix. Classification of states. Ergodic properties. Random walk problems. Illustrative examples from engineering and the physical, biological, and behavioral sciences.

d. Stochastic processes. (10 lessons) Types of processes. Markov processes: Poisson, birth and death. Applications, for example, to theory of queues. Branching processes. Brownian motion.

References:

Bailey, N.T.J. The Elements of Stochastic Processes with Applications to the Natural Sciences. New York, J. Wiley and Sons, 1964.

Bharucha-Reid, A.J. Elements of the Theory of Markov Processes and Their Applications. New York, McGraw-Hill Book Company, 1960.

Feller, W. An Introduction to Probability Theory and its Applications, 2nd ed. New York, J. Wiley and Sons, 1957.

Fisz, M. Probability Theory and Mathematical Statistics. New York, J. Wiley and Sons, 1964.

Kemeny, J.G. and Snell, L. Finite Markov Chains. Princeton, New Jersey, D. Van Nostrand Company, 1959.

Papoulis, A. Probability, Random Variables, and Stochastic Processes. New York, McGraw-Hill Book Company, 1965.

Parzen, E. Modern Probability Theory and its Applications. New York, J. Wiley and Sons, 1960.

Parzen, E. Stochastic Processes. San Francisco, California, Holden-Day, Inc., 1962.

Rosenblatt, M. Random Processes. New York, Oxford University Press, 1962.

Saaty, T.L. Elements of Queueing Theory with Applications. New York, McGraw-Hill Book Company, 1961.

Mathematics 8. Numerical Analysis. (3 semester hours)

a. Solution of equations. (6 lessons) Functional iteration of (nonlinear) equations, including convergence theorems, error effects, analysis of special methods such as the methods of false position and of Newton; iteration for systems of equations, methods of Bernoulli, Sturm, Graeffe, etc. for finding roots of polynomial equations.

b. Polynomial approximations; interpolation and quadrature. (18 lessons) Weierstrass theorem, Bernstein polynomials, Lagrange interpolation with error formulas, least squares, orthonormal systems relative to given weight functions; concepts and analysis of best approximation relative to given criteria--Chebychev polynomials, trigonometric approximations. Differencing, interpolation schemes, and formal difference calculus; quadrature formulas of the interpolation and Gaussian types with an analysis of error; numerical quadrature for improper integrals.

c. Initial value problems for ordinary differential equations. (9 lessons) Reduction to first order systems, Runge-Kutta, Adams, and other predictor-corrector methods, elementary considerations of stability and roundoff.

d. Matrix inversion and matrix eigenvalues. (6 lessons) A first treatment of such problems, to include Gaussian elimination and some iterative methods for inversion; Rayleigh quotients and power methods for obtaining the eigenvalues of symmetric matrices, including an analysis of convergence.

References:

Faddeev, D.K. and Faddeeva, V.N. Computational Methods of Linear Algebra, trans. by R. C. Williams. San Francisco, California, W. H. Freeman and Company, 1963; authorized trans. by Curtis Benster, New York, Dover Publishing Company, 1959.

Hamming, R.W. Numerical Methods for Scientists and Engineers. New York, McGraw-Hill Book Company, 1962.

Henrici, P. Discrete Variable Methods in Ordinary Differential Equations. New York, J. Wiley and Sons, 1962.

Householder, A.S. Principles of Numerical Analysis. New York, McGraw-Hill Book Company, 1953.

Stiefel, E.L. An Introduction to Numerical Mathematics, trans. by W. C. and C. J. Rheinboldt. New York, Academic Press, 1963.

Todd, J. Survey of Numerical Analysis. New York, McGraw-Hill Book Company, 1962.

Mathematics 9. Geometry. (3 semester hours) (Classical geometry version)

This course follows rather closely the recommendations of the Teacher Training Panel, for the first half of the year-course in geometry for secondary teachers. We recommend that priority be given to Euclidean geometry; as this takes a semester in itself, no formal treatment of hyperbolic geometry appears in the outline below. Much of the material below may look like high school work, but it is intended that the treatment by much more thorough and much more exact than any high school course now taught.

a. Incidence and separation properties of planes and space. (5 lessons) The idea of convexity; separation of planes by lines, and of space by planes; the postulate of Pasch, and the theorems based on it, which are needed in a deductive treatment of the later material. The treatment of betweenness (and hence of segments) is easier if betweenness is defined in terms of distance. For this we need to anticipate some of the rudiments of the following topic.

b. The metric apparatus. (10 lessons) Distance functions and angular measurement functions, and the postulates governing them. Metric definitions of congruence-relations, for segments, angles, and triangles. The elementary theory of triangle-congruences, based on one new postulate (not three).

c. The synthetic approach. (3 lessons) An alternative treatment, as in Hilbert, in which congruence is taken as undefined, for both segments and angles, subject to the usual postulates.

d. Geometric inequalities. (2 lessons) A brief and elementary treatment.

e. Models for Riemannian and hyperbolic geometry. (2 lessons) This portion of the course should be informal. The parallel postulate is about to be introduced; and the models are intended to elucidate its meaning by indicating the sort of "geometries" that it rules out. (At this stage, it would be very hard to show that the Klein model, or the Poincaré model, for hyperbolic geometry really do satisfy all of the postulates for Euclidean geometry, except for the parallel postulate.)

f. The parallel postulate. (4 lessons) Applications to the theory of similarity.

g. Area-theory. (4 lessons) The most elementary form of the theory, based on postulates, for polygonal regions only. The main purposes are (1) to clarify the rudiments and (2) to permit easy alternative treatments of the Pythagorean theorem, and of similarity theory.

h. Circles in a plane. (2 lessons) A minimal treatment, merely to permit a deductive treatment of the theory of constructions, to follow.

i. <u>Constructions with ruler and compass</u>. (8 lessons) The elementary constructions, plus perhaps some more difficult ones, followed by the algebraic theory which leads to the impossibility proofs for the trisection of angles and the duplication of the cube.

The pace of this course will depend on its style, as well as on the class. If deletions are necessary, then (c) or (g) (or both) can be omitted without damage to later developments. But note the recommendation in Chapter III: a year of geometry is desirable if it is possible.

References:


Borsuk, K. and Szmielew, W. Foundations of Geometry. New York, Interscience, 1960.

Coxeter, H.S.M. Introduction to Geometry. New York, J. Wiley and Sons, 1961.

Eves, Howard. A Survey of Geometry, Vol. I. Boston, Massachusetts, Allyn and Bacon, Inc., 1963.

Fishback, W.T. Projective and Euclidean Geometry. New York, J. Wiley and Sons, 1962.

Hilbert, David. Foundations of Geometry, trans. by E. J. Townsend. Chicago, Illinois, Open Court Publishing Company, 1959.

Moise, Edwin. Elementary Geometry from an Advanced Standpoint. Reading, Massachusetts, Addison-Wesley Publishing Co., Inc., 1963.

Mathematics 9. Differential Geometry. (3 semester hours) (Alternative to classical geometry)

The geometry of curves and surfaces in Euclidean 3-space, with extensive use of linear algebra. The aim is to motivate and demonstrate some of the most widely useful methods of differential geometry in the context of 2 and 3 dimensions.

a. Curves in 3-space. (5 lessons) Review of vector calculus for curves, dot and cross product. Frenet equations. Geometrical significance of curvature and torsion.

b. Geometry of 3-space. (5 lessons) Informal introduction to differential forms. Orthogonal coordinates and frame fields; covariant derivatives. Euclidean motions, invariance.

c. Calculus on a surface. (6 lessons) Review of elementary calculus for surfaces. (Coordinate systems, tangent vectors.) Differential forms on a surface, exterior derivative, Stokes' theorem. Introduction to the notion of differential manifold, in dimension 2.

d. Geometry of surfaces in 3-space. (14 lessons) The second fundamental form as a linear operator on each tangent plane. Normal curvature, principal curvatures and vectors, Gaussian and mean curvature. Techniques of computation. Surfaces of revolution and ruled surfaces. The Codazzi equations, flat surfaces, all-umbilic surfaces.

e. Intrinsic geometry of surfaces. (9 lessons) Geodesics and parallel translation; intrinsic distance. Isometries of surfaces; theorema egregium. Euler-Poincaré characteristic, Gauss-Bonnet theorem. Abstract surfaces (2-dimensional Riemannian manifolds). The hyperbolic plane, the projective plane. Euclidean geometry in retrospect: the parallel postulate.

The emphasis and rate of coverage will depend strongly on the level of preparation of the class. For well-prepared students, sections a, b, c, will be largely review and may be covered quite rapidly. For students with only average preparation, the topics in section e need not be covered in full detail. A prior knowledge of differential forms is not assumed.

References:

Flanders, Harley. Differential Forms, with Applications to the Physical Sciences. New York, Academic Press, 1963.

Willmore, T.J. Introduction to Differential Geometry. New York, Oxford University Press, 1959.

Mathematics 10. <u>Applied Mathematics.</u>

## On Applied Mathematics

At the heart of applied mathematics is the process of model building. What you need to do is to take a situation in another field - it doesn't matter whether this is engineering or physics or economics or biology or what have you - which you would like to understand better, and to invent a mathematical model that will (hopefully) help you to understand that situation. You then proceed to analyze this mathematical model, including particular numerical examples if they are relevant, and finally see what you have learned through the mathematical model about the original "physical" situation. Now a course in applied mathematics could be organized around either the field of human endeavor to which mathematics is being applied, or around the mathematical discipline being used in analyzing the model ("theoretical mechanics" and "methods of mathematical physics" are examples of each of these possibilities). Our purpose is to organize instead around the process of model building itself. A sequence of situations from various fields of applications could be chosen, for instance, to illustrate each of the following aspects of model building.

(1) A mathematical model of, say, a situation in physics, must be complicated enough so that it honestly represents the real world, without omitting any essential features of the physical situation, and yet be simple enough so that you have a fighting chance to do something with it mathematically. Typically these two don't meet at first try, and it is an exciting struggle to obtain a sufficiently simple mathematical model without losing the essence of the problem.

(2) When you have made a mathematical model, you have to consider all its consequences, those that you like because they agree with your physical intuition about the problem as well as those whose physical implications come as a real shock. This frequently leads to a refinement of the model, as well as to new problems that need to be formulated and analyzed.

(3) The analysis of the first mathematical formulation of, say, an engineering situation, may reveal that the engineer doesn't really know what it is he wants to understand, to build, or optimize. The mathematical model serves to focus on the question that actually should be asked - what are we trying to optimize, for instance, or when does the engineer wish to consider two mathematical solutions equivalent. The attempt to build a satisfactory mathematical model forces the right question about the original situation to come to the surface.

(4) A model is always an approximation to reality, and should therefore be stable with respect to perturbations in the less certain of its mathematical assumptions. If such changes in the assumptions cause a major upset of the mathematical conclusions, then the conclusions may be physically suspect, for we often cannot be completely sure of the precision of our assumptions. Attempts to obtain stable rather than unstable mathematical models are a very interesting aspect of model building.

It should be possible to organize a course of the above kind which presupposes some given degree of mathematical sophistication and understanding of other disciplines. A particular choice must depend on the interest of the instructor and the preparation of the students.

The forthcoming report of the ad hoc Committee on Applied Mathematics, loc. cit. Chapter I, Section 2, will offer two source-synopses to aid in the construction of courses of this type, one in the area of physical sciences and one on models and optimizing methods for biological, management, and social sciences.

Mathematics 11-12. Real Variable Theory. (6 semester hours)

First semester - 39 lessons.

a. Real numbers. (6 lessons) The integers; induction. The rational numbers; order structure, Dedekind cuts. The reals defined as a Dedekind-complete field. Outline of the Dedekind construction. Least upper bound property. Nested interval property. Denseness of the rationals. Archimedean property. Inequalities ([7] is a good source of problems). The extended real number system.

b. Complex numbers. (3 lessons) The complex numbers introduced as ordered pairs of reals; their arithmetic and geometry. Statement of algebraic completeness. Schwarz inequality.

c. Set theory. (4 lessons) Basic notation and terminology: membership, inclusion, union and intersection, cartesian product, relation, function, sequence, equivalence relation, etc.; arbitrary unions and intersections. Countability of the rationals; uncountability of the reals.

d. Metric spaces. (6 lessons) Basic definitions: metric, ball, boundedness, neighborhood, open set, closed set, interior, boundary, accumulation point, etc. Unions and intersections of open or closed sets. Subspaces. Compactness. Connectedness. Convergent sequence, subsequence, uniqueness of limit. A point of accumulation of a set is a limit of a sequence of points of the set. Cauchy sequence. Completeness.

e. Euclidean spaces. (6 lessons) $R^n$ as a normed vector space over R. Completeness. Countable base for the topology. Bolzano-Weierstrass and Heine-Borel-Lebesgue theorems. Topology of the line. The open sets; the connected sets. The Cantor set. Outline of the Cauchy construction of R. Infinite decimals.

f. Continuity. (8 lessons) (Functions into a metric space) Limit at a point, continuity at a point. Continuity; inverses of open sets, inverses of closed sets. Continuous images of compact sets are compact. Continuous images of connected sets are connected. Uniform continuity; a continuous function on a compact set is uniformly continuous. (Functions into R) Algebra of continuous functions. A continuous function on a compact set attains its maximum. Intermediate value theorem. Kinds of discontinuities.

g. Differentiation. (6 lessons) (Functions into R) The derivative. Algebra of differentiable functions. Chain rule. Sign of the derivative. Mean value theorems. The intermediate value theorem for derivatives. L'Hospital's rule. Taylor's theorem with remainder. One-sided derivatives; infinite derivatives. (This material will be relatively familiar to the student from his calculus course, so it can be covered rather quickly.)

Second semester - 39 lessons

h. The Riemann-Stieltjes integral. (11 lessons) [Alternative: the Riemann integral.] Upper and lower Riemann integrals.

[Existence of the Riemann integral: for f continuous; for f monotonic.] Monotonic functions and functions of bounded variation. Riemann-Stieltjes integrals. Existence of $\int_a^b f d\alpha$ for f continuous and $\alpha$ of bounded variation. Reduction to the Riemann integral in case $\alpha$ has a continuous derivative. Linearity of the integral. The integral as a limit of sums. Integration by parts. Change of variable. Mean value theorems. The integral as a function of its upper limit. The fundamental theorem of calculus. Improper integrals. The gamma function ([11], 367-378; [10], 285-297).

i. Series of numbers. (11 lessons) (Complex) Convergent series. Tests for convergence (root, ratio, integral, Dirichlet, Abel). Absolute and conditional convergence. Multiplication of series. (Real) Monotone sequences; lim sup and lim inf of a sequence. Series of positive terms; the number e. Stirling's formula, Euler's constant ([11], 383-388). Again, see [7] for problems.

j. Series of functions. (7 lessons) (Complex) Uniform convergence; continuity of uniform limit of continuous functions. Equicontinuity; equicontinuity on compact sets. (Real) Integration term by term. Differentiation term by term. Weierstrass approximation theorem. Nowhere-differentiable continuous functions.

k. Series expansions. (10 lessons) Power series, interval of convergence, real analytic functions, Taylor's theorem. Taylor expansions for exponential, logarithmic, and trigonometric functions.

Fourier series: orthonormal systems, mean square approximation, Bessel's inequality, Dirichlet kernel, Fejér kernel, localization theorem, Fejér's theorem. Parseval's theorem.

References:


1. Apostol, T.M. Mathematical Analysis. Reading, Massachusetts, Addison-Wesley Publishing Co., Inc., 1957.

2. Bartle, R.G. The Elements of Real Analysis. New York, J. Wiley and Sons, Inc., 1964.

3. Buck, R.C. Advanced Calculus, 2nd ed. New York, McGraw-Hill Book Company, 1964.

4. Eggleston, H.G. Introduction to Elementary Real Analysis. New York, Cambridge University Press, 1962.

5. Gelbaum, B.R. and Olmsted, J.M.H. Counterexamples in Analysis. San Francisco, California, Holden-Day, Inc., 1964.

6. Goldberg, R. Methods of Real Analysis. New York, Blaisdell Publishing Company, 1964.

7. Kazarinoff, N.D. Analytic Inequalities. New York, Holt, Rinehart and Winston, 1961.

8. Rankin, R.A. Introduction to Mathematical Analysis. New York, Pergamon Press, 1962.

9. Rudin, W. Principles of Mathematical Analysis. New York, McGraw-Hill Book Company, 1964.

10. Spiegel, M.R. Theory and Problems of Advanced Calculus. Schaum's Outline Series, New York, Schaum Publishing Co., 1963.

11. Widder, D.V. Advanced Calculus. Englewood Cliffs, New Jersey, Prentice Hall, Inc., 1961.

Mathematics 11. Real Variable Theory. (3 semester hours) (One semester version)

a. Real numbers. (3 lessons) Describe various ways of constructing them but omit details. Least upper bound property, nested interval property, denseness of the rationals.

b. Set theory. (4 lessons) Basic notation and terminology: membership, inclusion, union and intersection, cartesian product, relation, function, sequence, equivalence relation, etc.; arbitrary unions and intersections. Countability of the rationals; uncountability of the reals.

c. Metric spaces. (4 lessons) Material of topic d in Mathematics 11-12, condensed.

d. Euclidean spaces. (4 lessons) $R^n$ as a normed vector space over R. Completeness. Bolzano-Weierstrass and Heine-Borel-Lebesgue theorems. Topology of the line. Outline of the Cauchy construction of R. Infinite decimals.

e. Continuity. (5 lessons) (Functions into a metric space) Limit at a point, continuity at a point, inverses of open or closed sets. Uniform continuity. (Functions into R) A continuous function on a compact set attains its maximum. Intermediate value theorem.

f. Differentiation. (3 lessons) Review of previous information, including sign of the derivative, mean value theorem, L'Hospital's rule, Taylor's theorem with remainder.

g. Riemann-Stieltjes, or Riemann integration. (5 lessons) Functions of bounded variation (if the Riemann-Stieltjes integral is covered), basic properties of the integral, the fundamental theorem of the calculus.

h. Series of numbers. (8 lessons) Tests for convergence, absolute and conditional convergence. Monotone sequences, lim sup, series of positive terms.

i. Series of functions. (3 lessons) Uniform convergence, continuity of uniform limit of continuous functions, integration and differentiation term by term.

Mathematics 13. Complex Analysis. (3 semester hours)

This course is suitable for students who have completed work at the vector analysis and ordinary differential equation level. The development of skills in this area is very important in the sciences, and the course must exhibit many examples which illustrate the influence of singularities and which require varieties of technique for finding conformal maps, for evaluating contour integrals (especially those with multivalued integrands), and for using integral transforms.

a. Introduction. (4 lessons) The algebra and geometry of complex numbers. Definitions and properties of elementary functions, e.g., $e^z$, sin z, log z.

b. Analytic functions. (2 lessons) Limits, derivatives, Cauchy-Riemann equations.

c. Integration. (6 lessons) Integrals, functions defined by integrals. Cauchy's theorem and formula, integral representation of derivatives of all orders. Maximum modulus, Liouville's theorem, fundamental theorem of algebra.

d. Series. (5 lessons) Taylor and Laurent series. Uniform convergence, term-by-term differentiation, uniform convergence in general. Domain of convergence and classification of singularities.

e. Contour integration. (3 lessons) The residue theorem. Evaluation of integrals involving singlevalued functions.

f. Analytic continuation and multivalued functions. (6 lessons) Analytic continuation, multivalued functions, and branch points. Technique for contour integrals involving multivalued functions.

g. Conformal mapping. (6 lessons) Conformal mapping. Bilinear and Schwarz-Christoffel transformations, use of mapping in contour integral evaluation. Some mention should be made of the general Riemann mapping theorem.

h. Boundary value problems. (3 lessons) Laplace's equation in two dimensions and the solution of some of its boundary value problems, using conformal mapping.

i. Integral transforms. (4 lessons) The Fourier and Laplace transforms, their inversion identities, and their use in boundary value problems.

References:

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Knopp, K. Theory of Functions. New York, Dover Publications, Inc., 1945.

Nehari, Z. Introduction to Complex Analysis. Boston, Allyn and Bacon, Inc., 1961.

Conformal Mapping

Nehari, Z. Conformal Mapping. New York, McGraw-Hill Book Company, 1952.

Contour Integration

MacRobert, T.M. Functions of a Complex Variable. London, The Macmillan Company, 1933.

Whittaker, E.T. and Watson, G.N. A Course in Modern Analysis. New York, Cambridge University Press, 1958.

Integral Transforms

Sneddon, I.N. Fourier Transforms. New York, McGraw-Hill Book Company, 1951.

Steepest Descent

DeBruijn, N.G. Asymptotic Methods in Analysis. Amsterdam, North-Holland Publishing Company, 1958.

Jeffreys, H. and Jeffreys, B.S. Methods in Mathematical Physics. New York, Cambridge University Press, 1956.

Morse, P.M. and Feshbach, H. Methods of Theoretical Physics. New York, McGraw-Hill Book Company, 1953.