units 1,2,3 & 4

an experimental course in

MATHEMATICS

for the ninth year

THE UNIVERSITY OF THE STATE OF NEW YORK / THE STATE EDUCATION DEPARTMENT
BUREAU OF SECONDARY CURRICULUM DEVELOPMENT / ALBANY 1964

An Experimental Course

in

M A T H E M A T I C S

for the

Ninth Year

Unit 1. Sets

Unit 2. Algebraic Expressions

Unit 3. The Set of Integers

Unit 4. Open Sentences

BUREAU OF SECONDARY CURRICULUM DEVELOPMENT
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H733-My64-15,000

AN EXPERIMENTAL COURSE IN MATHEMATICS FOR THE NINTH YEAR

Mathematics 9X

CONTENTS

SYLLABUS OUTLINE

Mathematics 9X

Unit	Topics	Time Allotment (days)
	Optional topics are indicated by an asterisk (*).	
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2.	Algebraic Expressions Algebraic symbols Addition, subtraction, multiplication, and division of algebraic expressions Value of an expression	9 - 11
3.	The Set of Integers Properties of the natural numbers Operations in the set of integers Properties of the integers Absolute value	5 - 6
4.	Open Sentences Equations Identities Equations with no solution Inequalities Solution of equations Solving problems by use of equations Solution of inequalities Solving problems by use of inequalities Solution of equations and inequalities involving absolute value	30 - 35
5.	Algebraic Problems Formula problems Motion problems Value problems Mixture problems Business problems Work problems Geometric problems	25 - 30
6.	The Set of Real Numbers The set of rational numbers Irrational numbers Properties of the real numbers The real number line	9 - 11

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   - Non-negative exponents
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   - Operating with expressions containing exponents
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   - Operating with expressions containing radicals
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8. Polynomial Expressions — 10 - 12
   - Addition, subtraction, multiplication, and division of polynomial expressions
   - Factoring polynomial expressions

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   - *Solution by quadratic formula
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   - Simple proofs

10. Open Sentences in Two Variables — 9 - 10
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      - (addition and subtraction of equations)
      - (substitution)
    - Solution by graphing
    - Solution of inequalities

11. Relations and Functions — 7 - 9
    - Relations
    - Functions
    - Range and domain
    - Graphing relations and functions
    - Slope and intercept

*12. Trigonometric Functions
    - The unit circle in coordinate geometry
    - Sine, cosine, and tangent defined in terms of unit circle

FOREWORD

In April 1961, an advisory committee on secondary school mathematics convened at the Department to discuss the direction that secondary mathematics curriculum revision should take. This committee consisted of college and secondary school teachers, supervisors, administrators, and a consultant from one of the national curriculum programs. As a result of this meeting, the recommendation was made that a revision of the mathematics 7-8-9 program be undertaken immediately.

This is the first of several experimental editions containing materials and methods in the teaching of a revised mathematics program in grade 9. These materials will be tested in the schools of the State and carefully revised when administrators and teachers have had an opportunity to evaluate them while in operation in their schools. The material will serve as an optional alternative course of study for mathematics in grade 9 until such time as the necessary revisions have been completed.

The materials in the 9X experimental syllabus are based upon the foundations laid in the 7X and 8X experimental syllabuses. Therefore, it is to be understood that the 7X and 8X experimental courses are a prerequisite to the 9X experimental course. As in the 7X and 8X syllabuses, the chief emphasis is placed upon the understanding of basic mathematical concepts as contrasted with the all-too-frequently used program in which the mechanics of mathematics receives the greatest stress. The general approach and content used is that agreed upon by leading mathematical authorities as the most desirable. In the actual teaching of the program major emphasis is placed upon the "discovery process." The principal function of the teacher is to carefully set the stage for learning in an organized fashion such that the pupils will "discover" for themselves the fundamental concepts involved.

The materials in the mathematics 7X, 8X, and 9X experimental syllabuses include much of what today are called the basic ideas and concepts of mathematics. These concepts are those which the pupils will use throughout their study in mathematics. With this material the teacher should be able to aid the pupils to see the beauty of mathematics in terms of the fundamental structure found in mathematical systems. The important unifying concepts included in the new course of study for the ninth grade are:

- Algebraic Expressions and Open Sentences
- Analysis of Algebraic Problems
- The Set of Real Numbers
- Properties of Exponents and Radicals
- Operations with Polynomial Expressions
- Quadratic Equations
- Open Sentences in Two Variables
- Relations and Functions
- Trigonometric Functions

A new mathematical curriculum is not the sole answer to the improvement of mathematics instruction. Most important perhaps is the method of presenting the material. If the teacher develops lesson plans that will allow the pupils to discover concepts for themselves, the teaching and learning of mathematics will become excitingly different and no longer remain the dissemination of rules and tricks.

Unit 1, Sets, is essentially a review of the fundamentals presented in the 7X and 8X material. This unit should be presented for background since the basic algebra content is developed using the concepts involved with set notation. Units 2, 3, and 4: Algebraic Expressions, The Set of Integers, and Open Sentences introduce concepts in the study of algebra which form the essential foundation for the ninth year course.

A special committee was formed to review the 9X syllabus and to make recommendations for the writing of materials. This committee consisted of the following: David Adams, Liverpool High School; Benjamin Bold, Coordinator of Mathematics, High School Division, New York City Board of Education; Mary Challis, Plattsburgh High School, Francis Foran, Garden City Junior High School; Eleanor Maderer, Coordinator of Mathematics, Board of Education, Utica; William Mooar, Benjamin Franklin Junior High School, Kenmore; Verna Rhodes, Corning Free Academy; Leonard Simon, Curriculum Center, New York City; Joan Vodek, Chestnut Hill Junior High School, Liverpool; Frank Wohlfort, Coordinator of Mathematics, Junior High School Division, New York City Board of Education.

The materials for the 9X syllabus were written by Charles Burdick, coordinator and teacher of mathematics, Oneida Junior High School, Schenectady. The project has been developed under the joint supervision of this Bureau and the office of Frank Hawthorne, supervisor of mathematics education, who guided the planning. Georgette MacLean produced the drawings. Aaron Buchman, associate in mathematics education, reviewed and revised the original manuscript. Herbert Bothamley, acting as temporary curriculum associate, edited and prepared the final manuscript for publication.

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Chief, Bureau of Secondary
Curriculum Development

Warren W. Knox
Assistant Commissioner for
Instructional Services
(General Education)

UNIT 1: SETS

PART 1: BACKGROUND MATERIAL FOR TEACHERS

1.1 INTRODUCTION

This unit is a quick review of the topic of sets as covered in the Mathematics 8X course and is an extension of the basic concepts of sets to include matching sets, complements of sets, Cartesian sets, and further work with Venn diagrams and Euler circles. It is not intended that a great amount of time be devoted to this unit. A day or two devoted to review and a few days for the new material should prove sufficient. It is assumed that the pupils have completed the Mathematics 8X course and are familiar with the basic concepts of sets included in that course.

1.2 REVIEW OF SETS

This section is a review of the concepts of sets contained in the Mathematics 8X course. Before beginning this review it may be advisable for the teacher to read Part 1: Background Material for Teachers, Unit 1 Mathematics 8X. In this section, more use is made of set notation symbols than in the Mathematics 8X course. Such symbols include: $\in$, $\notin$, $\subset$, $\not\subset$, $\subseteq$, $\not\subseteq$, and $\{x \mid x \in A\}$.

The symbol $\in$ is used to identify an element as being a member of a certain set. " $a \in A$" is read "a is an element of set A." The capital letters are used to represent sets and the lower case letters are used to represent elements in a set.

The symbol $\subseteq$ is used to identify a set which is a subset of a set. $A \subseteq B$ means that set A is a subset of set B. Every element in set A is also an element in set B. Set A is a proper subset of set B if set B contains every element that is in set A plus at least one additional element. If A is a proper subset of set B, the symbols $A \subset B$ are used. The symbols $\notin$, $\not\subset$, and $\not\subseteq$ are the negations of the symbols $\in$, $\subset$, $\subseteq$ respectively.

The symbols $\{x \mid x \in A\}$ describe a certain set. The braces $\{\}$ are a symbol for a set. $\{x \mid x \in A\}$ is read "all x such that x is an element in set A." $\{x \mid x > 2\}$ is one way of identifying the set of all x greater than the number 2. If the universal set is the set of all real numbers, then $\{x \mid x > 2\}$ is the set of x, where x is a real number, such that x is greater than 2.

The distinction between the use of the symbol $\in$ and the symbol $\subset$ should be made clear to the pupils. The symbol $\in$ is used in reference to an element; the symbol $\subset$ is used in reference to a subset. Sometimes pupils get these two symbols confused.

1.3 MATCHING SETS

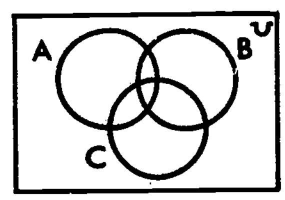
Two sets are called matching sets if for every element in the first set there is a corresponding element in the second set and for every element in the second set there is a corresponding element in the first set. When the elements of two sets can be matched in this way, the relationship is called a one-to-one correspondence between the two sets.

Two sets with the same number of elements are matching sets. Also, many infinite sets are matching sets. For example, the set of natural numbers and the set of even numbers are matching sets. There is a one-to-one-correspondence between the elements of the two sets.

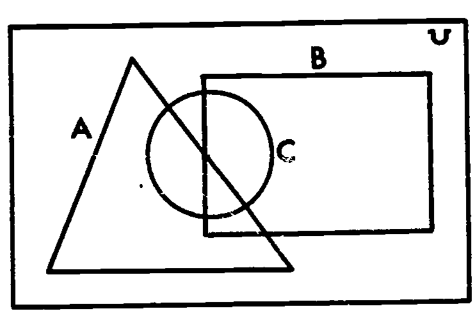
$$\begin{array}{ccccc} \{1, & 2, & 3, & 4, & 5,\ldots\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \{2, & 4, & 6, & 8, & 10,\ldots\} \end{array}$$

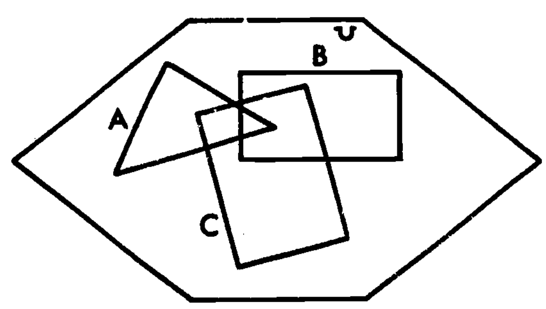
1.4 PICTORIAL REPRESENTATION OF SETS

The most common method of representing sets pictorially is by the use of Euler circles and Venn diagrams. The eighteenth-century Swiss mathematician Euler introduced the idea of using circles to represent sets and the nineteenth-century British logician Venn introduced further refinements, notably the use of three intersecting circles enclosed in a rectangle to illustrate all possible intersections of the sets. The rectangle represents the universal set and the three circles the sets A, B, and C.



Since the pupils have already had experience with Euler circles and Venn diagrams, it may be more interesting and challenging for them to use geometric figures such as triangles to represent sets. Below are two diagrams showing how various geometric figures may be used to represent sets.



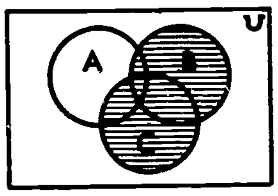


The union and intersection of sets may be shown in Euler circles by shading in appropriate portions of the diagrams. This method may be used to demonstrate the following properties of union and intersection.

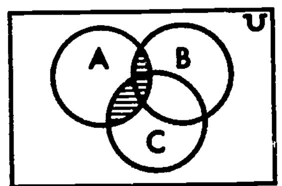
(a) $A \cup B = B \cup A$ Commutative property of union

(b) $A \cap B = B \cap A$ Commutative property of intersection
(c) $(A \cup B) \cup C = A \cup (B \cup C)$ Associate property of union
(d) $(A \cap B) \cap C = A \cap (B \cap C)$ Associate property of intersection
(e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Union distributive over intersection
(f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ Intersection distributive over union

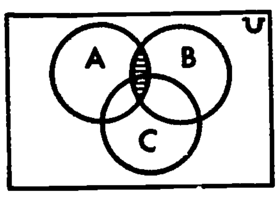
For example, to demonstrate $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



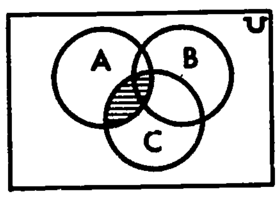
$B \cup C$



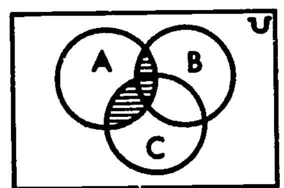
$A \cap (B \cup C)$



$A \cap B$



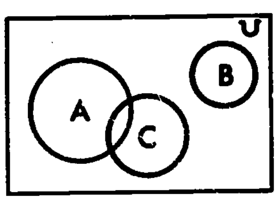
$A \cap C$

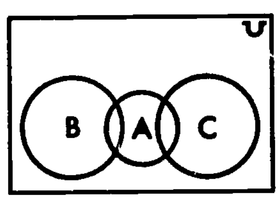


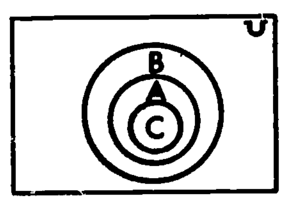
$(A \cap B) \cup (A \cap C)$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

This is not a proof that intersection is distributive over union; it merely demonstrates the property. The proof would have to consider all possible arrangements of three subsets of a universal set. Such a proof would be quite an undertaking due to the large number of possible arrangements of three subsets in a universal set. There are over 20 different possible arrangements, three of which are illustrated by the following Euler diagrams.







One interesting fact is that there are two distributive laws. Union is distributive over intersection and intersection is distributive over union. For this reason, union is not analogous to addition and intersection is not analogous to multiplication. Union is distributive over intersection but addition is not distributive over multiplication; $2 + (3 \cdot 4) \neq (2+3) \cdot (2+4)$.

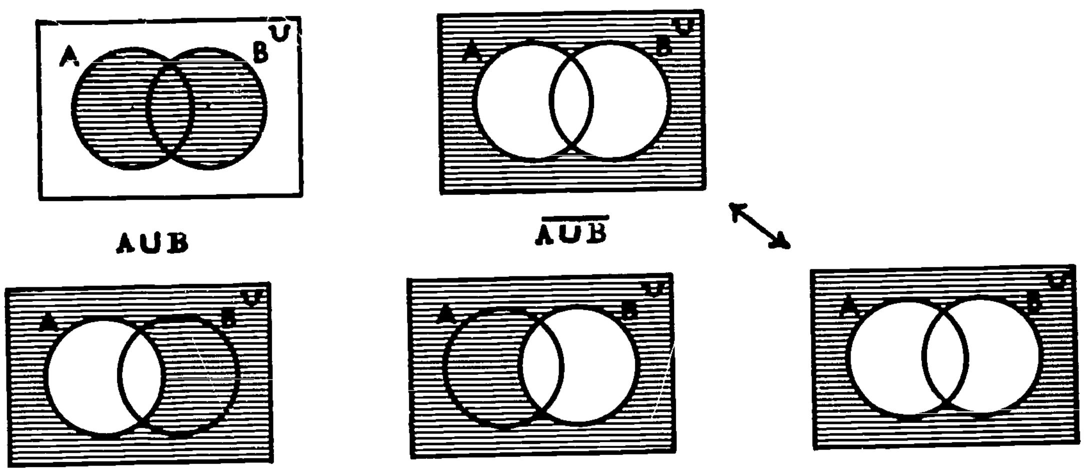
The complement of a set with respect to a given universe is the set of all elements in the universe that are not in the given set. The complement of set A is the set of all elements in the universal set that are not in set A. Symbols used for the complement of set A are $\bar{A}$, A', and ~A. In this unit the notation $\bar{A}$ is used but A' and ~A may be used if desired. The reason for the use of $\bar{A}$ is that this type of notation makes complements of intersections and unions easily recognizable. The complement of $A \cup B$ is $\overline{A \cup B}$. This may be more easily recognizable to the pupil than $(A \cup B)'$. The use of ~A is usually restricted to negations of propositions in logic. However, this notation may also be used to indicate a complement of a set.

Euler circles may be used to demonstrate two interesting relationships involving complements. The complement of the union of two sets is equal to the intersection of the complements of the two sets. The complement of the intersection of two sets is equal to the union of the complements of the two sets.

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

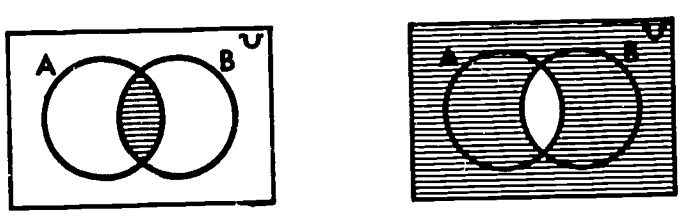
These two equations express what are called de Morgan's laws.



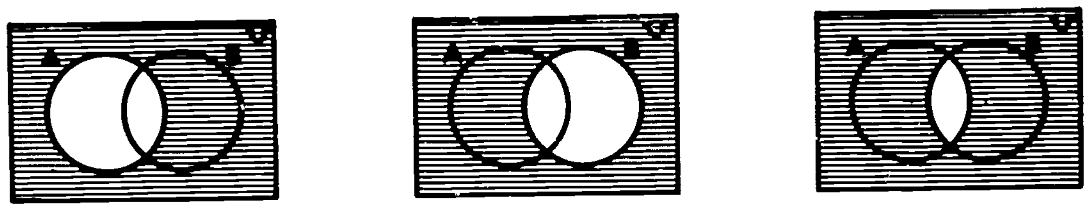
$A \cup B$ $\quad$ $\overline{A \cup B}$

$\bar{A}$ $\quad$ $\bar{B}$ $\quad$ $\bar{A} \cap \bar{B}$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$



$A \cap B$ $\quad$ $\overline{A \cap B}$



$\overline{A}$ $\overline{B}$ $\overline{A}\cup\overline{B}$

$$\overline{A\cap B} = \overline{A}\cup\overline{B}$$

1.5 CARTESIAN SETS

The set of all possible ordered pairs formed from the elements of two given sets is called the Cartesian product of the two sets or the Cartesian set.

For example, $A = \{2, 3, 4\}$
$B = \{1, 5, 9, 11\}$

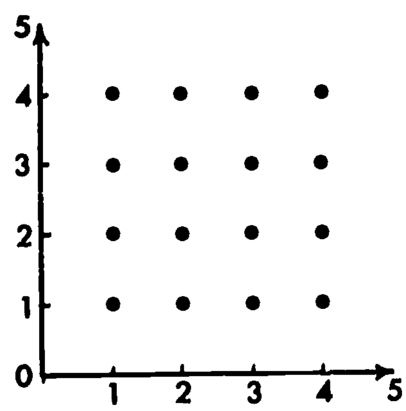
The Cartesian product, written $A \times B$, is the following set of ordered number pairs.

(2, 1)	(3, 1)	(4, 1)
(2, 5)	(3, 5)	(4, 5)
(2, 9)	(3, 9)	(4, 9)
(2,11)	(3,11)	(4,11)

The elements in set A are the first numbers in the ordered number pairs.

$A \times B$ is read A cross B or A crossed onto B. A set may also be crossed onto itself. For example, the Cartesian set $A \times A$ where $A = \{1, 2, 3, 4\}$ is

(1, 1)	(2, 1)	(3, 1)	(4, 1)
(1, 2)	(2, 2)	(3, 2)	(4, 2)
(1, 3)	(2, 3)	(3, 3)	(4, 3)
(1, 4)	(2, 4)	(3, 4)	(4, 4)



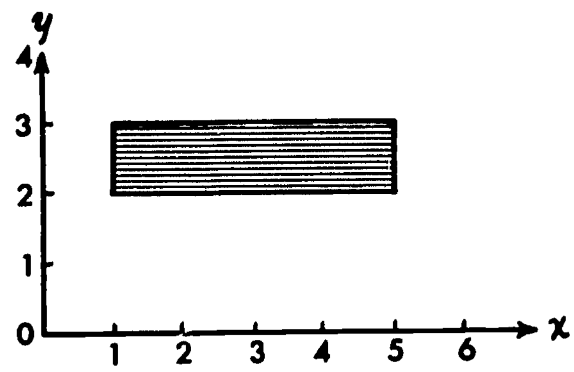
Cartesian sets may be graphed using a set of coordinate axes. The graph $A \times A$ where $A = \{1, 2, 3, 4\}$ is shown at the left.

The set or sets being crossed may also be infinite sets.

$$A = \{x \mid 1 \leq x \leq 5\}$$

$$B = \{y \mid 2 \leq y \leq 3\}$$

It is impossible to list all the ordered number pairs in such a Cartesian set but it may be graphed as shown at the right.



The Cartesian product A × B may also be described as:
$A \times B = \{(x, y) \mid x \in A, y \in B,\}$ or

$A \times B = \{(x, y) \mid 1 \leq x \leq 5 \text{ and } 2 \leq y \leq 3\}$

1.6 SOLUTION SETS

The answer or answers to a problem may be called the solution set to the problem. For example, the solution set of the equation $x^2 = 16$ is $\{4, -4\}$. Until quadratic equations are introduced in unit 9, the solution set of almost all problems will consist of a single element. It may take some effort to develop the idea that a solution set may consist of only one element. The solution set may also consist of the null set. Pupils must be careful not to confuse the set $\{0\}$ with the set $\emptyset$. The solution set of $x + 1 = 1$ is $\{0\}$. The solution set of $x = x + 1$ is $\emptyset$. The pupils must also be careful not to confuse $\{0\}$ with {all real numbers}. The solution set of the equation $x + 1 = x + 1$ is not $\{0\}$; the solution set is {all real numbers}.

The topic of solution sets is covered in much more detail in later units. In this unit, the term should be introduced and used so that the pupils will be familiar with the concept when it is used in later units.

<u>Teacher Notes</u>

UNIT 1: SETS

PART 2. QUESTIONS AND ACTIVITIES FOR CLASSROOM USE

1.1 INTRODUCTION

The questions and activities in this unit may require the use of ruler and compasses for drawing Euler circles, Venn diagrams, and other pictorial representation of sets. The graphs to be drawn are simple enough not to require the use of graph paper. Questions and activities contained in Unit 1, Mathematics 8X and questions and activities from other texts may be used to supplement the material in this unit.

1.2 REVIEW OF SETS

Concept: Set notation.

(1) Answer the following questions.

(a) What is a set?
(b) What name is given to the things or concepts that comprise a set?
(c) Name two ways of describing a set.
(d) How may the set of numbers 6, 8, 11, and 14, be expressed in set notation?
(e) If the set of numbers 6, 8, 11, and 14 is identified by the letter A, how may this be expressed in set notation?
(f) How may the fact that 6 is an element in set A be expressed in set notation?

Answers:

(a) A set is a group or collection of objects or concepts which have some property or characteristic in common.
(b) The things or concepts comprising a set are called the members or elements of the set.
(c) A set may be described by listing each element in the set or by describing the common property of all the elements in the set in such a way that there is no doubt as to which elements are in the set.
(d) $\{6, 8, 11, 14\}$
(e) $A = \{6, 8, 11, 14\}$
(f) $6 \in A$

(2) Indicate whether each of the following is true or false.

(a) $6 \in \{1, 3, 5, 7, 9\}$
(b) $a \notin \{a, b, c, d, e\}$
(c) square $\in$ {all parallelograms}
(d) $17 \notin$ {all prime numbers}
(e) $4 \in$ {all even numbers}

Answers:

(a) False (b) False (c) True (d) False (e) True

(3) Sets may also be expressed in notation such as $\{x \mid x \text{ is a book}\}$. This notation is read "the set of all x such that x is a book." The set of all numbers greater than 10 may be expressed as $\{x \mid x > 10\}$. <u>Express each of the following in this type of notation</u>.

(a) The set of all numbers that are real numbers.
(b) The set of all y such that y is a planet.
(c) The set of all x such that x is greater than 5 and less than 12.
(d) The set of all b such that b is red.

<u>Answers</u>:
(a) $\{x \mid x \text{ is a real number}\}$
(b) $\{y \mid y \text{ is a planet}\}$
(c) $\{x \mid 5 < x < 12\}$
(d) $\{b \mid b \text{ is red}\}$

<u>Concept</u>: Finite and infinite sets.

(4) <u>Answer the following questions</u>.
(a) What is a finite set?
(b) What is an infinite set?
(c) Identify each of the following as being either a finite or an infinite set:
1. The set of all molecules of gases in the atmosphere.
2. The set of all natural numbers less than 100.
3. The set of all real numbers greater than 5 and less than 10.
4. $A = \{6, 8, 10, 12...68\}$
5. $B = \{2, 4, 6, 8...\}$

<u>Answers</u>:
(a) A finite set is a set which contains a finite number of elements.
(b) An infinite set is a set which contains an endless number of elements.
(c) 1. A finite set — 2. A finite set — 3. An infinite set — 4. A finite set — 5. An infinite set

<u>Concept</u>: Equal sets and null set.

(5) <u>Answer the following questions</u>.
(a) What is meant by equal sets?
(b) If two sets contain the same elements but in different order, are the two sets equal sets?
(c) If two or more sets contain no elements are they equal sets?
(d) What is the name and symbol for a set which contains no elements?
(e) How many null sets are there?

<u>Answers</u>:
(a) Equal sets are sets which contain the same elements.
(b) Yes, two sets are equal sets if they contain the

same elements even though the elements are in different orders in the two sets.

(c) Sets with no elements are equal sets.

(d) A set with no elements is called the empty set or the null set. The symbol for the null set is $\emptyset$ or $\{\}$.

(e) There is only one null set. It is always referred to as <u>the</u> null set.

<u>Concept</u>: Subsets.

(6) <u>Answer the following questions.</u>

(a) What is the name given to the largest set under consideration?

(b) The elements of a universal set may be grouped in various ways to form several sets. What are these new sets called?

(c) How may the fact that set A is a subset of B be expressed in set notation?

(d) If a set A is a subset of set B, and set B contains one or more elements not contained in set A, what special type of subset is set A?

(e) How may the fact that set A is a proper subset of set B be expressed in set notation?

<u>Answers</u>:

(a) The universal set

(b) Subsets of the universal set

(c) $A \subseteq B$

(d) Set A is a proper subset of set B

(e) $A \subset B$

(7) <u>Answer the following questions?</u>

(a) What is the definition of a subset?

(b) Is any set A a subset of set A?

(c) Is the null set a subset of every set?

(d) List all possible subsets of the set $\{1, 2, 3\}$.

<u>Answers</u>:

(a) If every element in set A is also an element in set B, then set A is a subset of set B.

(b) Every set is a subset of itself.

(c) The null set is a subset of every set.

(d) $\{1, 2, 3\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\emptyset$

(8) <u>Insert in the blank in each expression below a proper symbol from the following</u>: $\in$, $\notin$, $\subset$, $\subseteq$, $\not\subset$, $\not\subseteq$, $=$, $\neq$.

(a) d ___ $\{a, b, c, d, e, f\}$

(b) $\{d\}$ ___ $\{a, b, c, d, e, f\}$

(c) d ___ $\{d\}$

(d) $\{d\}$ ___ $\{d\}$

(e) $\{a, b\}$ ___ $\{a, b, c\}$

(f) $\{k\}$ ___ $\{a, b, c\}$

(g) k ___ $\{a, b, c\}$

Answers:

(a) $\in$
(b) $\subseteq$ or $\neq$
(c) $\in$
(d) $=$ or $\subseteq$
(e) $\subset$ or $\neq$
(f) $\neq$, $\not\subset$, or $\not\subseteq$
(g) $\notin$

(9) **Below are four sets. Indicate which sets are subsets of one or more of the other sets.**

A = {1, 2, 3, 4}   C = {2, 4}   E = ∅
B = {1, 3}   D = {3, 1}

Answer: Set E is a subset of each of the other sets. Set B is a subset of set A and of set D. Set D is a subset of set B and of set A. Set C is a subset of set A. Each of the sets is a subset of itself.

(10) **Given the following sets:**

A = {1, 3, 5, 7}   B = {3, 5}   C = {1, 3, 5, 7, 9, 11}

Label each of the following statements as true or false.

(a) $B \subseteq A$	(d) $\emptyset \subseteq B$	(g) $B \subset A$
(b) $B \subseteq C$	(e) $A \subseteq C$	(h) $C \subseteq A$
(c) $A \subseteq B$	(f) $A \subset C$	(i) $B \subseteq B$

Answers:

(a) True	(d) True	(g) True
(b) True	(e) True	(h) False
(c) False	(f) True	(i) True

(11) H = {w, x, y}   I = {k, l, m, w, x, y, z} $K \subset H$

Indicate whether each of the following is true or false.

(a) $I \subset K$	(d) $H \subset K$
(b) $K \subset I$	(e) $I \subset H$
(c) $H \subset I$	

Answers:

(a) False	(d) False
(b) True	(e) False
(c) True	

(12) **Given** $A \subset B$, $B \subset C$, **and** $C \subset D$ **answer the following questions.**

(a) Is $B \subset D$?
(b) Is $A \subset D$?
(c) If $x \in A$, is $x \in D$?

Answers:

(a) Every element in set B is also in set C. Every element in set C is also in set D. Therefore, every element in set B must also be in set D. $B \subset D$.
(b) Every element in set A is also in set B. Every element in set B is in set D. Therefore, every element in set A is also in set D. $A \subset D$.
(c) Yes, because $A \subset D$.

Concept: Relationships of sets.

(13) <u>Answer the following questions.</u>

(a) What is meant by the union of two sets?
(b) How may the union of sets A and B be expressed?
(c) What is meant by the intersection of two sets?
(d) How may the intersection of sets A and B be expressed?
(e) If the union of set A and set B is expressed in set notation as $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$, how may the intersection of set A and set B be expressed?
(f) What is meant by disjoint sets?

<u>Answers:</u>

(a) The union of two sets A and B is the third set C formed by grouping the elements in set A with the elements in set B without repeating elements which are in both sets.
(b) $A \cup B$.
(c) The intersection of two sets A and B is the third set C composed of every element that is an element in both set A and set B.
(d) $A \cap B$.
(e) $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
(f) Disjoint sets are sets that have no elements in common. Their intersection is the null set.

(14) <u>If</u> $A = \{11, 13, 15\}$, $B = \{13, 15, 17, 19\}$, <u>and</u> $C = \{14, 16, 18, 20\}$, <u>find</u>:

(a) $A \cup B$
(b) $A \cup C$
(c) $A \cap B$
(d) $A \cap C$
(e) $B \cup C$
(f) $B \cap C$

<u>Answers:</u>

(a) $A \cup B = \{11, 13, 15, 17, 19\}$
(b) $A \cup C = \{11, 13, 14, 15, 16, 18, 20\}$
(c) $A \cap B = \{13, 15\}$
(d) $A \cap C = \emptyset$
(e) $B \cup C = \{13, 14, 15, 16, 17, 18, 19, 20\}$
(f) $B \cap C = \emptyset$

(15) <u>If</u> $A = \{4, 6\}$, $B = \{6, 12\}$, <u>and</u> $C = \{4, 6, 20\}$, <u>find</u>:

(a) $(A \cup B) \cup C$
(b) $A \cup (B \cup C)$
(c) $(A \cap B) \cap C$
(d) $A \cap (B \cap C)$
(e) $(A \cup B) \cap C$
(f) $A \cup (B \cap C)$
(g) $(A \cup B) \cap (A \cup C)$
(h) $(A \cap B) \cup C$
(i) $A \cap (B \cup C)$
(j) $(A \cap B) \cup (A \cap C)$

<u>Answers:</u>

(a) $(A \cup B) \cup C = \{4, 6, 12, 20\}$
(b) $A \cup (B \cup C) = \{4, 6, 12, 20\}$
(c) $(A \cap B) \cap C = \{6\}$
(d) $A \cap (B \cap C) = \{6\}$
(e) $(A \cup B) \cap C = \{4, 6\}$
(f) $A \cup (B \cap C) = \{4, 6\}$
(g) $(A \cup B) \cap (A \cup C) = \{4, 6\}$
(h) $(A \cap B) \cup C = \{4, 6, 20\}$
(i) $A \cap (B \cup C) = \{4, 6\}$
(j) $(A \cap B) \cup (A \cap C) = \{4, 6\}$

1.3 MATCHING SETS

Concept: Definition of matching sets.

(1) Given two sets, if for each element in the first set there is one and only one corresponding element in the second set and if for each element in the second set there is one and only one corresponding element in the first set, the two sets are called matching sets. There is no particular order in which the elements are to be paired. For example, the elements in the sets {1, 2, 3} and {A, B, C} may be matched in a number of ways.

{1, 2, 3}
↕ ↕ ↕
{A, B, C}

{1, 2, 3}
↕ ↕ ↕
{B, C, A}

{1, 2, 3}
↕ ↕ ↕
{C, B, A}

{1, 2, 3}
↕ ↕ ↕
{A, C, B}

{1, 2, 3}
↕ ↕ ↕
{B, A, C}

{1, 2, 3}
↕ ↕ ↕
{C, A, B}

When the elements of two sets can be matched in this manner, the relationship is called a one-to-one correspondence between the two sets. When a one-to-one correspondence exists between two sets, they are matching sets.

For each of the following, indicate whether or not each pair of sets is a pair of matching sets. For each pair of matching sets, indicate at least one way the elements in the two sets may be matched.

(a) A = {16, Texas, 32, 7, Bill}
B = {2, 4, 6, 8, 10}

(b) C = {1, 2, 3, 4, 5}
D = {2, 4, 6, 8, 10}

(c) W = {3, A, 4, B, 5, C}
X = {1, 2, 3,...}

(d) Y = {K, M, N}
Z = {Bill, Mary, Jane, Joan}

(e) S = {1, 2, 3,...}
T = {102, 104, 106, 108, 110,...}

Answers:

(a) They are matching sets.
A: {16, Texas, 32, 7, Bill}
↕ ↕ ↕ ↕ ↕
B: {6, 8, 10, 2, 4 }

(b) They are matching sets.
C: {1, 2, 3, 4, 5}
↕ ↕ ↕ ↕ ↕
D: {2, 4, 6, 8, 10}

(c) They are not matching sets.

(d) They are not matching sets.

(c) They are matching sets.

S: {1, 2, 3, 4, 5,...}
↕ ↕ ↕ ↕ ↕
T: {102, 104, 106, 108, 110,...}

(3) <u>Are the set of natural numbers and the set of even natural numbers matching sets?</u>

<u>Answer:</u>

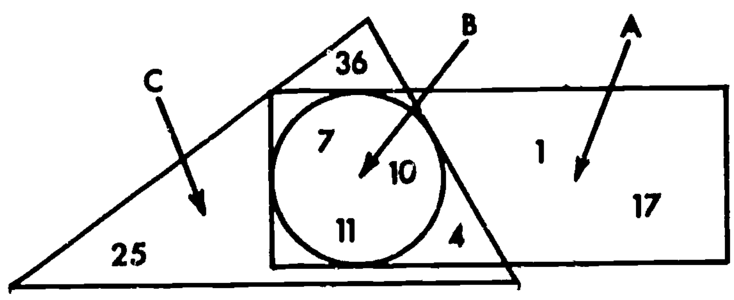
Yes they are matching sets.

{1, 2, 3, 4, 5...}
↕ ↕ ↕ ↕ ↕
{2, 4, 6, 8, 10...}

1.4 PICTORIAL REPRESENTATION OF SETS

<u>Concept:</u> Use of geometric figures to represent sets.

(1) Circles, rectangles, and other geometric figures may be used to represent sets and the relationships among the sets. For example, if A = {1, 4, 7, 10, 11, 17}, B = {7, 10, 11}, and C = {4, 7, 10, 11, 25, 36}, a rectangle may be used to represent set A, a circle used to represent set B, and a triangle used to represent set C. The relationships among the three sets may be represented as shown in the following diagram.

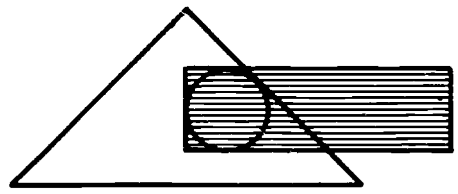


<u>With reference to the above diagram, do the following:</u>

(a) Shade in A ∪ B.
(b) List the elements in A ∪ B.
(c) Shade in A ∪ C.
(d) List the elements in A ∪ C.
(e) Shade in A ∩ B.
(f) List the elements in A ∩ B.
(g) Shade in (A ∩ B) ∪ C.
(h) List the elements in (A ∩ B) ∪ C.
(i) Shade in (A ∩ B) ∩ C.
(j) List the elements in (A ∩ B) ∩ C.

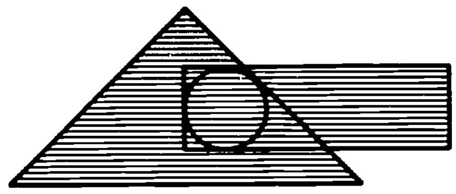
Answers:

(a)



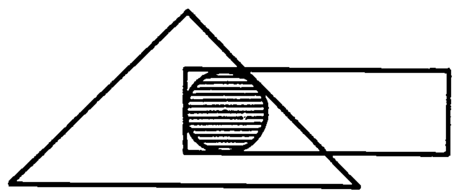
(b) $A \cup B = \{1, 4, 7, 10, 11, 17\}$

(c)



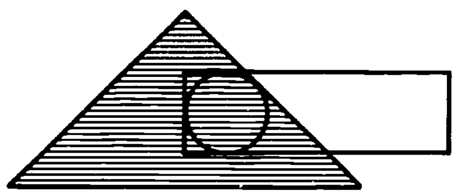
(d) $A \cup C = \{1, 4, 7, 10, 11, 17, 25, 36\}$

(e)



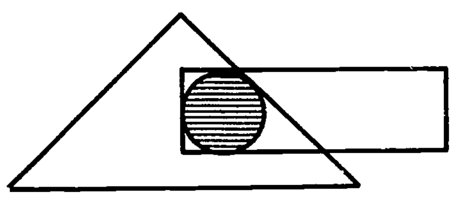
(f) $A \cap B = \{7, 10, 11\}$

(g)



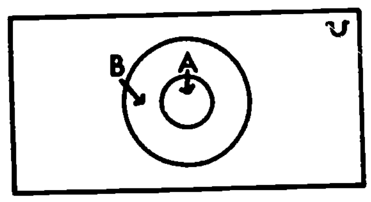
(h) $(A \cap B) \cup C = \{4, 7, 10, 11, 25, 36\}$

(i)

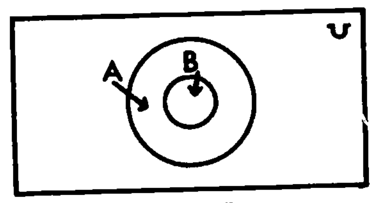


(j) $(A \cap B) \cap C = \{7, 10, 11\}$

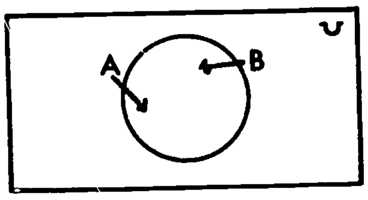
(2) It is common practice to use a rectangle to represent the universal set and circles to represent subsets of this universal set. These are called Euler circles. On the following page are five possible relationships between two sets that can be represented by Euler circles.



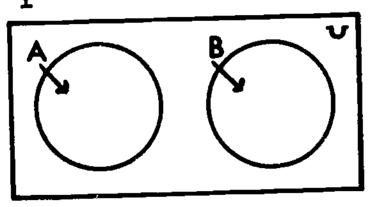
No. 1



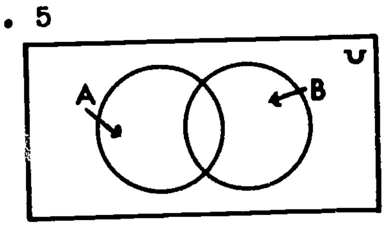
No. 5



No. 2



No. 3

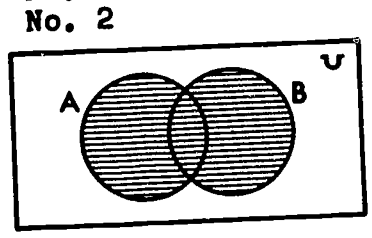


No. 4

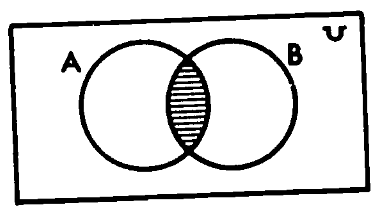
<u>Indicate which of the above diagrams illustrates each of the following:</u>

(a) $A \subset B$
(b) $A \cap B = \emptyset$
(c) $A \subseteq B$
(d) $A = B$
(e) $A \cup B = A \cap B$
(f) $A \cap B = A$
(g) $A \cap B \neq A$
(h) $A \cap U = B$
(i) In diagram No. 4 shade in $A \cup B$.
(j) In diagram No. 4 shade in $A \cap B$.

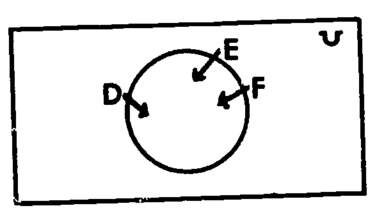
<u>Answers</u>:
(a) No. 1 (b) No. 3 (c) No. 1 and No. 2 (d) No. 2
(e) No. 2 (f) No. 1 (g) No. 3, No. 4, and No. 5
(h) No. 2
(i)



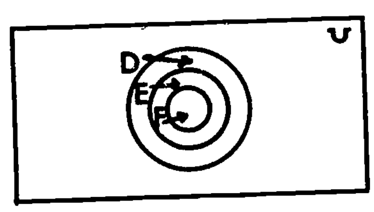
(j)



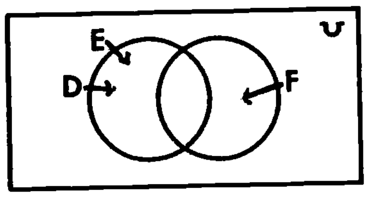
(3) Below are representations of three sets D, E, and F and some possible relationships among them. Diagram No. 5 which shows the three sets each having an intersection with the other two is called a Venn diagram.



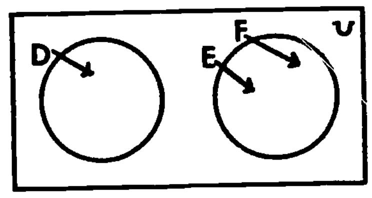
No. 1



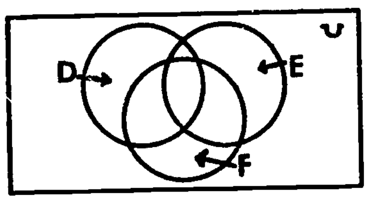
No. 2



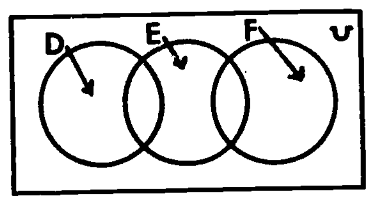
No. 3



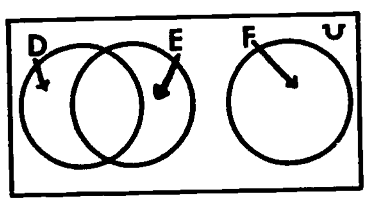
No. 4



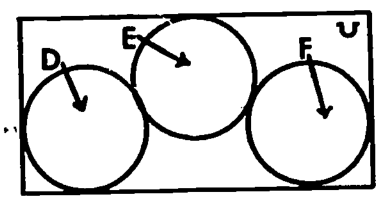
No. 5



No. 6



No. 7



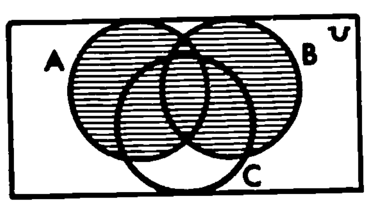
No. 8

<u>Shade in diagram No. 5 to show each of the following</u>:

(a) $A \cup B$ (b) $A \cup C$ (c) $B \cup C$ (d) $A \cap B$ (e) $A \cap C$ (f) $B \cap C$

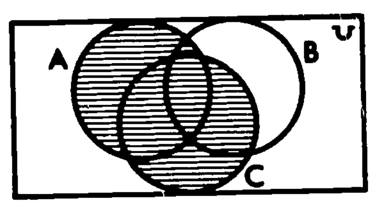
<u>Answers</u>:

(a)



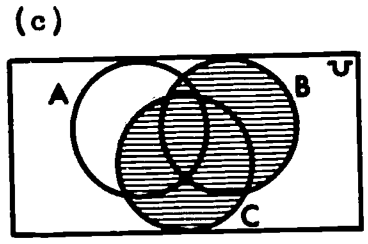
$A \cup B$

(b)



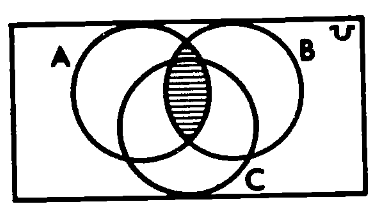
$A \cup C$

(c)

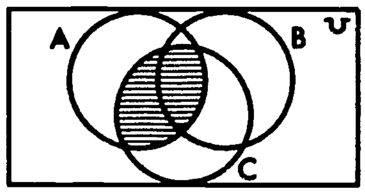


$B \cup C$

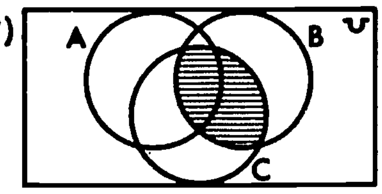
(d)



$A \cap B$

(e) 

$A \cap C$

(f) 

$B \cap C$

<u>Concept</u>: Properties of union and intersection.

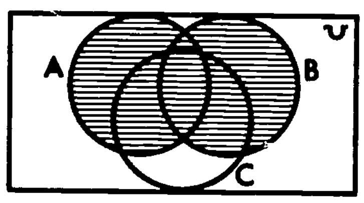
(4) <u>Darken a portion of a Venn diagram to represent (a) through (d) and answer (e).</u>

(a) $A \cup B$ (b) $B \cup A$ (c) $B \cup C$ (d) $C \cup B$

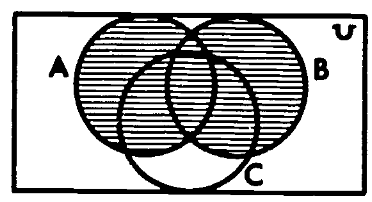
(e) Does union appear to have the commutative property?

<u>Answers</u>:

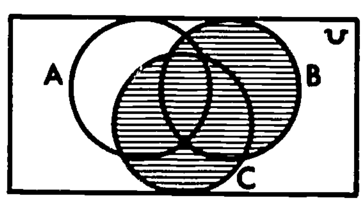
(a)



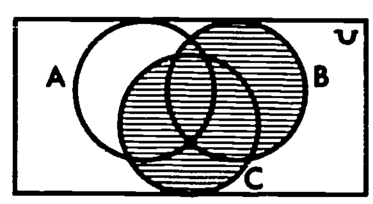
(b)



(c)



(d)



(e) Union appears to have the commutative property, because $A \cup B = B \cup A$ and $B \cup C = C \cup B$.

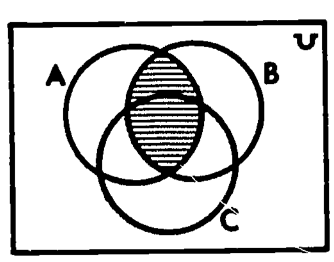
(5) <u>Darken in a portion of a Venn diagram to represent (a) through (d) and answer (e).</u>

(a) $A \cap B$ (b) $B \cap A$ (c) $B \cap C$ (d) $C \cap B$

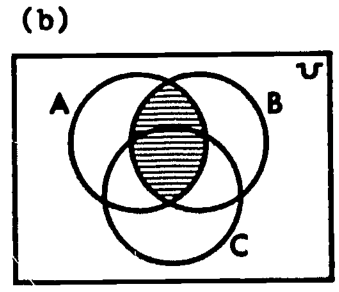
(e) Does intersection appear to have the commutative property?

<u>Answers</u>:

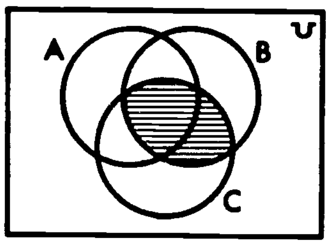
(a)

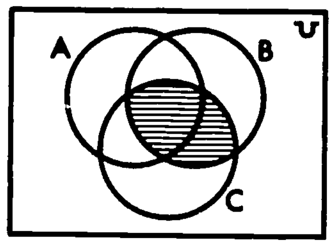


(b)



(c) (d)





(e) Intersection appears to have the commutative property, because A ∩ B = B ∩ A and B ∩ C = C ∩ B.

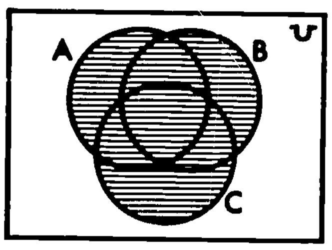
(6) Darken a portion of a Venn diagram to represent (a) and (b) and answer (c).

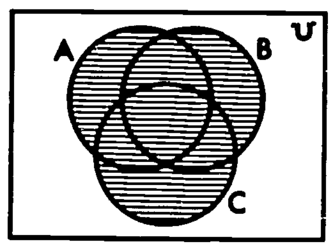
(a) (A ∪ B) ∪ C (b) A ∪ (B ∪ C)

(c) Does union appear to be associative?

Answers:

(a) (b)





(c) Union appears to be associative, because (A ∪ B) ∪ C = A ∪ (B ∪ C).

Concept: Associativity of intersection.

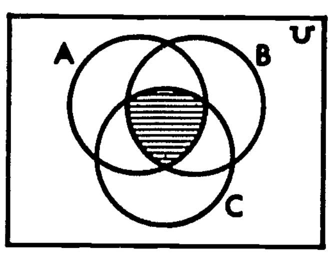
(7) Use a Venn diagram to represent (a) and (b) and answer (c).

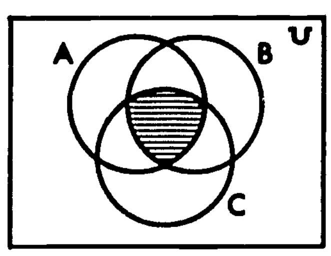
(a) (A ∩ B) ∩ C (b) A ∩ (B ∩ C)

(c) Does intersection appear to be associative?

Answers:

(a) (b)





(c) Intersection appears to be associative because $(A \cap B) \cap C = A \cap (B \cap C)$.

Concept: Union distributive over intersection.

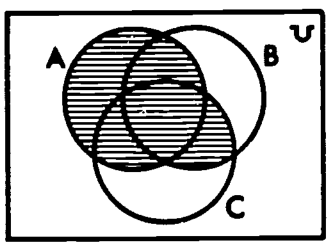
(8) Use a Venn diagram to represent (a) and (b) and answer (c).

(a) $A \cup (B \cap C)$ (b) $(A \cup B) \cap (A \cup C)$

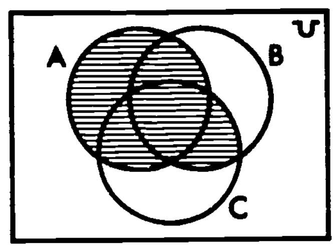
(c) Does union appear to be distributive over intersection?

Answers:

(a)



(b)



(c) Union appears to be distributive over intersection because $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Concept: Intersection distributive over union.

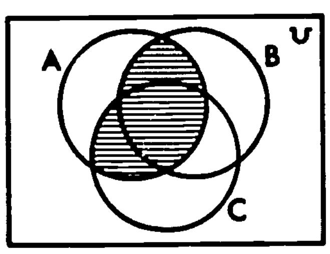
(9) Use a Venn diagram to represent (a) and (b) and answer (c).

(a) $A \cap (B \cup C)$ (b) $(A \cap B) \cup (A \cap C)$

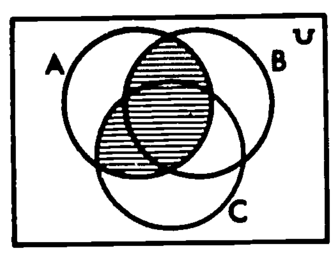
(c) Does intersection appear to be distributive over union?

Answers:

(a)



(b)



(c) Intersection appears to be distributive over union because $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Concept: Complement of a set.

(10) The complement of set A is the set of all the elements in the universal set U that are not in set A. The symbol for the complement of set A is $\bar{A}$. Sometimes the symbol A' is used, or the symbol ~A.

Answer the following.

(a) To what is the intersection of set A and set $\overline{A}$ equal?

(b) To what is the union of set A and set $\overline{A}$ equal?

Answers:

(a) $A \cap \overline{A} = \emptyset$ (b) $A \cup \overline{A} = U$

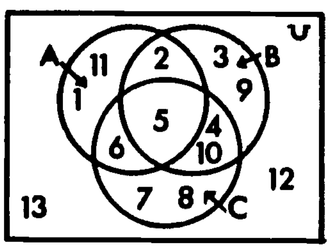
(11) If $U = \{1, 2, 3, \ldots 10\}$, $B = \{1, 3, 5, 7, 9\}$, and $C = \{2, 4, 6, 7, 8\}$, determine the following sets.

(a) $\overline{B}$	(e) $\overline{B} \cap \overline{C}$	(i) $\overline{B} \cup C$
(b) $\overline{C}$	(f) $\overline{B \cup C}$	(j) $B \cap \overline{C}$
(c) $B \cap C$	(g) $\overline{B} \cup \overline{C}$	(k) $\overline{B} \cap C$
(d) $\overline{B \cap C}$	(h) $B \cup \overline{C}$	(l) $B \cap \overline{B}$

Answers:

(a) {2, 4, 6, 8, 10}
(b) {1, 3, 5, 9, 10}
(c) {7}
(d) {1, 2, 3, 4, 5, 6, 8, 9, 10}
(e) {10}
(f) {10}
(g) {1, 2, 3, 4, 5, 6, 8, 9, 10}
(h) {1, 3, 5, 7, 9, 10}
(i) {2, 4, 6, 7, 8, 10}
(j) {1, 3, 5, 9}
(k) {2, 4, 6, 8}
(l) $\emptyset$

(12)



At the left is a Venn diagram representing the following sets.

$U = \{1, 2, 3, \ldots 13\}$

$A = \{1, 2, 5, 6, 11\}$

$B = \{2, 3, 4, 5, 9, 10\}$

$C = \{4, 5, 6, 7, 8, 10\}$

List the elements in each of the following sets.

(a) $\overline{A}$	(e) $\overline{A \cap C}$	(i) $U \cap \overline{A}$
(b) $\overline{A \cup B}$	(f) $\overline{(A \cup B) \cup C}$	(j) $\overline{A} \cap \overline{C}$
(c) $\overline{A \cap B}$	(g) $\overline{A} \cap \overline{B}$	(k) $A \cap (B \cap C)$
(d) $\overline{A \cup C}$	(h) $A \cup \overline{B}$	(l) $\overline{(A \cup B)} \cap C$

Answers:

(a) {3, 4, 7, 8, 9, 10, 12, 13}
(b) {7, 8, 12, 13}
(c) {1, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13}
(d) {3, 9, 12, 13}

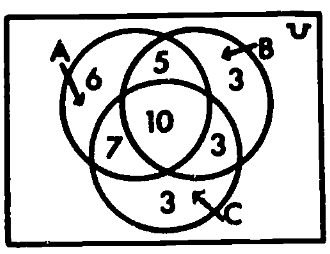
(e) {1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13}
(f) {12, 13}
(g) {7, 8, 12, 13}
(h) {1, 2, 5, 6, 7, 8, 11, 12, 13}
(i) {3, 4, 7, 8, 9, 10, 12, 13}
(j) {3, 9, 12, 13}
(k) {5}
(l) {7, 8}

(13) Members of an English class had been assigned three books (A, B, C) to read during the semester. After eight weeks, a poll was taken. It showed that each pupil in the class had read at least one of the books. The poll also gave the following data.

Ten pupils had read all three books.
Fifteen pupils had read both book A and book B.
Seventeen pupils had read both book A and book C.
Thirteen pupils had read both book C and book B.
Twenty-eight pupils had read book A.
Twenty-one pupils had read book B.
Twenty-three pupils had read book C.

Draw a Venn diagram to analyze the problem and use it to determine how many pupils were in the class.

Answer:



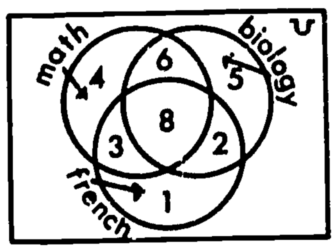
There were thirty-seven pupils in the class.

(14) A group of pupils were interviewed to determine how many of them would be studying mathematics, biology, and French the following year.

All the pupils plan on taking one of the courses.
Eight pupils plan on studying all three subjects.
Six pupils plan on studying mathematics and biology but not French.
Three pupils plan on studying mathematics and French but not biology.
Two pupils plan on studying biology and French but not mathematics.
Five pupils plan on studying biology but not mathematics nor French.
Four pupils plan on studying mathematics but not biology nor French.
One pupil plans on studying French but not mathematics nor biology.

Draw a Venn diagram and use it to determine how many pupils were interviewed.

Answer:



Twenty-nine pupils were interviewed.

1.5 CARTESIAN SETS

Concept: Ordered number pairs.

(1) The topic of ordered number pairs was covered previously in the study of coordinate geometry. Ordered number pairs were used to identify points on a number plane. Consider the set of natural numbers less than 6. $U = \{1,2,3,4,5\}$.

List all the possible ordered number pairs that may be formed from such a set of elements.

Answers:

(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)
(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)
(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)
(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)
(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)

Concept: Cartesian Sets.

(2) The set of all ordered number pairs that may be formed from a given set U is called the Cartesian set of U. This is written $U \times U$, and is read U cross U. It is often called the Cartesian product.

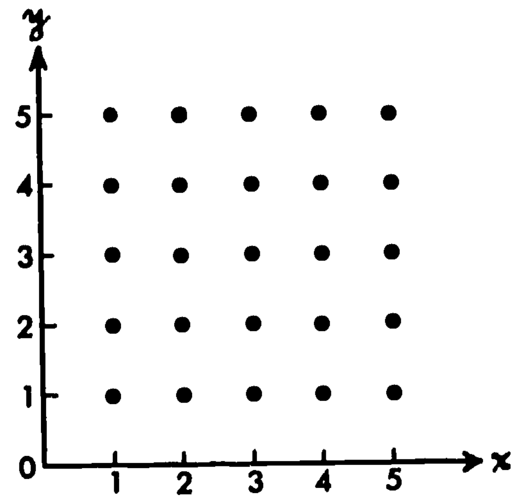
Draw the following:

(a) A graph of $U \times U$ where $U = \{1, 2, 3, 4, 5\}$. (The number pairs are (x, y) with x and y belonging to U.)

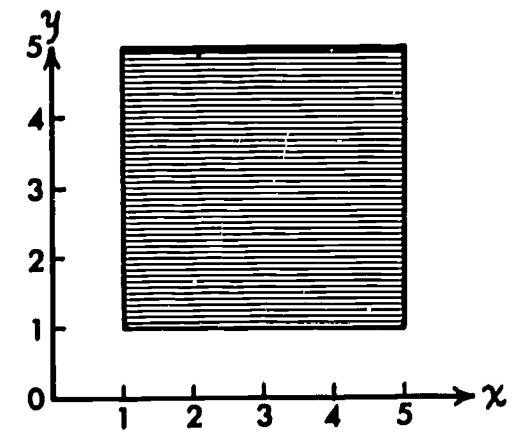
(b) A graph of $U \times U$ where the universal set U is the set of all real numbers from 1 to 5, including 1 and 5. (The graph will be the graph of all ordered number pairs (x, y) for which x and y are such that $1 \leq x \leq 5$ and $1 \leq y \leq 5$.)

Answers:

(a)



(b)



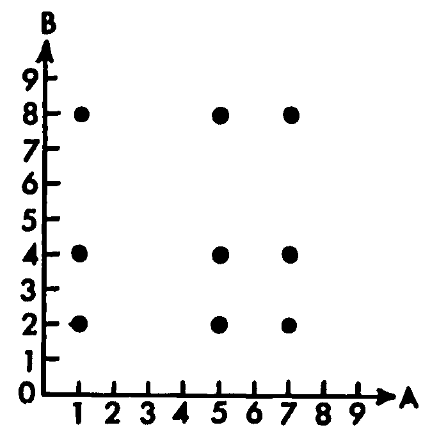
(3) Given the following sets: U = {1, 2, 4, 5, 7, 8, 10}, A = {1, 5, 7}, B = {2, 4, 8}, and C = {1, 2, 10}, answer the following.

(a) List the elements in the Cartesian set A × C.
(b) List the elements in the Cartesian set B × C.
(c) Graph the elements in the Cartesian set A × B.

Answers:

(a)	(1,1)	(5,1)	(7,1)
	(1,2)	(5,2)	(7,2)
	(1,10)	(5,10)	(7,10)
(b)	(2,1)	(4,1)	(8,1)
	(2,2)	(4,2)	(8,2)
	(2,10)	(4,10)	(8,10)

(c)



1.6 SOLUTION SETS

<u>Concept</u>: Solution set notation.

(1) The set of answers to a given problem or equation may be called the solution set. Consider the problem "What natural numbers are greater than 6 and less than 11?" The solution set to this problem would be $\{7, 8, 9, 10\}$.

<u>Indicate the solution sets to the following problems and equations.</u>

(a) In the set of natural numbers, the set of all x such that $18 \leq x \leq 21$.

(b) $x + 3 = 6$ in the set of natural numbers.

(c) $x^2 = 9$ in the set of integers.

(d) $x + 6 = 3$ in the set of natural numbers.

<u>Answers</u>:

(a) $\{18, 19, 20, 21\}$ (b) $\{3\}$ (c) $\{-3, +3\}$ (d) $\emptyset$

<u>Teacher Notes</u>

UNIT 2: ALGEBRAIC EXPRESSIONS

PART 1. BACKGROUND MATERIAL FOR TEACHERS

2.1 INTRODUCTION

The purpose of this unit is to introduce the concept of variable and the use of variables in algebraic expressions. The pupils have had experience in previous years in the use of placeholders or symbols to represent numbers in equations. In the elementary grades some pupils are given experience with equations such as $\square + 4 = 7$ or $m - 6 = 3$. However, in this unit the approach to the use of variables is somewhat more formal.

Sometimes a pupil may ask the teacher, "Why do I have to learn this algebra?" or "What value will algebra be to me in my daily living?" Such questions are an expression of the concern many pupils have when they first begin the study of algebra and realize that the course pertains to a body of mathematics not usually required in ordinary daily activities such as shopping, banking, and computing income tax. It may be advisable for the teacher to explain right at the beginning of the course the purposes of algebra and the reasons for mastering the material contained in the course.

One of the most important purposes of algebra is to prepare the pupil for high school courses that require a knowledge of algebra, such as chemistry and physics. Many pupils do not realize that the solution to a large number of scientific problems even at the high school level requires the use of algebra. Science courses at the college level very often require the knowledge of calculus. A knowledge of algebra is necessary in almost all mathematics courses at the senior high school level. These courses in turn are necessary for the study of analytic geometry and calculus required in many scientific, engineering, and mathematical courses at the college level.

In addition to its value in terms of preparation for the sciences and for further mathematics, the study of algebra has important values of its own. It gives pupils an opportunity to study and explore an important and intrinsically interesting structure. It gives pupils further practice in the techniques of problem solving, including such areas as organization of data and plans of attack. In this course, pupils become familiar with many concepts of variation and various relationships between numerical quantities. And algebra gives pupils a talking acquaintanceship with mathematical vocabulary and concepts important in the background of a well-grounded member of society.

Rather than being mainly a preparation for ordinary daily activities, the study of algebra is for many pupils one of the first important steps on the road to college or preparation for some lifetime occupation. It is an important element of a well-rounded education and its importance should be made clear to the pupils taking the course.

2.2 VARIABLES

After the concept of set has been developed, it becomes necessary to develop some convenient method for referring to arbitrary elements of the set. This may be done verbally, but it is far easier to use symbols and apply the concept of variable. A variable is a symbol and is usually a letter of the alphabet. The symbol occurs in a mathematical phrase or sentence and acts as a spaceholder or placeholder in that it holds a space or place in the phrase or sentence which can be filled by some one or more of the numbers in a given set, called the replacement set. For example, consider the set of natural numbers and the phrase 7x. The symbol x is a variable. The set of natural numbers is the replacement set for x, in that x may be replaced by any element in the set of natural numbers. The expression $(7)\cdot(5)$ shows a particular replacement of x from its replacement set. The number 35 is the value of the expression 7x when the variable is replaced by 5.

There are several concepts in this earlier material that serve as an introduction to algebra. There is the concept of a symbol, usually a letter, being used as a placeholder for a number. The phrase in which the symbol or symbols occur is called an expression. 3x and 5x + 3y are expressions. There is the concept of variable. A symbol in a phrase or sentence is a variable only if it is replaceable by a number belonging to a defined set of numbers called the replacement set of the variable. If the replacement set is not given, it is usually assumed to be the set of real numbers. There is the concept of the value of an expression when the variable is replaced by an element in the replacement set.

The next concept introduced is that of open sentences and phrases. A sentence that contains one or more variables so that it is neither true nor false as it stands, and becomes either true or false only when the variable is replaced by a number from the replacement set, is called an open sentence. The equation $y + 3 = 19$ is an open sentence. It is neither true nor false as it stands. If the variable y is replaced by the number 5 from the set of real numbers, the equation becomes $5 + 3 = 19$, and this is a false sentence. If the variable y is replaced by the number 16 from the set of real numbers, the equation becomes $16 + 3 = 19$, and this is a true statement. The statements $x = 3$ and $3x + 2y = 19$ are examples of open sentences.

An algebraic expression that does not contain the operation addition nor the operation subtraction is called an algebraic term. 9abc is a term. An expression may contain several terms. Thus, $x + y$ is an expression consisting of two terms.

An expression consisting of only one term is called a monomial. An expression consisting of two terms is called a binomial. A trinomial is an expression consisting of three terms. Any expression involving two or more terms may also be called a polynomial. Thus 16xy is a monomial. Also 2x + y is a binomial and may also be called a polynomial.

Algebraic terms may also be classified as like and unlike terms. If two terms have the same identical variable or variables and the exponents of corresponding variables are the same, these terms are called like terms; otherwise, they are called unlike terms. The terms 24ab and 6ab are like terms. The terms 9a and 9x are unlike terms. The terms 3x and $3x^2$ are unlike terms.

2.3 OPERATING WITH ALGEBRAIC TERMS

The pupils should be given experience at translating English phrases into algebraic phrases. This experience should include phrases involving addition, subtraction, multiplication, and division. Most difficulty is usually experienced in translating phrases involving the noncommutative operations of subtraction and division. In noncommutative operations, the order of the terms is of great importance and often the pupils place the terms in reverse order. This is particularly true of subtraction.

The phrases "a diminished by b" and "a less than b," where a and b are variables, are sometimes difficult. The first phrase is written as the algebraic expression a - b, and the second phrase is written as b - a. The phrase "the number of miles in f feet" is also often written erroneously as 5280f. It is correctly written as $\frac{f}{5280}$.

The pupils should be given experience at translating such English phrases into algebraic phrases in sufficient number and variety as to make them aware of the common errors made in such translation.

Almost the entire remainder of this portion of the unit is devoted to the development of the concepts that the sum, difference, product, and quotient of two like terms can be expressed as a single term and also that the product and quotient of two unlike terms can be expressed as a single term. The sum and the difference of two unlike terms cannot be expressed as a single term. One method of showing that the sum of two like terms, such as 3x and 4x, can be expressed as a single term is to first express 3x as x + x + x and 4x as x + x + x + x. The sum of 3x + 4x is therefore x + x + x + x + x + x + x or 7x. The second method of arriving at the sum of 3x + 4x is by application of the distributive and commutative principles, ab + ac = a(b + c) or (b + c)a. Applying this principle, 3x + 4x = (3 + 4)x and is equal to 7x.

The distributive principle is also used for demonstrating how to find the difference of two like terms. By applying the distributive and commutative principles, ab - ac = a(b - c) or (b - c)a, the difference 6x - 3x can be shown to equal (6 - 3)x or 3x.

If the pupils have not had subtraction of integers, examples of subtraction of like terms must be restricted to examples where the coefficient of the minuend is greater than the coefficient of the subtrahend so that the coefficient of the difference is not a negative number. In unit 3, subtraction of integers is introduced for those pupils who did not have this optional topic in Mathematics 8X and is reviewed for those pupils who did cover the topic the previous year. After this topic has been covered, examples of subtraction of like terms need not be restricted to those resulting in differences with positive coefficients.

The application of the distributive and commutative principles in collecting like terms is used in developing the concept of simplifying the sum of a series of terms if the series includes two or more like terms. Applying the commutative principle, the terms in the expression $3x + 4y + 5x + 3z$ may be rearranged to read $3x + 5x + 4y + 3z$ so that the two terms $3x$ and $5x$ may be combined into the single term $8x$, resulting in the simplified expression $8x + 4y + 3z$. The process of collecting and combining like terms is restricted to the operation addition in this material. The discussion of collecting like terms involving subtraction is not covered until after the topic of subtraction is covered in the next unit.

The concept of multiplying two like or unlike terms includes the ideas that the numerical coefficients are written first and then the resulting letters are arranged in alphabetical order. If two numerical coefficients are contained in the final product, they are combined into a single numerical coefficient. For example, $(3abc)(4d) = 3 \cdot 4abcd = 12abcd$. The pupils may be given some experience at expressing the product of two like factors by the use of exponents such as $x \cdot x = x^2$ and $y \cdot y \cdot y = y^3$. It is not intended that the topic of exponents be pursued very far at this point as this topic is introduced in a later unit.

The quotient of two like or unlike quantities can be expressed as a single term and expressed as the ratio of the dividend to the divisor. The quotient may be reduced to lowest terms by the application of the principle of equality of rational expressions and the principle that $\frac{ac}{bc} = \frac{a}{b}$. Any factor common to the numerator and denominator may be dropped or canceled in reducing the expression to lowest terms. Such factors include numbers and variables; for example, $\frac{4x^2y}{12xz} = \frac{4 \cdot x \cdot x \cdot y}{4 \cdot 3 \cdot x \cdot z} = \frac{xy}{3z}$.

One important fact to emphasize is that division by zero is undefined and not permitted. The denominator of the expression $\frac{a}{x - x}$ is zero and it is therefore not a valid expression. This topic is discussed further in later units where expressions such as $\frac{3}{x - 1}$ are considered. Such an expression is a valid expression only if $x \neq 1$. If $x = 1$, then $x - 1$ is zero. This is not permitted. Therefore, the replacement set of the variable x does not include the element 1.

UNIT 2: ALGEBRAIC EXPRESSIONS

PART 2. QUESTIONS AND ACTIVITIES FOR CLASSROOM USE

2.1 INTRODUCTION

This unit has been placed before the units on the integers and the real numbers in order to involve the pupils in the study of actual algebra as quickly as possible. The questions and activities pertaining to variables, open sentences, and operating with algebraic terms have been restricted in this unit so as not to include negative integers or quadratic expressions other than the very simplest. The concepts contained in this unit may be extended to include negative integers after Unit 3, The Integers, has been completed. These concepts have been extended to include quadratic expressions in Unit 7, Exponents and Radicals and in other later units.

2.2 VARIABLES

Concept: Definition of a variable.

(1) The following statement appeared on a history test. "The second president of the United States was □." Possible replacements are: Madison, Monroe, Washington, Adams, and Jefferson.

Answer the following questions.

(a) What is the purpose of the rectangle?
(b) The rectangle is a placeholder for one of five names. Which are the five names?
(c) May the name Monroe be substituted for the rectangle? Will the resulting sentence be true or false?
(d) May the name Adams be substituted for the rectangle? Will the resulting sentence be true or false?

Answers:

(a) The rectangle is a placeholder for one of the possible replacements.
(b) Madison, Monroe, Washington, Adams, Jefferson
(c) The name Monroe may be substituted for the rectangle, but the resulting sentence is false.
(d) The name Adams may be substituted for the rectangle, and the resulting sentence is true.

(2) Consider the following statement.
In our algebra class, △ sits nearest the door.
Possible replacements are: any pupil in the classroom.

Answer the following questions.

(a) Describe the set of possible answers by listing each element in the set.
(b) Which element in the set of possible answers is the element which, when substituted for the triangle, results in a true sentence?

Answers:

(a) The answer will be the set in which the name of each pupil in the classroom is listed.

(b) The correct answer will be the name of the pupil who sits nearest the door.

(3) Answer the following questions.

(a) The number of grains of sand in a quart jar is $\square$. Can a negative number be a possible replacement? Describe the set of possible replacements to this question.

(b) Given $x + 3 < 10$. Possible replacements are any non-negative integers. What number may be substituted for x? What numbers, when substituted for x, will result in true sentences?

Answers:

(a) The replacement cannot be negative. The set of possible replacements is the set of non-negative integers.

(b) Any non-negative integer may be substituted for x in the inequality. Any of the numbers 0, 1, 2, 3, 4, 5, or 6 may be substituted for x and the resulting statement is true.

(4) Any symbol, including frames and letters, that is used as a placeholder for some word, phrase, or number is called a variable. The set of possible replacements is called the replacement set of the variable. The set of elements in the replacement set that will, when substituted for the variable, result in a true statement is called the solution set.

For each of the following identify the variable, the replacement set, and the solution set.

(a) $\square$ was the thirty-fourth president of the United States. Possible replacements are: Truman, Eisenhower, Hoover, Kennedy, Roosevelt.

(b) The boy next door owns 13 marbles. I own more marbles than he does, but I do not own 15 marbles. I own $\triangle$ number of marbles.

(c) $y + 9 < 13$, where y is a non-negative integer.

Answers:

(a) The variable is $\square$.
The replacement set is {Truman, Eisenhower, Hoover, Kennedy, Roosevelt}.
The solution set is {Eisenhower}.

(b) The variable is $\triangle$.
The replacement set is {any natural number}.
The solution set is {14}.

(c) The variable is y.
The replacement set is {0, 1, 2, ...}.
The solution set is {0, 1, 2, 3}.

Concept: Open sentences and phrases.

(5) The sentence "George Washington was the fifth president of the United States" is false. The statement $2 + 5 = 7$ is true. What can you say about the truthfulness of the statement "He is a teacher?"

Answer: The statement cannot be determined to be true or false until we know to whom the pronoun, he, refers.

(6) Under what conditions is the statement $x + 4 = 9$ true? Under what conditions is it false?

Answer: The statement $x + 4 = 9$ is true if and only if the placeholder x is replaced by the number 5. If x is replaced by any number other than 5, the statement is false.

(7) A sentence that contains a variable is called an open sentence. What can you say about the truthfulness of an open sentence?

Answer: An open sentence is neither true nor false as it stands.

(8) When does an open sentence become true or false?

Answer: An open sentence becomes true or false only when the variable is replaced by an element in the replacement set.

(9) For each of the following statements, indicate whether the sentence is true or false when the variable is the placeholder for the number 5.

(a) $\square + 11 > 16$

(b) $\frac{3\triangle + 5}{2} < 15$

(c) $5(x + 3) = 13x - 25$

Answers:
(a) False (b) True (c) True

(10) What would be the name given to a phrase that contains a variable?

Answer: An open phrase

(11) Using the letter n as the variable or placeholder, write an expression for each of the following.
(a) Six more than the variable
(b) Seventeen less than the variable
(c) Three times the variable
(d) The quotient of the variable divided by 3

Answers:
(a) $n + 6$ (c) $3n$
(b) $n - 17$ (d) $\frac{n}{3}$

(12) Write an expression for each of the following open phrases.

(a) The number of feet in y yards
(b) The number of miles in f feet
(c) The number of feet in m miles

Answers:
(a) $3y$ (b) $\frac{f}{5280}$ (c) $5,280m$

Concept: Value of open phrases.

(13) Determine the number represented by each of the following open phrases when the variable ▭ is replaced by 2, the variable △ replaced by 3, and the variable □ replaced by 5.

(a) ▭ + △ - □

(b) ▭ (△ + □)

(c) ▭ · △ + □

(d) (6▭ - 4△) · $\frac{△}{□}$

(e) □ · □ - 6△ + ▭ · ▭

Answers:
(a) 0 (b) 16 (c) $\frac{6}{5}$ (d) 0 (e) 11

Concept: Algebraic terms.

(14) Below is a list of algebraic expressions some of which are labeled "expression consisting of one term" and the others are labeled "expression consisting of ... terms." By observing which operations are contained in each open phrase, what do you conclude to be the characteristics of an algebraic term?

(a)	$5xy$	expression consisting of one term
(b)	$3abc - 7x$	expression consisting of two terms
(c)	156	expression consisting of one term
(d)	$17w + 5z$	expression consisting of two terms
(e)	$\frac{5x}{2}$	expression consisting of one term
(f)	$4a + 3b - 2c$	expression consisting of three terms
(g)	$x + 1$	expression consisting of two terms

Answer: An algebraic term is an expression which contains neither the operation addition nor the operation subtraction.

Concept: Like and unlike terms.

(15) The column below contains pairs of terms that are called like terms.

(a) 6xy and 9xy
(b) 16ab and 5ba
(c) 14x and $\frac{9x}{2}$
(d) 167 and 42
(e) 17a and 4a

The column below contains pairs of terms that are called unlike terms.

(a) 6xy and 9x
(b) 7y and 5c
(c) 142 and 17a
(d) 4k and 5
(e) x and x^2

What appears to be the difference between like and unlike terms?

Answer: Like terms contain the same powers of the same variable or variables. Unlike terms do not contain the same powers of the same variables.

2.3 OPERATING WITH ALGEBRAIC TERMS

Concept: Algebraic expressions.

(1) Using the letters a and b as placeholders for two numbers, express each of the following in algebraic symbols.

(a) The sum of the two numbers
(b) The difference of the two numbers, the letter, a, representing the minuend and the letter, b, the subtrahend
(c) The product of the two numbers
(d) The quotient of the two numbers, the letter, a, representing the dividend and the letter, b, representing the divisor
(e) The sum of the squares of the two numbers
(f) The quotient obtained by dividing the sum of the two numbers by the product of the two numbers
(g) a diminished by b
(h) b more than a
(i) b less than a
(j) a times b

Answers:

(a) a + b
(b) a - b
(c) ab
(d) $\frac{a}{b}$
(e) $a^2 + b^2$
(f) $\frac{a + b}{ab}$
(g) a - b
(h) a + b
(i) a - b
(j) ab

Concept: Addition of like and unlike terms.

(2) Answer the following questions.

(a) Write an expression equivalent to 6x + 7x, making application of the distributive principle.
(b) Perform the addition indicated within the parentheses. What is the resulting expression?
(c) Is 6x + 7x = 13x a true statement?

Answers:
(a) $(6 + 7)x$
(b) $13x$
(c) Yes

(3) By applying the distributive principle, find the simplest expression for each of the following sums and differences.

(a) $19a + 17a$
(b) $15xy + 21xy$
(c) $7xy + 2xy$
(d) $5kmp - 3kmp$
(e) $3 \square + 4 \square$

Answers:
(a) $19a + 17a = (19 + 17)a = 36a$
(b) $15xy + 21xy = (15 + 21)xy = 36xy$
(c) $7xy - 2xy = (7 - 2)xy = 5xy$
(d) $5kmp - 3kmp = (5 - 3)kmp = 2kmp$
(e) $3 \square + 4 \square = (3 + 4)\square = 7 \square$

(4) Is there any way that the distributive principle can be applied to find a simpler expression for the sum of the unlike terms $3x + 4y$?

Answer: No

(5) Is it permissible to rearrange the terms in the expression $8x + 9y + 7x + 7y$ so that like terms are adjacent to one another?

Answer: Yes, addition is commutative and the terms may be rearranged as $8x + 7x + 9y + 7y$.

(6) Applying the distributive principle, perform the addition of like terms to simplify the expression $8x + 7x + 9y + 7y$.

Answer:
$8x + 7x + 9y + 7y = (8 + 7)x + (9 + 7)y = 15x + 16y$

(7) The process in which terms in an expression are rearranged so that like terms are adjacent to each other and then the addition of the like terms is performed by applying the distributive principle is sometimes called collecting and combining like terms.

Determine a simpler expression for each of the following by collecting and combining like terms.

(a) $7ab + 6a + 4b + 3ab + 9b + 5a$

(b) $21 \triangle + 16 \square + 4 \triangle + 3 \square + 9 \triangle$

Answers:
(a) $7ab + 6a + 4b + 3ab + 9b + 5a$
$= 7ab + 3ab + 6a + 5a + 4b + 9b$
$= (7 + 3)ab + (6 + 5)a + (4 + 9)b$
$= 10ab + 11a + 13b$

(b) $21\triangle + 16\square + 4\triangle + 3\square + 9\triangle$

$= (21 + 4 + 9)\triangle + (16 + 3)\square$

$= 34\triangle + 19\square$

<u>Concept</u>: Multiplication of like and unlike terms.

(8) The product of x and 9 is written 9x.
The product of 3b and ac is written 3abc.
The product of y and 9k and p is written 9kpy.
<u>By observing the pattern in the above three examples, describe what appears to be the general rule for writing the product of unlike terms.</u>

<u>Answer</u>: The product of unlike terms is written with the number first. (The variables are usually arranged in alphabetical order.)

(9) <u>Write the following products in simplest form.</u>
(a) (3x)(4y)
(b) (16x)(ab)
(c) (3x)(4y)(5z)

<u>Answers</u>:
(a) $(3x)(4y) = 3 \cdot 4 \cdot xy = 12xy$
(b) $(16x)(ab) = 16abx$
(c) $(3x)(4y)(5z) = 3 \cdot 4 \cdot 5 \cdot xyz = 60xyz$

(10) $x \cdot x = x^2$ and $b \cdot b \cdot b = b^3$.
<u>Complete the following</u>:
(a) $w \cdot w \cdot w \cdot w =$
(b) $y \cdot y \cdot y \cdot y \cdot y =$

<u>Answers</u>:
(a) $w \cdot w \cdot w \cdot w = w^4$
(b) $y \cdot y \cdot y \cdot y \cdot y = y^5$

(11) <u>Express the product of each of the following in simplest form.</u>
(a) (3y)(4y)
(b) (16ab)(3a)(4b)(2c)
(c) (3a)(4abc)(6ac)

<u>Answers</u>:
(a) $(3y)(4y) = 3 \cdot 4 \cdot y \cdot y = 12y^2$
(b) $(16ab)(3a)(4b)(2c) = 16 \cdot 3 \cdot 4 \cdot 2 \cdot aabbc$
$= 384a^2b^2c$
(c) $(3a)(4abc)(6ac) = 3 \cdot 4 \cdot 6 \cdot aaabcc$
$= 72a^3bc^2$

<u>Concept</u>: Division of like and unlike terms.

(12) <u>Express the following as a ratio of the dividend to the divisor.</u>

(a) $2 \div 3$
(b) $17 \div 5$
(c) $13x \div 15$
(d) $5a \div 7b$
(e) $15 \div x$
(f) $y \div 3a$
(g) $3abc \div 5xyz$

Answers:

(a) $\frac{2}{3}$ (b) $\frac{17}{5}$ (c) $\frac{13x}{15}$ (d) $\frac{5a}{7b}$ (e) $\frac{15}{x}$

(f) $\frac{y}{3a}$ (g) $\frac{3abc}{5xyz}$

(13) Under what conditions are two rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ equivalent?

Answer: The two rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if and only if $ad = bc$. The cross products must be equal.

(14) Is the statement $\frac{ac}{bc} = \frac{a}{b}$ always true?

Answer: Yes, $\frac{ac}{bc} = \frac{a}{b}$ is always true because the cross products are equal; $acb = bca$.

(15) What is the common name given to the principle $\frac{ac}{bc} = \frac{a}{b}$? Why is it called this?

Answer: The cancellation law. Any factor common to the numerator and denominator may be dropped or canceled in forming a new rational expression equivalent to the given expression.

(16) What is meant by "reducing a rational expression to lowest terms?"

Answer: By "reducing to lowest terms" is meant that an equivalent expression is formed which has no factor common to the numerator and denominator, except the factor 1.

(17) Express the quotient in each of the following as a rational expression reduced to lowest terms.

(a) $14x \div x$
(b) $21ab \div 7b$
(c) $36x^2 \div 6xy$

Answers:

(a) $14x \div x = \frac{14x}{x} = 14$

(b) $21ab \div 7b = \frac{21ab}{7b} = 3a$

(c) $36x^2 \div 6xy = \frac{36xx}{6xy} = \frac{6x}{y}$

(18) If it is possible to do so, express each of the following as a single term reduced to lowest terms.

(a) $\frac{17ab + 13ab}{5a}$

(b) $\frac{(6x)(4y) - 10xy}{12xy + (3x)(3y)}$

(c) $\frac{5ab + 3ab}{6ab - (3a)(2b)}$

Answers:

(a) $\frac{17ab + 13ab}{5a} = \frac{30ab}{5a} = 6b$

(b) $\frac{(6x)(4y) - 10xy}{12xy + (3x)(3y)} = \frac{24xy - 10xy}{12xy + 9xy} = \frac{14xy}{21xy} = \frac{2}{3}$

(c) $\frac{5ab + 3ab}{6ab - (3a)(2b)} = \frac{8ab}{6ab - 6ab} = \frac{8ab}{0} =$ meaningless.

This algebraic expression is meaningless because division by zero is undefined and not permitted, so no valid expression can be written.

Teacher Notes

UNIT 3: THE SET OF INTEGERS

PART 1. BACKGROUND MATERIAL FOR TEACHERS

3.1 INTRODUCTION

The properties of the set of integers and the operations in this set of numbers were presented in detail in the Mathematics 8X course. However, some topics pertaining to negative integers were indicated as optional and were to be introduced at the discretion of the teacher. Due to the limited time that can be devoted to the study of each of the units in Mathematics 9X, the concepts of the properties of and operations in the set of integers cannot be presented in such detail or with such emphasis as was done in the Mathematics 8X course.

In this unit the concepts are presented in such a way so as to be used as a review for those pupils who studied the concepts in Mathematics 8X and to be used as an introduction to these concepts for those pupils who did not study the negative integers in the Mathematics 8X course. Care should be taken to cover the topics in this unit within the suggested time allotment so as to leave sufficient time to cover properly the other units in the course.

3.2 THE SET OF NATURAL NUMBERS

This section is a concise review of the properties of the set of natural numbers and definition of the operations in this set of numbers. The concepts pertaining to these properties and operations were presented in Mathematics 7X and Mathematics 8X. In these courses it was recommended that the discovery approach be used as much as was practical. This section is simply a summary and review, with care being taken to define the operations of addition, subtraction, multiplication, and division so that these operations may be defined in the set of integers with as little modification as possible.

The first operation to be performed in the set of natural numbers is counting. Counting may be carried out indefinitely in the direction of greater numbers, but can be carried out in the direction of smaller numbers only as far as the number 1. The number 1 is the least natural number.

The next operation considered is addition. Addition may be defined in terms of counting. If the letters a and b represent any natural numbers, the sum a + b may be determined by counting "b" successive numbers from a in the direction of greater numbers. The last number so counted is the sum a + b. For example, the sum 3 + 5 is determined by counting five numbers from the number 3 in the direction of greater numbers. The last number so counted is 8; therefore, the sum 3 + 5 is 8. In addition of natural numbers, the counting is always carried out in the direction of greater numbers. Counting in this direction may be carried out indefinitely so there is no restriction on such counting. The sum of any two natural numbers may be determined in this manner. Therefore, the set of natural numbers has the property of closure under addition.

Subtraction is defined as the inverse operation of addition. The subtraction $a - b = \square$ is carried out by performing the corresponding addition $\square + b = a$. For example, to find the difference $12 - 8 = \square$, the corresponding addition is $\square + 8 = 12$. The problem is to determine what number must be added to 8 to result in a sum of 12; the answer is 4. Subtraction requires the use of an addition table. Difficulties in performing subtraction sometimes indicate lack of proficiency in performing addition or failure to memorize an addition table properly. Subtraction may also be defined in terms of counting. The difference $a - b$ may be determined by counting "b" successive numbers from a in the direction opposite to that used in performing addition. That is, the counting is carried out in the direction of smaller numbers.

Of these two definitions of subtraction, it is essential that the pupils master the concept of subtraction as being the inverse operation of addition. It is this concept which is generalized for operating with all numbers in the real number system.

In performing the subtraction $8 - 9 = \square$, the corresponding addition is $\square + 9 = 8$. There is no natural number that, when added to 9, will result in a sum 8. If the subtraction is performed by the counting method, the counting cannot be carried out. In attempting to count 9 numbers from the number 8 in the direction of smaller numbers, it is soon discovered that there are only 7 such numbers to count. The difference $8 - 9$ is not in the set of natural numbers and so the set of natural numbers does not have the property of closure under subtraction.

Multiplication in the set of natural numbers is defined in terms of addition. To find the product of c and d, where c and d are any natural numbers, add "c" addends each of which is d. The product of 4 and 5 is found by performing the addition $5 + 5 + 5 + 5$; the answer is 20. Short cuts in performing multiplication, with which the pupils are already familiar, involve the use of a memorized multiplication table and application of the distributive principle. This is discussed later. In the set of natural numbers, multiplication is defined in terms of addition. There is no restriction on addition in this set of numbers and so there is no restriction on multiplication. The set of natural numbers has the property of closure under multiplication.

Division is defined as the inverse operation of multiplication. The quotient $a \div b = \square$ is determined by solving the corresponding multiplication problem $(\square)(b) = a$. For example, the quotient $12 \div 4 = \square$ is determined by solving the multiplication problem $(\square)(4) = 12$; the answer is 3. Division requires the knowledge of a multiplication table. In the set of natural numbers, division may also be defined in terms of repetitive subtraction. The quotient 12 ÷ 4 is the number of successive times that the number 4 can be subtracted from 12 and the resulting differences such that the final remainder is zero. The number 4 may be subtracted from 12 and the resulting differences 3 times with a final remainder of zero. The quotient 12 ÷ 4 is therefore 3.

Of these two definitions of division, the one of greatest value is that of division being the inverse operation of multiplication. This definition is valid in all the sets studied, the integers, the rationals, and the real numbers.

For some pairs of natural numbers there is no quotient in the set. For example, in determining the quotient 5 ÷ 3 = □ the corresponding multiplication is (□)(3) = 5. There is no natural number that, when multiplied by 3, will result in the product 5. Therefore, the set of natural numbers does not have the property of closure under division.

In this set of numbers, multiplication is distributive over addition and it is also distributive over subtraction. If a, b, and c represent any natural numbers, then a(b + c) = ab + ac and a(b - c) = ab - ac. This distributive principle may also be written ab + ac = a(b + c) and ab - ac = a(b - c) or even as ab + ac = (b + c)a and ab - ac = (b - c)a. Thus multiplication is distributive over addition from the left or from the right. This principle has been used by the pupils for several years in performing multiplications such as (567)(234). The number 234 may be considered to be the sum 200 + 30 + 4. The multiplication is (567)(200 + 30 + 4) = (567)(200) + (567)(30) + (567)(4). This is written as:

567		567
234		234
2268	or as	2268
17010		1701
113400		1134
132678		132678

Addition is not distributive over multiplication and subtraction is not distributive over multiplication.

$$a + (bc) \neq (a + b)(a + c)$$
and
$$a - (bc) \neq (a - b)(a - c)$$

Multiplication is not distributive over division and division is not distributive over multiplication.

$$a (b \div c) \neq (ab) \div (ac)$$
and
$$a \div (bc) \neq (a \div b)(a \div c)$$

Division is not distributive over addition nor subtraction from the left, but division is distributive over addition and subtraction from the right, when the replacement set of the variables is such that the indicated quotients are all natural numbers.

$$a + (b + c) \neq (a + b) + (a + c)$$

and

$$a + (b - c) \neq (a + b) - (a + c)$$

But $(a + b) + c = (a + c) + (b + c)$

and $(a - b) + c = (a + c) - (b + c)$

This can be demonstrated by substituting natural numbers for the letters a, b, and c and performing the indicated operations.

The set of natural numbers also has the property of the existence of an identity element under multiplication. The product of any natural number a and the number 1 is the number a. Thus $(a)(1) = a$.

The natural numbers are commutative and associative under the operations addition and multiplication. $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$. The properties of the natural numbers such as closure, commutative property, associative property, and distributive property are actually axioms of the system. They are not proven to be true; they are accepted as true for the purpose of establishing a useful number system. The teacher may wish to establish this fact that the properties of the natural numbers are accepted axioms at the beginning of the algebra course so that the pupils will have an understanding of axioms and their purpose in our number system when other sets of numbers are studied.

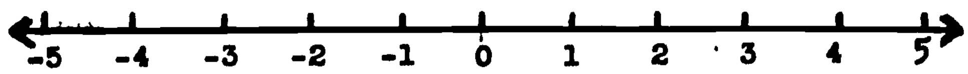
The pupils may also benefit from a discussion of inverse operation. Addition and subtraction are inverse operations of each other, and multiplication and division are inverse operations of each other. Using the symbols ● and # to represent operations and the letters a, b, and c to represent numbers, "● is the inverse of #" means that if a ● b = c, then c # b = a. Applying this concept of inverse operation, the pupils may be given the task of showing that certain pairs of operations are not inverse operations, such as addition and division, subtraction and multiplication; subtraction and division, addition and multiplication. For example, it can be shown that addition and division are not inverse operations because $12 + 4 = 16$ but $16 + 4 \neq 12$.

3.3 ADDITION IN THE SET OF INTEGERS

Addition in the set of integers, like addition in the set of natural numbers, may be defined in terms of counting. Counting in the set of integers can be carried out indefinitely in the positive direction or indefinitely in the negative direction. Before addition is defined, certain terms and symbols should be discussed. The positive integers may be written with a "+" sign in front of the number such as +3 or with the "+" raised slightly such as $^{+}3$ so as not to be confused with the symbol for the operation addition. However, it is common practice to drop the "+" sign and write the positive integers in the same manner as the natural numbers. That is, the integer +5 may be written as 5.

The negative integers are written with a "-" sign in front such as -3 or with the sign raised slightly such as $^{-}3$. Zero is neither positive nor negative.

The integers may be represented by points on a number line such as:



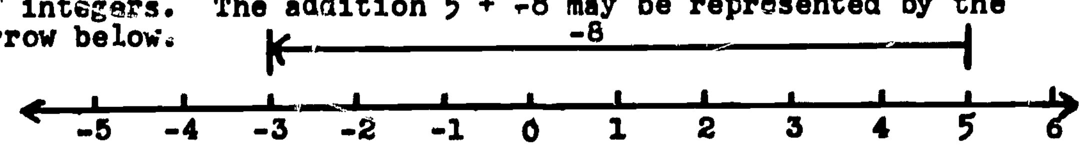
In such a graph any integer is greater than any integer to its left. The integer 3 is greater than the integer -31. This is indicated in symbolic form by writing $3 > -31$.

The absolute value of an integer is the number of units an integer is from zero on a number line. The integer +6 is six units from zero so its absolute value is 6. The integer -8 is eight units from zero so its absolute value is 8. The symbol for absolute value is two short vertical parallel line segment one on each side of the number such as $|-8|$. If the letter b represents a positive integer, $|b| = b$ and $|-b| = b$. The absolute value of zero is zero, thus $|0| = 0$.

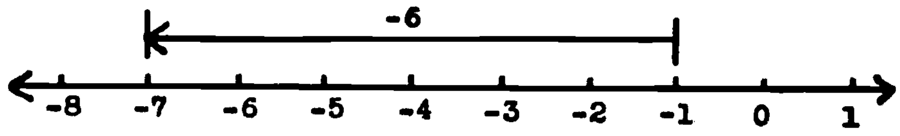
When the integers are represented by points on a number line, any two integers which are the same distance from zero are called opposites or inverses of each other. For every integer a there is an integer b such that their sum is zero. $a + b = 0$. The integer b is the additive inverse or opposite of a. If an integer is represented by the letter x, its inverse is represented by -x. The symbol -x is read "the inverse of x." The letter x may represent any integer. If x is a positive integer, -x is a negative integer. If x is zero, -x is zero. If x is negative, -x must be positive. The symbol -(-3) is read "the inverse of negative three." The inverse of a negative integer is the corresponding positive integer, thus $-(-3) = 3$.

The sum a + b in this set may be determined by counting. If b is a positive integer, the counting begins at the integer a and is carried out "b" consecutive integers in the positive direction. If b is a negative integer, the counting begins at the integer a and is carried out in the negative direction a number of consecutive integers equal to the absolute value of b. If b is zero, the sum is the integer a.

A number line is a very useful tool in teaching addition of integers. The addition $5 + {}^{-}8$ may be represented by the arrow below.



The addition is performed by counting eight consecutive integers in the negative direction from 5; the sum is -3. The addition $-1 + -6$ may be represented as:



The sum is -7.

The use of such a number line clearly demonstrates that the sum of two positive integers must be a positive integer and the sum of two negative integers must be a negative integer. In both instances, the absolute value of the sum is equal to the sum of the absolute values of the integers being added. The pupils can be led to discover this by doing a few such addition problems. The sum of two integers of opposite signs, one positive integer and one negative integer, is a little more difficult to calculate. However, by doing a number of such addition problems most pupils can soon discover that the absolute value of the sum of two integers is equal to the difference of the absolute values of the integers being added, and the sign of the sum of the two integers is the same as the sign of the addend that has the larger absolute value.

These rules for adding integers are sometimes stated as follows:

(1) To add two integers with the same sign, add the absolute values of the two numbers. The sum will have the common sign.

(2) To add two integers having unlike signs, find the difference between their absolute values. The sum will have the sign of the addend having the larger absolute value.

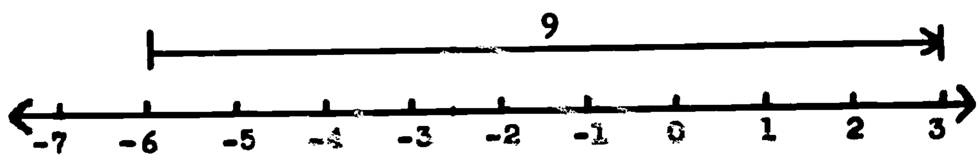
These rules are very convenient but emphasis must be placed on having the pupils discover these rules for themselves after having mastered the basic concept of addition, instead of having the teacher give them the rules.

One interesting relation that may be discussed is that the sum of absolute values of two integers does not always equal the absolute value of the sum of the integers. That is, $|a| + |b| = |a + b|$ is not true for all integers a and b. For example, $|-6| + |4| \neq |-6 + 4|$. $|-6| + |4| = 10$ while $|-6 + 4| = |-2| = 2$, and $10 \neq 2$.

Under addition, the set of integers has all the properties of the set of natural numbers under the same operation, plus one additional property. Under the operation addition, the set of integers has an identity element. The sum of any integer a and the integer zero is the integer a. Thus $a + 0 = a$. The number zero is the additive identity element.

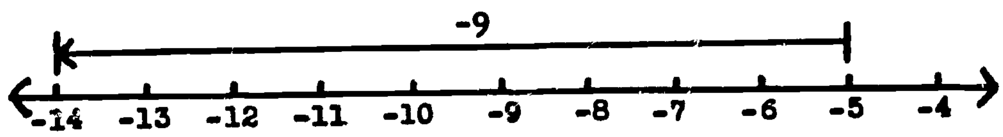
3.4 SUBTRACTION IN THE SET OF INTEGERS

Subtraction in the set of integers, like subtraction in the set of natural numbers, is defined as the inverse operation of addition. If $a + b = c$, then $c - b = a$ and if $d - e = f$ then $f + e = d$. The difference $a - b$ may be found by determining what integer must be added to b to give a sum of a. For example, (-6) - (-9) = □ has the same solution set as □ + (-9) = -6. The integer 3 must be added to -9 to give a sum of -6; therefore, (-6) - (-9) = 3. Subtraction may also be defined in terms of counting. The difference a - b may be found by starting at the integer a and counting in the direction opposite to that used in addition a number of consecutive integers equal to $|b|$. If b is a positive integer, counting is carried out in the negative direction. If b is a negative integer, counting is carried out in the positive direction. This can be demonstrated by use of the number line. For example, to find the difference (-6) - (-9), the counting begins at -6 and is carried out nine consecutive integers in the positive direction.



The difference (-6) - (-9) is 3.

Counting nine numbers from -6 in the positive direction is the same as adding 9 to -6. The difference (-6) - (-9) and the sum (-6) + (9) are the same number. Therefore, (-6) - (-9) = (-6) + (9). Subtracting a negative integer -b from an integer a gives the same result as adding the inverse of -b to a. a - (-b) = a + b. The difference -5 - 9 is determined by counting nine numbers from -5 in the negative direction. Counting nine numbers in the negative direction from -5 gives the same result as adding -9 to -5; that is, -5 - 9 = -5 + (-9).



The answer is -14. -5 - 9 = -14 and -5 + (-9) = -14. Subtracting a positive integer b from any integer a gives the same result as adding the inverse of b to a. Thus a - b = a + (-b).

These two rules may be summarized as "subtracting any integer b from any integer a is equivalent to adding the inverse of b to a; a - b = a + (-b)." Again the emphasis is on having the pupils discover this rule for themselves by applying the basic concept of subtraction in the set of integers.

In <u>Mathematics 8X</u>, subtraction of negative integers was an optional topic. Those pupils who studied subtraction of negative integers last year will need only a quick review of the topics. Those pupils who did not study this topic will need a more careful development of the concept of subtraction in the integers.

The difference of the absolute values of two integers is not always equal to the absolute value of the difference of the two numbers. The statement $|a| - |b| = |a - b|$ is not always true. $|9| - |-6| \neq |9 - (-6)|$. $|9| - |-6| = 3$. $|9 - (-6)| = 15$, and $3 \neq 15$.

3.5 MULTIPLICATION IN THE SET OF INTEGERS

Multiplication of two positive integers is defined in the same manner as multiplication of two natural numbers. The product of a and b, where a and b are any two positive integers, is found by adding a addends each of which is b, just like in the set of natural numbers. The sum of any number of positive integers is a positive integer so the product of two positive integers is a positive integer. Its absolute value is equal to the product of the absolute values of the two numbers being multiplied. The product of any integer and zero is therefore zero.

There are several ways of presenting the concept of multiplication of two integers, one of which is positive and one negative. Such a multiplication may be defined in terms of addition. If +a represents a positive integer and -b represents a negative integer, the product (+a)(-b) may be determined by adding a addends each of which is -b. The product (5)(-4) may be found by performing the addition (-4) + (-4) + (-4) + (-4) + (-4); the answer is -20. The integers have all the properties that the natural numbers have including the commutative property under multiplication. The product (-3)(2) is equal to the product (2)(-3). Regardless of whether the negative integer comes first or second, the product may be found by performing the appropriate addition of equal negative addends. The sum of any number of negative integers is a negative integer. Therefore, the product of any two integers, one negative and one positive, is a negative integer. The absolute value of the product is equal to the product of the absolute values of the integers being multiplied.

Another method of presenting the concept of multiplication of two integers with unlike signs is by making use of the distributive principle. There are two ways of performing the multiplication $5[4 + (-4)]$. One way is to first perform the addition within the brackets and then perform the multiplication. $5[4 + (-4)] = (5)(0) = 0$; therefore $5[4 + (-4)] = 0$. A second way of performing the indicated operations is to apply the distributive principle.

$$5[4 + (-4)] = 0$$
$$(5)(4) + (5)(-4) = 0$$
$$20 + (5)(-4) = 0$$

The only number that when added to 20 will result in the sum zero, is -20; therefore, (5)(-4) must equal -20.

The product of any positive integer (+a) and any negative integer (-b) may be found in this way.

$$(+a)[(+b) + (-b)] = 0$$
$$(+a)(+b) + (+a)(-b) = 0$$
$$+ab + (+a)(-b) = 0$$

Thus, the additive inverse of +ab is (+a)(-b). But the additive inverse of +ab is -ab. Therefore, (+a)(-b) must equal -ab.

The product is negative and its absolute value is equal to the product of the absolute values of the numbers being multiplied.

A third method of demonstrating the product of two integers with unlike signs is by the use of a number pattern. To demonstrate the product (3)(-2) the following may be used.

$$(3)(3) = 9$$
$$(3)(2) = 6$$
$$(3)(1) = 3$$
$$(3)(0) = 0$$
$$(3)(-1) = \square$$
$$(3)(-2) = \triangle$$

The pattern of the product is pointed out to the pupils. They can recognize this pattern and see that if the pattern is to continue, the product of (3)(-1) must be -3 and the product of (3)(-2) must be -6. They can be shown in this way that the product of two integers with unlike signs must be negative. This number pattern method may also be used to show that the product of two negative integers must be a positive integer. For example,

$$(-3)(3) = -9$$
$$(-3)(2) = -6$$
$$(-3)(1) = -3$$
$$(-3)(0) = 0$$
$$(-3)(-1) = \square$$
$$(-3)(-2) = \triangle$$

The pupils can be led to see the pattern of the products and if the pattern of these numbers is to continue, the product of (-3)(-1) must be +3 and the product of (-3)(-2) must be +6. They can see that the product of two negative integers must be a positive integer.

The concept of multiplication of two negative integers may be presented by making use of the distributive principle.

$$-6\,[5 + (-5)] = 0$$
$$(-6)(5) + (-6)(-5) = 0$$
$$-30 + (-6)(-5) = 0$$

Thus, the additive inverse of -30 is (-6)(-5). But the additive inverse of -30 is +30. The only number that will result in the sum zero when added to -30 is +30. Therefore, (-6)(-5) must equal +30. If -a and -b represent any negative integers, then

$$-a\,[b + (-b)] = 0$$
$$(-a)(b) + (-a)(-b) = 0$$
$$-ab + (-a)(-b) = 0$$

Thus, the additive inverse of -ab is (-a)(-b). But the additive inverse of -ab is +ab; therefore, (-a)(-b) must equal +ab.

The product of two negative integers is a positive integer and its absolute value is equal to the product of the absolute values of the integers being multiplied. The set of integers has an identity element under the operation multiplication.

$$(+a)(+1) = +a$$
$$(-a)(+1) = -a$$
$$(0)(+1) = 0$$

The product of any integer a and the integer +1 is the integer a. The integer +1 is the identity element under multiplication.

Very often teachers attempt to select some common every-day living experience with which the pupil is familiar and use this experience as an example of multiplication by a negative integer. There are very few, if any, common every-day experiences which are actual examples of multiplication by a negative integer. Such multiplication is a mathematical operation and it is probably best to discuss it in terms of pure mathematics rather than attempt to relate it to some common experience when such an experience actually does not involve multiplication by a negative integer. This will lessen the risk of confusing the pupil about this operation. The product of the absolute values of any two integers is equal to the absolute value of the product of the two integers. If +a and +b represent positive integers and -a and -b represent negative integers, then

$$|+a| \cdot |+b| = |(+a)(+b)|$$
$$|+a| \cdot |-b| = |(+a)(-b)|$$
$$|-a| \cdot |+b| = |(-a)(+b)|$$
$$|-a| \cdot |-b| = |(-a)(-b)|$$

3.6 DIVISION IN THE SET OF INTEGERS

Division in the set of integers is defined as the inverse operation of multiplication. For all allowable replacements the division a ÷ b = c has the same solution set $\{(a_1, b_1, c_1), ...\}$ as the corresponding multiplication cb = a, and the multiplication de = f has the same solution set as the corresponding division f ÷ e = d, division by zero being excluded. The division -14 ÷ -7 = □ may be solved by first writing the corresponding multiplication (□)(-7) = -14 and determining what integer multiplied by -7 results in the product -14; the answer is +2. After the pupils have mastered this concept of division, they can then be led to discover the common short cuts in performing division.

If the quotient of two positive integers is an integer, it is a positive integer. If +a and +b represent positive integers and -a and -b represent negative integers, the corresponding multiplication for the division +a ÷ +b = □ is (□)(+b) = +a. The symbol □ must be a placeholder for a positive integer because if it were replaced by a negative integer, the product would be negative and if it were replaced by zero, the product would be zero. The product is neither negative nor zero, so □ must be a placeholder for a positive integer.

If the division is possible, the corresponding multiplication for the division +a ÷ -b = □ is (□)(-b) = +a. If □ were to represent a positive integer, the product would be negative. If □ were to represent zero, the product would be zero. If □ represents a negative integer, the product is positive. The product is positive so □ must represent a negative integer, if the division is possible. In this same way it can be demonstrated that the quotient -a ÷ +b must be a negative integer, if the division is possible. Therefore, the quotient of two integers, one of which is positive and one negative, is a negative integer, if the division is possible in the field of integers.

If the division is possible, the quotient of two negative integers is a positive integer. The corresponding multiplication for the division $-a \div -b = \square$ is $(\square)(-b) = -a$. The number which replaces □ cannot be negative because if it were, the product would be positive. It cannot be zero because the product would then be zero. It can be positive because the product of a positive integer and a negative integer is a negative integer. Therefore, the quotient of two negative integers is a positive integer, if the division is possible in the field of integers.

If the dividend is zero, the quotient is zero. $0 \div a = \square$ where a is any integer except zero. The corresponding multiplication is $(\square)(a) = 0$. Since a is not zero, the number which replaces □ must be zero in order for their product to be zero. Therefore, the quotient of $0 \div a$ must be zero.

The rules for division of integers may be summarized as follows:

(1) The quotient of two integers with like signs is a positive integer, if the division can be performed in the field of integers.
(2) The quotient of two integers with unlike signs is a negative integer, if the division can be performed in the field of integers.
(3) The quotient, which results when zero is divided by any integer other than zero, is zero.
(4) Division by zero is undefined and not permitted.
(5) The absolute value of the quotient is equal to the quotient of the absolute values of the dividend and divisor.

If the quotient of the absolute values of two integers is an integer, it is equal to the absolute value of the quotient of the two integers.

$$|+a| \div |+b| = |+a \div +b|$$
$$|+a| \div |-b| = |+a \div -b|$$
$$|-a| \div |+b| = |-a \div +b|$$
$$|-a| \div |-b| = |-a \div -b|$$

Here +a and +b represent any positive integers and -a and -b represent any negative integers such that the indicated quotients are also integers.

3.7 THE PROPERTIES OF THE SET OF INTEGERS

The set of integers has all the properties of the set of natural numbers plus some properties in addition to those of the natural numbers. The set of integers has the property of closure under subtraction. Subtraction may be defined in terms of counting, and counting may be carried out indefinitely in either the positive or the negative direction. There is no limitation on counting in the set of integers so subtraction will have closure in this set of numbers. The set of integers does not have the property of closure under division. There is no quotient in the set of integers for the division -7 ÷ -5. There is no integer y such that (y)(-5) = -7. The set of integers has the property of having an additive inverse for every element in the set. For every integer a there exists an integer -a such that a + (-a) = 0. If a is positive, -a is negative. If a is negative, -a is positive. If a is zero, -a is zero. The symbol -a is the symbol for the inverse of the integer a. The symbol -a may represent a positive integer, a negative integer, or zero. This sometimes causes confusion on the part of the pupils. They often tend to consider -a as representing only a negative integer. The symbol -a should not be read "negative a;" it should be read "the inverse of a." This will lessen the chance for confusion about this symbol.

The set of integers has an additive identity element, the integer zero. The sum of zero and any integer a is the integer a. Thus a + 0 = a.

The set of integers has three properties not found in the set of natural numbers.

(1) Closure under subtraction
(2) The existence of the identity element under addition
(3) The existence of an inverse element for every element under addition

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<u>Teacher Notes</u>

UNIT 3: THE SET OF INTEGERS

PART 2. QUESTIONS AND ACTIVITIES FOR CLASSROOM USE

3.1 INTRODUCTION

The questions and activities in this unit have been selected to permit coverage of the essential concepts of the properties of the set of natural numbers, the set of integers, and the operations in these sets of numbers in the length of time suggested in the course outline. This means that the concepts must be presented with an efficient use of class time. If additional questions and activities are needed or if the teacher desires a slower, more systematic presentation of any of the concepts it is suggested that pertinent material from Mathematics 8X be used.

3.2 THE SET OF NATURAL NUMBERS

Concept: Use of the natural numbers.

(1) For what purpose was the natural numbers developed?

Answer: The natural numbers were developed for counting and indicating quantities.

Concept: The properties of natural numbers.

(2) Answer the following questions.
(a) Is there a least natural number?
(b) Is there a greatest natural number?

Answers:
(a) Yes, the least natural number is 1.
(b) No, there is no greatest natural number.

(3) How may the sum of any two natural numbers such as 5 and 7 be obtained?

Answer: The sum of the two natural numbers 5 and 7 may be obtained by counting 7 consecutive numbers from 5 in the direction of the larger numbers. The sum would be 12.

(4) What properties does the set of natural numbers have under addition?

Answer: Under addition, the set of natural numbers has the closure property, the commutative property, and the associative property.

(5) What is meant by the closure property under addition?

Answer: The closure property for addition in the set of natural numbers, means that the sum of any two elements in the set of natural numbers is an element in the set. The sum of any two natural numbers is a natural number.

(6) What is meant by the commutative property under addition in the set of natural numbers?

Answer: The sum of any two natural numbers is the same regardless of the order in which the addition is performed.

(7) What is meant by the associative property under addition in the set of natural numbers?

Answer: The sum of any three natural numbers is the same regardless of whether the sum of the last two numbers is added to the first or whether the sum of the first two is added to the last.

(8) How may the product of any two natural numbers such as 3 and 5 be obtained?

Answer: The product of 3 times 5 is obtained by adding three 5's. $3 \times 5 = 5 + 5 + 5 = 15$.

(9) What properties do the set of natural numbers have under multiplication?

Answer: The natural numbers have the closure property, the commutative property, and the associative property under multiplication. Also, the set has the identity element, 1, under multiplication.

(10) What is meant by an identity element under multiplication?

Answer: For every element a in the set of natural numbers, $a \cdot 1 = a$.

(11) Describe the concept of subtraction in the set of natural numbers.

Answer: Subtraction is the inverse of addition. Where a, b, and c are natural numbers, $a - b = c$, if and only if, $c + b = a$.

(12) How may this concept of subtraction be used to calculate the difference 16 - 4?

Answer: $16 - 4 = \square$ has the same solution set as

$$\square + 4 = 16.$$

The problem is solved by determining what number must be added to 4 to result in a sum of 16; the answer is 12.

(13) Does the subtraction problem 7 - 9 = ☐ have a solution in the set of natural numbers?

Answer: No, this subtraction problem has no solution in the set of natural numbers. 7 - 9 = ☐ has the same solution set as ☐ + 9 = 7. There is no natural number that can be added to 9 to give a sum 7.

(14) What restriction is there on the operation subtraction in the set of natural numbers?

Answer: The minuend must be greater than the subtrahend.

(15) Does the set of natural numbers have the property of closure for subtraction?

Answer: No, if the minuend is equal to or less than the subtrahend, the difference is not in the set of natural numbers.

(16) Is the statement 7 - 5 = 5 - 7 true?

Answer: No, 7 - 5 ≠ 5 - 7. 7 - 5 is equal to the natural number 2. The difference 5 - 7 is not in the set of natural numbers.

(17) Does the set of natural numbers have the commutative property under subtraction?

Answer: No

(18) Is the statement 8 -(5 - 2) = (8 - 5) - 2 true?

Answer: No. 8 -(5 - 2) = 8 - 3 = 5
(8 - 5) - 2 = 3 - 2 = 1;
therefore, 8 - (5 - 2) ≠ (8 - 5) - 2

(19) Does the set of natural numbers have the associative property under subtraction?

Answer: No

(20) What is the concept of division in the set of natural numbers?

Answer: Division is the inverse operation of multiplication.

(21) How may this concept of division be applied to calculating the quotient of 18 ÷ 6 = ☐?

Answer: 18 ÷ 6 = ☐ has the same solution set as (☐)(6) = 18. The problem is to determine what number multiplied by 6 gives a product of 18; the answer is 3.

(22) *Does the division problem* $3 \div 5 = \square$ *have an answer in the set of natural numbers?*

Answer: No, this does not have an answer in the set of natural numbers. $3 \div 5 = \square$ has the same solution set as $(\square)(5) = 3$. There is no natural number that will produce a product of 3 when multiplied by 5.

(23) *What restriction is there on the operation division in the set of natural numbers?*

Answer: Division in the set of natural numbers is restricted to dividends that are multiples of the divisor.

(24) *Does the set of natural numbers have the property of closure under division?*

Answer: No, if the dividend is not a multiple of the divisor the quotient is not in the set of natural numbers.

(25) *Is the statement* $12 \div 4 = 4 \div 12$ *true?*

Answer: No. $12 \div 4 = 3$. The quotient $4 \div 12$ is not in the set of natural numbers; therefore, $12 \div 4 \neq 4 \div 12$.

(26) *Does the set of natural numbers have the commutative property under division?*

Answer: No

(27) *Is the statement* $48 \div (12 \div 4) = (48 \div 12) \div 4$ *true?*

Answer:

No
$48 \div (12 \div 4) = 48 \div 3 = 16$
$(48 \div 12) \div 4 = 4 \div 4 = 1;$
therefore, $48 \div (12 \div 4) \neq (48 \div 12) \div 4$.

(28) *Does the set of natural numbers have the associative property under division?*

Answer: No

(29) *What property of the natural numbers involves both multiplication and addition and both multiplication and subtraction?*

Answer: The distributive property. Multiplication is distributive over addition and multiplication is distributive over subtraction.

(30) *Summarize the properties of the natural numbers.*

Answer:
(1) Closure property under addition and multiplication
(2) Commutative property under addition and multiplication
(3) Associative property under addition and multiplication
(4) Property that multiplication is distributive over addition and subtraction
(5) Existence of an identity element under multiplication

(31) Name the property of which each of the following is an example.

(a) $3\cdot(4\cdot5) = (3\cdot4)\cdot5$
(b) $5 + 9 = 9 + 5$
(c) $11\cdot1 = 11$
(d) $1 + (8 + 17) = (1 + 8) + 17$
(e) $8\cdot17 = 17\cdot8$
(f) $6(4 + 3) = 6\cdot4 + 6\cdot3$
(g) $7(4 - 2) = 7\cdot4 - 7\cdot2$

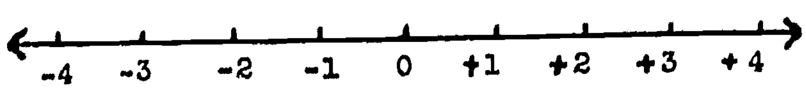
Answers:
(a) Associative property under multiplication
(b) Commutative property under addition
(c) Identity element under multiplication
(d) Associative property under addition
(e) Commutative property under multiplication
(f) Distributive property
(g) Distributive property

3.3 ADDITION IN THE SET OF INTEGERS

Concept: Counting in the intégers.

(1) Represent the set of integers by the use of a number line on which equally spaced points represent the integers. How does counting in the set of integers differ from counting in the set of natural numbers?

Answer:



Counting in the set of natural numbers can be performed indefinitely in the direction of greater numbers, but can be performed only as far as the number 1 in the direction of smaller numbers. In the set of integers, counting may be performed indefinitely in either the direction of greater numbers or the direction of smaller numbers.

Concept: Order in the set of integers.

(2) Given any two integers, how can the greater number be determined?

Answer: When the integers are represented as shown on the previous page with the positive integers to the right of zero and the negative integers to the left of zero, any integer is greater than any other integer to its left.

(3) In each of the following, indicate which in the pair is the greater number.
(a) 0, -6 (b) -13, +1 (c) -5, +5

Answers:
(a) $0 > -6$ (b) $+1 > -13$ (c) $+5 > -5$

Concept: Absolute value of an integer.

(4) Answer the following questions.
(a) When the integers are represented on a number line, how many units from zero is the integer -6?
(b) How many units from zero is the integer +6?

Answers:
(a) The integer -6 is six units from zero.
(b) The integer +6 is six units from zero.

(5) The symbol $|+6|$ is read "the absolute value of +6." The symbol $|-6|$ is read "the absolute value of -6."
By observing the following, what would you think is the meaning of absolute value?

$|+6| = 6$
$|-6| = 6$
$|+17| = 17$
$|-24| = 24$
$|0| = 0$

Answer: It appears that the absolute value of an integer is the number of units from zero the integer is on a number line.

(6) Would it be possible for an integer to have a negative absolute value? Why?

Answer: No, the distance of an integer from the integer zero can be a positive number of units or zero units. The integer may be located in the negative direction from zero but its distance from zero cannot be a negative number of units.

(7) For each of the following, indicate whether the statement is true or false.
(a) $|-16| > 15$ (c) $-1 < |-1|$
(b) $|0| > -1$ (d) $|4 + 7| \neq |-4| + |-7|$

Answers:
(a) True (b) True (c) True (d) False

Concept: Addition of two positive integers.

(8) The positive integers closely resemble what set of numbers studied previously?

Answer: The set of natural numbers

(9) The operations with two positive integers are identical to the same operations with the natural numbers.
Describe how the addition (+5)+(+7) could be performed.

Answer: The addition (+5)+(+7) could be performed by counting seven numbers from +5 in the direction of greater numbers; the sum would be +12.

(10) Would the sum of two positive integers always be a positive integer? Why?

Answer: The sum of two positive integers is always a positive integer. To add two positive integers, counting begins at a positive integer and is carried out in the positive direction. The sum, therefore, must be a positive integer.

(11) Describe the absolute value of the sum of two positive integers in terms of the absolute values of the integers.

Answer: The absolute value of the sum of two positive integers is the sum of their absolute values.

Concept: Addition of two integers with opposite signs.

(12) If the sum of (+5) + (+7) is found by counting seven consecutive integers in the positive direction from +5, how could the sum (+5) + (-7) be found?

Answer: The sum (+5) + (-7) could be found by counting seven consecutive integers from +5 in the negative direction.

(13) Using the above method of addition, what would be the sum of (+5) + (-7)?

Answer: -2

(14) The set of integers has all the properties of the set of natural numbers. This means the set of integers must have which properties under addition?

Answer: Closure, commutative property, and associative property

(15) <u>By making use of the commutative principle, how could the sum (-9) + (+6) be determined?</u>

<u>Answer</u>: (-9) + (+6) is equal to the same sum as (+6) + (-9). The sum may be found by counting 9 consecutive integers from +6 in the negative direction.

(16) <u>Find the sum of each of the following.</u>

(a) (+17) + (-16) (c) (-642) + (+600)
(b) (-21) + (+10)

<u>Answers</u>:
(a) +1 (b) -11 (c) -42

(17) Examine the three examples below.

(+17) + (-16) = +1
(-21) + (+10) = -11
(-642) + (+600) = -42

<u>How does the absolute value of the answer compare to the absolute value of the integers being added?</u>

<u>Answer</u>: The absolute value of the sum is equal to the difference of the absolute values of the integers being added.

(18) <u>When adding two integers with opposite signs, how can you tell whether the answer will be positive or negative?</u>

<u>Answer</u>: The answer will have the same sign as that addend that has the greater absolute value.

(19) The positive integers have all the properties of the natural numbers and are often written without the "+" sign. For example, +5 is often written simply as 5. This simpler notation is used throughout most of this unit.

<u>Find the sum of each of the following.</u>

(a) (-1449) + (1644) (c) (7733) + (-8389)
(b) (-8765) + (9341)

<u>Answers</u>:
(a) 195 (b) 576 (c) -656

<u>Concept</u>: Addition of two negative integers.

(20) <u>Answer the following questions.</u>

(a) How is the sum of 3 + (-6) found?
(b) How do you think the sum of (-3) + (-6) could be found?
(c) What would be the sum (-3) + (-6)?

<u>Answers</u>:
(a) The sum of 3 + (-6) is found by counting six consecutive integers in a negative direction from 3.

(b) The sum (-3) + (-6) could be found by counting six consecutive integers from -3 in the negative direction.

(c) -9

(21) Find the sum of each of the following.

(a) (-8) + (-1)   (c) (-1423) + (-1583)

(b) (-16) + (-24)

Answers:

(a) -9   (b) -40   (c) -3006

(22) Examine the following three examples.

(-8) + (-1) = -9

(-16) + (-24) = -40

(-1423) + (-1583) = -3006

Now answer the following questions.

(a) When adding two negative integers, how does the absolute value of the answer compare with the absolute values of the integers being added?

(b) When adding two negative integers, how can you tell whether the sum will be positive or negative?

Answers:

(a) The absolute value of the answer will be equal to the sum of the absolute values of the integers being added.

(b) The answer will always be negative because the counting begins at a negative integer and is carried out in a negative direction.

(23) Answer the following questions.

(a) Describe briefly how addition is performed by counting in the set of integers.

(b) Describe briefly how the sum of any two integers with like signs can be determined without using the counting process.

(c) Describe briefly how the sum of any two integers with opposite signs can be determined without using the counting process.

Answers:

(a) When adding a positive integer to any integer a, counting is carried out in the positive direction from a. When adding a negative integer to any integer a, counting is carried out in the negative direction from a. The counting is carried out a number of consecutive integers equal to the absolute value of the second given integer.

(b) The absolute value sum of two integers with like signs is equal to the sum of the absolute values of the integers being added. The sum of two positive integers is positive. The sum of two negative integers is negative.

(c) The absolute value of the sum will equal the difference of the absolute values of the integers being added. The answer will have the same sign as that addend that has the greater absolute value.

<u>Concept</u>: Additive identity element.

(24) <u>Does there exist an integer b such that the sum of any integer a and b is a? That is, for every integer a, a + b = a?</u>

<u>Answer</u>: Yes, the integer zero. Zero is the additive identity element.

<u>Concept</u>: Additive inverse or opposite.

(25) <u>Answer the following questions.</u>

(a) How many units from zero is the integer 13 on the number line?
(b) In counting from 13 to zero, in what direction is the counting performed?
(c) Counting 13 units from 13 in the negative direction is the same as adding what integer to 13?
(d) What is the sum of 13 + (-13)?
(e) In what direction and for how many units must counting be performed to count from -17 to zero?
(f) This is the same as adding what number to -17?
(g) Therefore, what is the sum of -17 and 17?

<u>Answers</u>:
(a) 13 units
(b) The negative direction
(c) It is the same as adding -13 to 13.
(d) 13 + (-13) = 0.
(e) 17 units in the positive direction
(f) This is the same as adding 17 to -17.
(g) -17 + 17 = 0.

(26) <u>For every integer x is there an integer y such that the sum x + y is zero? Explain.</u>

<u>Answer</u>: Yes. If x is any given integer and y has the same absolute value as x but opposite sign, the sum x + y is equal to zero.

(27) <u>If x + y = 0, y is called the additive inverse or opposite of x. Indicate the opposite of each of the following.</u>

(a) 16 (b) -9 (c) 0

<u>Answers</u>:
(a) -16 (b) 9 (c) 0

(28) <u>Answer the following questions.</u>

(a) If the replacement set of the variable x is the set of integers and the symbol -x represents the opposite of x, what integer does -x represent when x is the placeholder for the number -3?

(b) Does the symbol -x always represent a negative number?
(c) The integer -3 is read "negative three." Should the symbol -x be read "negative x?"

Answers:
(a) If x = -3, then -x must be the opposite of -3, or 3; therefore, -x = -(-3) = 3.
(b) No, -x represents a positive integer when x is the placeholder for a negative integer.
(c) No, -x may be positive. -x should be read "opposite of x" or "inverse of x."

3.4 SUBTRACTION IN THE SET OF INTEGERS

Concept: Definition of subtraction.

(1) What is the concept of subtraction as used in the set of natural numbers?

Answer: Subtraction is the inverse of addition. If a + b = c, then c - b = a. Also, if a - b = c, then c + b = a.

(2) Applying this concept of subtraction to the set of integers, how could the difference (-6) - 3 = □ be determined?

Answer: If (-6) - 3 = □, then □ + 3 = -6. The problem is to determine what number added to 3 would give a sum of -6; the answer is -9.

(3) Applying the concept that subtraction is the inverse operation of addition, for each of the following write a corresponding equation involving addition and indicate the answer to the subtraction problem.

(a) -3 - 8 = □
(b) 3 - (-8) = □
(c) -9 - (-6) = □
(d) -89 - (-89) = □
(e) -100 - 194 = □

Answers:

(a)	□ + 8 = -3	□ = -11
(b)	□ + (-8) = 3	□ = 11
(c)	□ + (-6) = -9	□ = -3
(d)	□ + (-89) = -89	□ = 0
(e)	□ + 194 = -100	□ = -294

(4) Each of the following is a subtraction problem and a corresponding addition problem. Observe that the second term in the addition problem is the additive inverse or opposite of the subtrahend in the corresponding subtraction problem. Find the answer to each problem and compare the two answers in each pair of problems.

(a) -3 - 6 = □	-3 + (-6) = □
(b) 5 - 9 = □	5 + (-9) = □
(c) 6 - 4 = □	6 + (-4) = □
(d) -8 - (-7) = □	-8 + 7 = □
(e) -10 - (-17) = □	-10 + 17 = □

<u>Answers</u>:

(a) -3 - 6 = □ □ + 6 = -3 □ = -9	-3 + (-6) = □ □ = -9
(b) 5 - 9 = □ □ + 9 = 5 □ = -4	5 + (-9) = □ □ = -4
(c) 6 - 4 = □ □ + 4 = 6 □ = 2	6 + (-4) = □ □ = 2
(d) -8 - (-7) = □ □ + (-7) = -8 □ = -1	-8 + 7 = □ □ = -1
(e) -10 - (-17) = □ □ + (-17) = -10 □ = 7	-10 + 17 = □ □ = 7

In each of the pairs of examples the two answers are the same.

(5) The previous examples indicate that the subtraction x - y may be performed by carrying out a corresponding addition.
<u>What addition corresponds to the subtraction x - y</u>?

<u>Answer</u>: x + (-y)

(6) <u>Solve each of the following subtractions by carrying out the corresponding addition</u>.

(a) (-18) - 9 (c) 44 - (-50)
(b) (-35) - (-26) (d) 81 - 42

<u>Answers</u>:

(a) -18 + (-9) = -27 (c) 44 + 50 = 94
(b) -35 + 26 = -9 (d) 81 + (-42) = 39

<u>Concept</u>: Closure under subtraction.

(7) <u>Does every integer have an opposite</u>? <u>For every integer x is there a -x</u>?

<u>Answer</u>: Yes, for every integer x there is an integer -x such that x + (-x) = 0. Every integer has an opposite.

(8) If every subtraction x - y in the integers may be performed by carrying out the corresponding addition x + (-y), does the set of integers have closure under subtraction?

Answer: Yes, the set of integers has closure under subtraction.

(9) Answer the following questions.

(a) Is the statement (-3) - (-4) = (-4) - (-3) a true statement?

(b) Does the set of integers have the commutative property under subtraction?

(c) Is the statement (-3) - [4 - (-2)] = [(-3) - 4] - (-2) true?

(d) Does the set of integers have the associative property under subtraction?

Answers:

(a) No, (-3) - (-4) = (-3) + 4 = 1
(-4) - (-3) = -4 + 3 = -1
Therefore, (-3) - (-4) ≠ (-4) = (-3)

(b) No

(c) No. (-3) - [4 - (-2)] = -3 - [4 + 2] = -3 - 6 = -3 + (-6) = -9 But [(-3) - 4] - (-2) = [-3 + (-4)] - (-2) = -7 - (-2) = -7 + 2 = -5
Therefore, (-3) - [4 - (-2)] ≠ [(-3) - 4] - (-2)

(d) No

3.5 MULTIPLICATION IN THE SET OF INTEGERS

Concept: Multiplication of two integers by addition.

(1) In the set of natural numbers, multiplication was defined in terms of what other operation?

Answer: Addition

(2) Applying the concept of multiplication as being a form of addition, describe how to find the products of the following.

(a) (3)(6) (b) (3)(-6)

Answers:

(a) The product (3)(6) is found by determining the sum 6 + 6 + 6; the answer is 18.

(b) The product (3)(-6) is found by determining the sum (-6) + (-6) + (-6); the answer is -18.

(3) The set of integers has all the properties that the set of natural numbers has, including the commutative property under multiplication.
How could this principle be applied to solving the multiplication (-6)(5)?

Answer: Because the set of integers has the commutative property under multiplication, (-6)(5) = (5)(-6).

This product can then be found by determining the sum (-6) + (-6) + (-6) + (-6) + (-6); the answer is -30.

(4) Answer the following questions.

(a) The product of two positive integers will always be what type of integer? Why?

(b) The product of a positive integer and a negative integer is always what type of integer? Why?

Answers:

(a) The product of two positive integers will always be a positive integer because the product is found by adding a number of positive integers. The sum of any number of positive integers is a positive integer.

(b) The product will be a negative integer because the product is found by adding a number of negative integers. The sum of any number of negative integers is a negative integer.

Concept: Multiplication of two negative integers.

(5) Find the product of each of the following.

(a) (-3)(4)
(b) (-3)(3)
(c) (-3)(2)
(d) (-3)(1)
(e) (-3)(0)

Answers:

(a) -12 (b) -9 (c) -6 (d) -3 (e) 0

(6) Complete the table below, assuming that the number pattern established in the first five products at the left continues for the three products at the right.

(-3)(4) = -12	(-3)(-1) = □
(-3)(3) = -9	(-3)(-2) = △
(-3)(2) = -6	(-3)(-3) = ▭
(-3)(1) = -3	
(-3)(0) = 0	

Answers:

(-3)(-1) = 3
(-3)(-2) = 6
(-3)(-3) = 9

(7) Answer the following questions.

(a) To what is the product (-3)(0) equal?

(b) Is 0 = (-3) + 3?

(c) In the equation (-3)(0) = 0, substitute (-3) + 3 for the first zero.

(d) Every property of the natural numbers is also a property of the integers. What property could be applied to the above multiplication?

(e) Apply the distributive property to the left side of the equation (-3)[(-3) + 3] = 0

(f) To what is (-3)(3) equal?

(g) Substitute -9 for (-3)(3) in the equation
$[(-3)(-3)] + [(-3)(3)] = 0$.

(h) What is the only number that, when added to -9, gives a sum of zero?

(i) Therefore, (-3)(-3) must be equal to what number?

(j) The product of two negative integers will always be what type of integer?

Answers:

(a) $(-3)(0) = 0$

(b) Yes

(c) $(-3)[(-3) + 3] = 0$

(d) The distributive property

(e) $(-3)[(-3) + 3] = 0$
$[(-3)(-3)] + [(-3)(3)] = 0$

(f) -9

(g) $(-3)(-3) + (-9) = 0$

(h) 9

(i) $(-3)(-3) = 9$

(j) The product will always be a positive integer.

(8) How does the absolute value of the product of two integers compare with the absolute values of the integers being multiplied?

Answer: The absolute value of the product is equal to the product of the absolute values of the integers being multiplied.

(9) Indicate in each of the following what integer must be used in replacing the variable to make the sentence a true statement.

(a) $(-21)(-30) = x$

(b) $(-12)(7) = y$

(c) $2a = -16$

(d) $(-6)(b) = 12$

(e) $(\triangle)(9) = -27$

(f) $(-11)(\square) = 99$

Answers:

(a) $x = 630$

(b) $y = -84$

(c) $a = -8$

(d) $b = -2$

(e) $\triangle = -3$

(f) $\square = -9$

(10) In each of the following, what word must be inserted in the blank to make the sentence a true statement?

(a) The product of any two positive integers is a (an) ______ integer.

(b) The product of any two negative integers is a (an) ______ integer.

(c) The product of any two integers, one positive and one negative, is a (an) ______ integer.

(d) The product of any integer and zero is ______.

Answers:

(a) Positive

(b) Positive

(c) Negative

(d) Zero

Concept: Identity element under multiplication.

(11) Answer the following questions.

(a) What was the identity element under multiplication in the set of natural numbers?

(b) What integer is the identity element under multiplication?

Answers:

(a) The natural number 1

(b) The integer +1

(12) Answer the following questions.

(a) What is meant by additive inverse?

(b) What would be meant by multiplicative inverse?

(c) Does there exist an integer u such that the product of -3 and u is the identity +1?

(d) Does the set of integers have the property of an inverse element for every element under multiplication?

Answers:

(a) If for every element a there exists an element b such that their sum is the additive identity, then b is the additive inverse of a.

(b) If for every element a there exists an element b such that their product is the multiplicative identity, then b is the multiplicative inverse of a.

(c) No, there is no integer u such that -3u = 1.

(d) No

3.6 DIVISION IN THE SET OF INTEGERS

Concept: Definition of division.

(1) Division in the set of integers is defined the same as in the set of natural numbers.

Finish the following sentence.

a ÷ b = c if and only if _____.

Answer: a ÷ b = c, if and only if bc = a.

(2) What restrictions are placed on the replacement set for the variables in the division a ÷ b = c in the set of integers?

Answer: The replacement set for the variable b does not contain the element zero. Division by zero is undefined and not permitted. The replacement set for the variable a includes only multiples of b.

Concept: Division involving two positive integers.

(3) Complete the following exercises.

(a) Rewrite the equation (+6) ÷ (+3) = x as a corresponding equation involving multiplication - one having the same solution set as the original equation.

(b) What integer must be used to replace x to make the equation a true statement.

(c) Rewrite the equation (+a) ÷ (+b) = x as a corresponding equation involving multiplication.

(d) If +a and +b represent any positive integers, what type of integer must be used to replace the placeholder x if the equation is to be a true statement?

(e) Is the quotient of two positive integers always a positive integer?

Answers:

(a) (x)(+3) = +6

(b) x = +2

(c) (x)(+b) = +a

(d) The placeholder x must be replaced by a positive integer. The placeholder x cannot be replaced by a negative integer because the product of two integers, one positive and one negative, is a negative integer. Also, x cannot be replaced by zero because the product of zero and any integer is zero. Therefore, x must be replaced by a positive integer.

(e) The quotient of two positive integers must be a positive number for the reason stated in (d).

Concept: Division of two integers, one positive and one negative.

(4) Complete the following exercises.

(a) Rewrite the equation (+18) ÷ (-3) = x as a corresponding equation involving multiplication.

(b) What is the only integer that can be used to replace x to make the equation a true statement?

(c) Rewrite the equation (+a) ÷ (-b) = x as a corresponding equation involving multiplication, assuming that a is a multiple of b.

(d) If +a represents a positive integer and -b represents a negative integer, what type of integer must x represent?

(e) Rewrite the equation (-a) ÷ (+b) = y as a corresponding equation involving multiplication, assuming that a is a multiple of b.

(f) If -a represents a negative integer and +b represents a positive integer, what type of integer must y represent?

(g) If the quotient of two integers with unlike signs is an integer, what type of integer is it?

Answers:

(a) (x)(-3) = +18

(b) x = -6

(c) (x)(-b) = +a

(d) The x must represent a negative integer. If x were the placeholder for a positive integer, the product of (x)(-b) would be a negative integer. If x were replaced by zero, the product would be zero. The product is a positive integer so x must be the placeholder for a negative integer.

(e) (y)(+b) = -a

(f) The y must represent a negative integer. If y were the placeholder for a positive integer, the product (y)(+b) would be a positive integer. If y were replaced by zero, the product would be zero. The product is a negative integer so y must be the placeholder for a negative integer.

(g) If the quotient of two integers with unlike signs is an integer, it is a negative integer.

<u>Concept</u>: The quotient of two negative integers.

(5) <u>Complete the following exercises</u>.

(a) Rewrite the equation (-18) ÷ (-3) = y as a corresponding equation involving multiplication.

(b) What integer must y represent?

(c) Rewrite the equation (-a) ÷ (-b) = x as a corresponding equation involving multiplication.

(d) Assuming that a is a multiple of b, what type of integer must x represent?

(e) If the quotient of two negative integers is an integer, what type of integer must it be?

<u>Answers</u>:

(a) (y)(-3) = -18

(b) y = +6

(c) (x)(-b) = -a

(d) The x must represent a positive integer. If x werethe placeholder for a negative integer, the product (x)(-b) would be a positive integer. If x were replaced by zero, the product would be zero. The product is a negative integer so x must be the placeholder for a positive integer.

(e) A positive integer

(6) <u>Complete the following exercises</u>.

(a) Rewrite the equation 0 ÷ (-6) = k as a corresponding equation involving multiplication.

(b) What integer must k represent?

(c) Rewrite the equation 0 ÷ (-a) = k as a corresponding equation involving multiplication.

(d) What integer must k represent?

(e) Rewrite the equation 0 ÷ (+a) = m as a corresponding equation involving multiplication.

(f) What integer must m represent?

(g) The quotient zero divided by any integer other than zero must be what integer?

<u>Answers</u>:

(a) (k)(-6) = 0

(b) k = 0

(c) (k)(-a) = 0

(d) k must represent zero. If k were the placeholder for a positive integer, the product (k)(-a) would be a negative integer. If k were the placeholder for a negative integer, the product would be a positive integer. The product is neither positive nor negative so k must be the placeholder for zero.

(e) $(m)(+a) = 0$
(f) $m = 0$
(g) Zero

(7) <u>What term must be inserted in each of the following blanks to make the sentence a true statement?</u>

(a) If the quotient of any two positive integers is an integer, it is a (an) ______ integer.
(b) If the quotient of any two negative integers is an integer, it is a (an) ______ integer.
(c) If the quotient of any two integers, one positive and one negative is an integer, it is a (an) ______ integer.
(d) The quotient of zero divided by either a positive integer or a negative integer is the integer ______.

<u>Answers</u>:
(a) Positive (b) Positive (c) Negative (d) Zero

(8) <u>Indicate what integer must be used to replace the variable in each of the following to make it a true statement</u>.

(a) $(-16) \div (-4) = a$	(e) $\square \div 6 = -36$
(b) $(-24) \div (3) = k$	(f) $-49 \div \triangle = 7$
(c) $(144) \div (-12) = m$	(g) $72 \div \square = 9$
(d) $\triangle \div (-5) = 25$	(h) $-102 \div x = -17$

<u>Answers</u>:

(a) $a = 4$	(e) $\square = -216$
(b) $k = -8$	(f) $\triangle = -7$
(c) $m = -12$	(g) $\square = 8$
(d) $\triangle = -125$	(h) $x = 6$

<u>Concept</u>: Properties of the integers under division.

(9) <u>Answer the following questions.</u>

(a) Does the division $-5 \div 3$ have a quotient in the set of integers?
(b) Does the set of integers have the property of closure under division?
(c) Is the following statement true? $-12 \div -4 = -4 \div -12$
(d) Is division commutative?
(e) Is the statement $-48 \div (12 \div 4) = (-48 \div 12) \div 4$ true?
(f) Is division associative?

<u>Answers</u>:
(a) No, there is no integer x such that $3x = -5$.
(b) No

(c) No, $-12 \div -4 = 3$
$-4 \div -12$ has no quotient in the set of integers
$-12 \div -4 \neq -4 \div -12$

(d) No

(e) No, $-48 \div (12 \div 4) = -48 \div 3 = -16$
$(-48 \div 12) \div 4 = -4 \div 4 = -1$
$-48 \div (12 \div 4) \neq (-48 \div 12) \div 4$

(f) No

3.7 THE PROPERTIES OF THE SET OF INTEGERS

The set of integers has all the properties of the set of natural numbers and it also has additional properties that are not valid in the set of natural numbers.

Below is a list of some possible properties of numbers.

Closure under addition
Closure under subtraction
Closure under multiplication
Closure under division
Commutative property under addition
Commutative property under subtraction
Commutative property under multiplication
Commutative property under division
Associative property under addition
Associative property under subtraction
Associative property under multiplication
Associative property under division
Multiplication distributive over addition
Multiplication distributive over subtraction
Identity element under addition
Identity element under multiplication
Inverse element for every element under addition
Inverse element for every element under multiplication

Answer the following questions.

(a) Which of the above properties are valid in the set of integers?
(b) Which of the above properties are not valid in the set of integers?
(c) Which of the above properties are valid in the set of integers but are not valid in the set of natural numbers?

Answers:

(a) Closure under addition, subtraction, and multiplication
Commutative property under addition and multiplication
Associative property under addition and multiplication
Multiplication distributive over addition and subtraction
Identity element under addition and multiplication
Inverse element for every element under addition

(b) Closure under division
Commutative property under subtraction and division

Associative property under subtraction and division
Inverse element for every element under multiplication

(c) Closure under subtraction
Identity element under addition
Inverse element for every element under addition

<u>Teacher Notes</u>

UNIT 4: OPEN SENTENCES

PART 1. BACKGROUND MATERIAL FOR TEACHERS

4.1 INTRODUCTION

This experimental course differs from the traditional course in several ways. One of the purposes of this course is to teach the why of the fundamental principles of algebra. For example, the fact that the product $(-x)(-y)$ is equal to xy is a simple fundamental algebraic principle which almost any algebra pupil can learn without difficulty. However, many pupils fail to master the mathematical concept upon which this principle is based. The pupil may know that the product $(-x)(-y)$ is equal to xy, but when he is required to justify his answer, he may reply "The product of $-x$ and $-y$ is xy because the product of two negative numbers is a positive number." Such an incorrect answer would indicate that the pupil has failed to master the concept of the meaning of the symbols $-x$ and $-y$ and why their product is xy. The symbol $-x$ represents the additive inverse of x. If x is positive, $-x$ is negative. If x is negative, $-x$ is positive. If x is zero, $-x$ is zero. Therefore, $-x$ may represent a positive number, negative number, or zero. The same is true for $-y$. Determining the product of $-x$ and $-y$ cannot be based on the assumption that they represent negative numbers or that xy is a positive number.

The concept that $(-x)(-y) = xy$ is developed from the principle that $-x = (-1)(x)$.

$$\begin{aligned}(-x)(-y) &= (-1)(x)(-1)(y)\\ &= (-1)(-1)(x)(y)\\ &= 1(xy)\\ &= xy\end{aligned}$$

Pupils can be taught to solve an algebraic equation simply by memorizing the mechanical operations involved in deriving the solution to such an equation. However, if algebra is to be at all meaningful to the pupil, he should master the mathematical concepts upon which the mechanical operations are based.

This experimental course also includes the topic of inequalities and the solution to inequalities. The presentation of this topic may answer for the teacher (or pupil) the question of why the topic of sets has been given so much emphasis in the so-called modern mathematics. One of the reasons for the stress on sets is the fact that the solution set of an inequality is usually the union or intersection of two sets of numbers. The pupil should have an understanding of set notation in order to present his solution to an inequality concisely.

The fundamental concepts of equations and inequalities and their solution are presented in this unit. However, these topics are also discussed further in later units.

4.2 EQUATIONS

Equations are classified as identities, conditional equations, or internally inconsistent equations.

An identity is an equation that is always true regardless of what element from the replacement set is substituted for the variable. The solution set and the replacement set are identical. $x + 3 = 3 + x$ is an identity. It is always true regardless of what number is substituted for the variable.

A conditional equation is an equation whose truthfulness depends on which element or elements from the replacement set are used to replace the variable. $y + 2 = 6$ is a conditional equation. When the number 4 is substituted for the variable, the equation is a true statement. When any number other than 4 is substituted for the variable, the equation is a false statement. The solution set of a conditional equation is a proper subset of the replacement set. The solution set contains at least one but not all of the elements in the replacement set. Care should be taken not to say that the solution set of a conditional equation is a subset of the replacement set. Every set is a subset of itself. A subset of a set may be the set itself. The solution set of a conditional equation is a proper subset of the replacement set. There must be at least one element in the replacement set that is not contained in the solution set.

An internally inconsistent equation is one whose solution set is the null set. $n - 3 = n$ is an internally inconsistent equation. There is no number that can be substituted for n to make the equation a true statement.

Equations are classified according to their solution sets.

Identity - the solution set is identical to the replacement set.
Conditional equation - the solution set is a proper subset of the replacement set.
Internally inconsistent equation - the solution set is the null set.

An identity is always true; an internally inconsistent equation is always false; the truthfulness of a conditional equation depends on what number is substituted for the variable. It might seem that there would be little chance of confusing an identity with an internally inconsistent equation. However, this sometimes happens. For example, in solving the equation $x + 4 = x + 6 - 2$, $x = x$. When pupils get to the place in the solution where they have shown $x = x$, some pupils indicate that the equation has no solution. Quite to the contrary, the equation is an identity. Its solution set is the replacement set. Giving the pupils adequate experience at recognizing identities helps prevent such errors.

The question arises as to how one can prove that an equation is or is not an identity. It is easy enough to prove it is not an identity. Just solve the equation to determine the solution set. Then substitute for the variable in the equation any element that is in the replacement set but not in the solution set. Such a substitution will result in a false statement. This proves the equation is not an identity. Often it will not be necessary to solve the equation, as it may be possible to determine a number that will result in a false statement when substituted for the variable just by inspecting the equation.

In attempting to prove that an equation is an identity, pupils sometimes suggest the process of substituting numbers for the variable. The only way that an equation can be proven to be an identity by substitution is to substitute every element in the replacement set for the variable. It must be demonstrated that each element in the replacement set results in a true statement when substituted for the variable. If just one element in the replacement set results in a false statement, then the equation is not an identity. This process could be used only if the replacement set contains a finite number of elements. And even then, the substitution method is not practical if the number of elements is large. If the replacement set is infinite, then the substitution method cannot be used.

By far the easiest method of proving an equation is an identity is by showing that the equation is an application of one of the axioms, postulates, or definitions of our number system. Examples are commutative property, associative property, distributive property, definition of equality and inequality. The equation $t + 11 = 11 + t$ is an identity because it is an application of the commutative property under addition. The equation $k = k$ is an identity because it is an application of the reflexive property of equations. From this it can be seen that it is necessary for the pupils to be familiar with the basic properties of equations, including the reflexive, symmetric, and transitive properties.

The definition of these three properties of equations are as follows:

Reflexive - for any expression a, $a = a$.
Symmetric - for any expressions a and b, if $a = b$ then $b = a$.
Transitive - for any expression a, b, and c, if $a = b$ and $b = c$ then $a = c$.

One of the classroom activities given in part 2 of this unit is to determine whether the relations "$\neq$", "$<$", and "$\perp$" are reflexive, symmetric, and transitive. The relation "$\neq$" is not reflexive. For any expression a, $a \neq a$ is never true. This is a negation of the reflexive property of equations $a = a$ which is an identity and is always true. The negation of a true statement is always a false statement.

The relation "$\neq$" is symmetric. The proof of this is simple. If $a \neq b$, there are only the two possibilities that either $b = a$ or $b \neq a$. The statement $b = a$ must be false because if $b = a$ then $a = b$. This is contradictory to the given statement that $a \neq b$; therefore, $b \neq a$. This method of proof involves showing that there are only two possibilities and then proving one of these is false; therefore, the other possibility must be true.

This material gives the teacher an excellent opportunity to bring up the topic of proof and methods of proof for discussion. The purpose of this is to begin laying the foundatio for the understanding of methods of proof which is so fundamental to the work in tenth year mathematics. Many classroom activities in this and later units involve one or more methods of proof.

The relation "$\neq$" is not transitive. The method of proof used is proof by counter example. A statement can be proven not to be true by demonstrating one example in which it is false. If "$\neq$" were transitive, then if $a \neq b$ and $b \neq c$ then $a \neq c$ for any expression a, b, and c; however $\frac{1}{2} \neq \frac{2}{3}$ and $\frac{2}{3} \neq \frac{8}{16}$, yet it is false that $\frac{1}{2} \neq \frac{8}{16}$. Therefore, "$\neq$" is not transitive.

The relation " $<$ " can be shown not to be reflexive or symmetric by use of counter examples. The statement $5 < 5$ is false so the relation " $<$ " is not reflexive. The statement if $4 < 5$ then $5 < 4$ is false so the relation " $<$ " is not symmetric. However, the relation " $<$ " is transitive. A discussion of the proof of this must be put off until the section on inequalities is introduced.

The relation "perpendicular to" is symmetric but not reflexive or transitive.

An equation can be proved to be an identity if it is an application of, or if an equivalent equation is an application of, one of the axioms, postulates, or definitions contained in our number system. An equation can be proved to be an internally inconsistent equation if it or an equivalent equation is a statement that is contradictory to any axiom, postulate, or definition of our number system. $x + 3 = x$ is an internally inconsistent equation. When $-x$ is added to each side of the equation, the equivalent equation $3 = 0$ results. This is contradictory to the fact that the integers are unique. No two integers are equal to each other; therefore, the statement $x + 3 = x$ is never true.

An equation can be proved to be a conditional equation by proving that at least one element in the replacement set is in the solution set and that at least one element in the replacement set is not in the solution set. This may be done by inspection or by solving the equation to determine the solution and then choosing an element from the replacement set that is not in the solution set, substituting it for the variable, and showing it is not in the solution set by showing it results in a false statement when substituted for the variable.

Attempting to prove that equations are or are not identities, conditional equations, or internally inconsistent equations leads to the problem of solving equations. The solving of equations is based on the additive property of equations and on the multiplicative property of equations. The additive property states that if the same number is added to both sides of an equation the result is an equivalent equation; that is, the solution set of the new equation is identical to the solution set of the given equation. Any number may be added to both sides of an equation. Adding a negative number is equivalent to subtracting a positive number. Adding a positive number is equivalent to subtracting a negative number. The additive property of equations may be expanded to include the fact that if any number is subtracted from both sides of an equation the result is an equivalent equation. Again, the solution set of the new equation is identical to the solution set of the given equality.

Equations may be solved by using either addition or the equivalent subtraction. For example:

$$x + 6 = 13 \qquad\qquad x + 6 = 13$$
$$x + 6 + (-6) = 13 + (-6) \qquad\qquad x + 6 - 6 = 13 - 6$$
$$x = 7 \qquad\qquad x = 7$$

Solution set is $\{7\}$. Solution set is $\{7\}$.
Check: $7 + 6 = 13$ true Check: $7 + 6 = 13$ true

Many teachers prefer having the pupils solve this type of equation by the use of addition as they find it is simpler. Subtraction often must be changed to the equivalent addition before the computation is performed.

The multiplicative property of equations is a statement that if both sides of an equation are multiplied by the same number, the result is an equivalent equation; that is, an equation with the same solution set as the original equation. For every division there is an equivalent multiplication. To divide by 3 is equivalent to multiplying by $\frac{1}{3}$. The multiplicative property therefore includes multiplying or dividing both sides by the same number, other than zero. The equation $14y = 28$ may be solved by multiplying each side by $\frac{1}{14}$ or dividing each side by 14. In both cases the solution set is $\{2\}$. The multiplicative property of equations may be stated as: for every a, b, and c ($c \neq 0$), $a = b$ if and only if $ac = bc$. The "if and only if" phrase means that if $a = b$ then $ac = bc$ and if $ac = bc$ ($c \neq 0$) then $a = b$.

The additive property of equations may be stated as: for every a, b, and c, $a = b$ if and only if $a + c = b + c$. The "if and only if" phrase means that if $a = b$, then $a + c = b + c$ and if $a + c = b + c$, then $a = b$.

The additive property also applies to adding or subtracting the same multiple of the same variable from each side of the equation. Usually this is the same as the variable which is already in the equation. In equations such as $3x = 2x + 9$, $-2x$ may be added to each side (or 2x subtracted) so that all variables are on the same side of the equation and then the two like terms are combined into a single term.

$$3x = 2x + 9$$
$$3x + (-2x) = 2x + 9 + (-2x)$$
$$x = 2x + (-2x) + 9$$
$$x = 9$$

check
$3 \cdot 9 = 2 \cdot 9 + 9$
$27 = 18 + 9$ true

Solution set is $\{9\}$.

The multiplicative property for equations does not always apply to multiplying both sides of an equation by a variable. Sometimes this will result in an equation whose solution set is different from the solution set of the original equation. In multiplying each side of the equation $x = 1$ by x, the new equation is $x^2 = x$. The solution set of the original equation is $\{1\}$. The solution set of the second equation is $\{0, 1\}$. Thus the equations $x = 1$ and $x^2 = x$ are not equivalent equations. This is discussed later in the unit on quadratic equations. However, both sides of an equation may be multiplied by a variable as long as the solution set is not changed. One example of the type of equation in which this is true is that in which the variable is in the denominator of a fraction.

$$\frac{4}{x} = -2$$

$$x\left(\frac{4}{x}\right) = x\,(-2)$$

$$4 = -2x$$

$$-2 = x$$

check

$$\frac{4}{-2} = -2 \text{ true}$$

Solution set is $\{-2\}$.

In this case both sides of the equation may be multiplied by the variable because the solution set of the resulting equation is the same as the solution set of the original equation.

When solving equations such as $3m + 4 = m + 7$ which involves use of both the additive property and the multiplicative property of equations, it is usually advisable to first apply the additive property and then apply the multiplicative property. This usually makes computation simpler because fewer fractions are involved.

The use of absolute value can make the solutions of simple linear equations challenging and interesting and affords an opportunity to give the pupil experience with identities and internally inconsistent equations.

The equation $|\square| = k$ where k is non-negative is true if $\square = k$ and $\square = -k$.

If k is negative, the solution set of $|\square| = k$ is always $\emptyset$, because there is no number whose absolute value is negative. For example, the solution set of $|x| = -6$ is $\emptyset$. This is an internally inconsistent equation.

If k is non-negative, an equation in the form $|\square| = k$ may be solved by determining what two equations without absolute value signs are equivalent to the given equation, and then these equations are solved. For example, the equation $|5x + 7| = 22$ may be solved in the following manner.

$$|5x + 7| = 22$$

$$5x + 7 = 22 \qquad\qquad 5x + 7 = -22$$
$$5x = 15 \qquad\qquad 5x = -29$$
$$x = 3 \qquad\qquad x = -\frac{29}{5}$$

Solution set is $\{3\}$. Solution set is $\left\{-\frac{29}{5}\right\}$.

The solution set of $|5x + 7| = 22$ is $\left\{3, -\frac{29}{5}\right\}$.

check

$$|5\cdot 3 + 7| \overset{?}{=} 22$$
$$|22| = 22 \text{ true}$$
$$\left|5\left(-\frac{29}{5}\right) + 7\right| \overset{?}{=} 22$$
$$|-29 + 7| \overset{?}{=} 22$$
$$|-22| = 22 \text{ true}$$

Care must be taken when solving equations as $|2y + 4| = y - 4$, in which there is a variable on each side of the equation. The left side of the equation can never be negative if the equation is to be a true statement. Therefore, the right side, the expression $(y - 4)$, cannot be negative, so y must be equal to or greater than 4. Any number less than 4 is not in the solution set.

Attempting to solve the equation by the usual method leads to the following.

$$2y + 4 = y - 4$$
$$2y + 4 + (-4) + (-y) = y + (-4) + (-4) + (-y)$$
$$y = -8$$

Solution set is $\{-8\}$.

$$2y + 4 = -(y - 4)$$
$$2y + 4 = -y + 4$$
$$2y + 4 + (-4) + y = -y + 4 + (-4) + y$$
$$3y = 0$$
$$y = 0$$

Solution set is $\{0\}$.

At first it might appear that the solution set of the original equation is $\{-8, 0\}$. However, substituting each of these numbers for the variable in the original equation results in false statements. Therefore, they are not elements of the solution set. The reason for this is that they are less than 4, and any number less than 4 will make the right side of the original equation negative. There is no solution, so the equation $|2y + 4| = y - 4$ is an internally inconsistent equation and its solution set is $\emptyset$.

Equations containing absolute values should first be inspected to see if there is any obvious restriction on what values may be used for the variable. Then any solutions to the two corresponding equations must be substituted in the original equation to determine whether or not they are elements in the solution set of the original equation.

This section contains the proofs of several of the fundamental theorems of algebra, a subset of which is necessary in the solution of most algebraic equations. Included among the theorems are the following:

(1) $(-1)(x) = -x$
(2) $ax - bx = (a - b)x$
(3) $(-1)(-x) = x$
(4) $\frac{x}{-1} = -x$
(5) $-(x + y) = -x - y$
(6) $-(x - y) = -x + y$
(7) $-(x + y - z) = -x - y + z$
(8) $(-x)(y) = -(xy)$
(9) $(-x)(-y) = xy$

The simple proofs of these theorems afford an excellent opportunity for the discussion of certain topics in algebra about which there is often some misunderstanding. One such topic is the meaning of the symbol -x. The symbol -x is read "the inverse of x." It should not be read "minus x" and it should be not read "negative x." If -x is read "minus x" or "negative x," it may tend to imply that it represents a negative number. If x represents a negative number, then -x represents a positive number. If x represents 0, then -x represents 0. The symbol -x is the inverse of x and it may represent a positive number, a negative number, or zero. Therefore, it should not be called "negative x" nor "minus x."

The proof of $(-1)(x) = -x$ shows that the inverse of an algebraic expression is equal to the product of that expression and -1. Therefore, $-(6x + 3y - 2z) = (-1)(6x + 3y - 2z)$. The inverse of any expression can be found by multiplying that expression by -1 and then applying the distributive principle.

Subtraction problems can then be solved by adding the inverse of the subtrahend to the minuend. For example,

$$\begin{aligned} 6x - (3y + 4z) &= 6x + [-(3y + 4z)] \\ &= 6x + (-1)(3y + 4z) \\ &= 6x + (-3y) + (-4z) \\ &= 6x - 3y - 4z \end{aligned}$$

The last topic in this section is that of collecting like terms and combining them under the operation of addition. In an expression such as $6k - 4m - 8k - 2m$ the terms cannot be rearranged because the expression involves subtraction and subtraction is not commutative. In order to rearrange the terms so that like terms are adjacent, the subtractions are written as the equivalent additions:
$6k - 4m - 8k - 2m = 6k + (-4m) + (-8k) + (-2m)$.
Since addition is commutative, the order of the terms may be rearranged so that like terms are adjacent to one another:
$6k + (-4m) + (-8k) + (-2m) = 6k + (-8k) + (-4m) + (-2m)$.
Like terms can then be combined under addition:
$6k + (-8k) + (-4m) + (-2m) = -2k + (-6m)$

The addition in the final answer may be written as the equivalent subtraction so that the final answer will contain no parentheses:
$-2k + (-6m) = -2k - 6m$.

The topic, equations and methods of solving equations, is discussed sufficiently in this section to give the pupil the tools necessary for solving the problems in unit 5 by means of algebraic equations. However, the topic is explored more extensively in later units.

4.3 INEQUALITIES

Open sentences are classified as equations and inequalities. Equations are classified as either identities, conditional equations, or internally inconsistent equations according to their solution sets. Inequalities are also classified as to their solution sets. If the solution set of an inequality is identical to the replacement set, the inequality is called an absolute inequality. An absolute inequality is always true, regardless of what element from the replacement set is substituted for the variable. The inequality $x + 3 > x$ is an absolute inequality. It is always true. No matter what element from the replacement set is substituted for the variable, the inequality will be a true statement.

If the solution set of an inequality is the null set, the inequality is an internally inconsistent inequality. $x < x$ is never true. No matter what element from the replacement set is substituted for the variable, the inequality will be a false statement. The solution set of $x < x$ is the null set so $x < x$ is an internally inconsistent inequality.

If the solution set of an inequality contains at least one, but not all, of the elements in the replacement set, the inequality is called a conditional inequality. $b - 2 < 5$ is a conditional inequality. If any number less than 7 is substituted for the variable, the inequality is a true statement. If any number equal to or greater than 7 is substituted for the variable, the inequality is a false statement. The solution set is a proper subset of the replacement set.

The discussion of the solution sets of inequalities leads quite naturally to the topic of the methods of solving an inequality. Solutions of inequalities are based on the following definition of inequality:

$$x < y \text{ if and only if } y - x > 0$$

This means that if x is less than y, then the difference $y - x$ is a positive number; and if the difference $y - x$ is a positive number, then x is less than y.

Inequalities are solved by applying the additive property of inequalities and the multiplicative property of inequalities.

The additive property of inequalities is stated as follows:

$$x < y \text{ if and only if } x + z < y + z$$

This property is developed in the following manner.

(1) $(y + z) - (x + z) = y - x$
(2) If $x < y$, then $y - x > 0$
(3) If $y - x > 0$, then $(y + x) - (x + z) > 0$
(4) If $(y + z) - (x + z) > 0$, then $x + z < y + z$
(5) If $x < y$ then $x + z < y + z$

This proves that if $x < y$, then $x + z < y + z$. The second part of the additive property of inequalities is that if $x + z < y + z$, then $x < y$. This property is developed in the following manner.

(1) $(y + z) - (x + z) = y - x$
(2) If $x + z < y + z$, then $(y + z) - (x + z) > 0$
(3) If $(y + z) - (x + z) > 0$, then $y - x > 0$
(4) If $y - x > 0$, then $x < y$
(5) If $x + z < y + z$, then $x < y$

The additive property of inequalities may be stated as "if the same number is added to or subtracted from both sides of an inequality, the solution set of the resulting inequality is the same as the solution set of the original inequality."

Inequalities such as $d + 4 < 14$ or $h - 9 > -8$ are solved in much the same manner as equations are solved, by using the additive property. For example,

$d + 4 < 14$
$d + 4 + (-4) < 14 + (-4)$
$d < 10$
Solution set is $\{d | d < 10\}$.

$h - 9 > -8$
$h + (-9) + 9 > -8 + 9$
$h > 1$
Solution set is $\{h | h > 1\}$.

Notice that the solution set may be written in set-builder notation.

The multiplicative property of inequalities is somewhat more complicated than the additive property. The multiplicative property of inequalities may be stated as follows.
If $z > 0$, $x < y$ if and only if $xz < yz$.
If $z < 0$, $x < y$, if and only if $xz > yz$.
If both sides of an inequality are multiplied or divided by the same positive number, the solution set of the new inequality is the same as the solution set of the original inequality. If both sides of an inequality are multiplied or divided by the same negative number and the inequality sign is reversed, the solution set of the resulting inequality is the same as the solution set of the original inequality.

The proofs of all parts of this property are not difficult, and the teacher may wish to have the pupils learn some of these proofs. The teacher may even assign one or more of these proofs as original exercises for the pupils to perform.

<u>TO PROVE</u>: If $x < y$ and $z > 0$ then $xz < yz$

(1) If $x < y$ then $y - x > 0$
(2) If $y - x > 0$ and if $z > 0$, then $(y - x)z > 0$
(3) If $x < y$ and $z > 0$ then $yz - xz > 0$
(4) If $yz - xz > 0$ then $xz < yz$
(5) If $x < y$ and $z > 0$ then $xz < yz$

TO PROVE: If $xz < yz$ and $z > 0$, then $x < y$

(1) If $xz < yz$, then $yz - xz > 0$
(2) If $yz - xz > 0$, then $(y - x) z > 0$
(3) If $(y - x) z > 0$ and $z > 0$, then $y - x > 0$
(4) If $xz < yz$ and $z > 0$, then $y - x > 0$
(5) If $y - x > 0$, then $x < y$
(6) If $xz < yz$ and $z > 0$ then $x < y$

TO PROVE: If $x < y$ and $z < 0$ then $xz > yz$

(1) If $x < y$ then $y - x > 0$
(2) If $z < 0$ and $y - x > 0$ then $(y - x) z < 0$
(3) If $(y - x) z < 0$ then $yz - xz < 0$
(4) If $x < y$ and $z < 0$ then $yz - xz < 0$
(5) If $yz - xz < 0$ then $-(yz - xz) > 0$
(6) If $-(yz - xz) > 0$ then $xz - yz > 0$
(7) If $xz - yz > 0$ then $yz < xz$
(8) If $yz < xz$ then $xz > yz$
(9) If $x < y$ and $z < 0$ then $xz > yz$

TO PROVE: If $xz > yz$ and $z < 0$ then $x < y$

(1) If $xz > yz$ then $yz < xz$
(2) If $yz < xz$ then $xz - yz > 0$
(3) If $xz - yz > 0$ then $-(xz - yz) < 0$
(4) If $-(xz - yz) < 0$ then $yz - xz < 0$
(5) If $yz - xz < 0$ then $(y - x) z < 0$
(6) If $z < 0$, and $(y - x) z < 0$, then $(y - x) > 0$
(7) If $(y - x) > 0$ then $x < y$
(8) If $xz > yz$ and $z < 0$ then $x < y$

The methods of solving inequalities may be summarized as:

- If the same number is added to or subtracted from each side of an inequality, the solution set of the resulting inequality is the same as the solution set of the original inequality.
- If both sides of an inequality are multiplied or divided by the same positive number, the solution set of the resulting inequality is the same as the solution set of the original inequality.
- If both sides of an inequality are multiplied or divided by the same negative number and the inequality sign is reversed, the solution set of the resulting inequality is the same as the solution set of the original inequality.

Below are examples of solutions to typical inequalities.

$$2x - 3 < 5$$
$$2x + (-3) + 3 < 5 + 3$$
$$2x < 8$$
$$x < 4$$

Solution set is $\{x | x < 4\}$.

$$-3x + 4 < 8$$
$$-3x + 4 + (-4) < 8 + (-4)$$
$$-3x < 4$$
$$\frac{-3x}{-3} > \frac{4}{-3}$$
$$x > \frac{-4}{3}$$

Solution set is $\left\{x | x > \frac{-4}{3}\right\}$.

The last topic in this section pertains to the solving of inequalities that contain absolute value.

The solution set of an inequality of the type $|a| > 8$ consists of the union of the sets $\{a|a > 8\}$ and $\{a|a < -8\}$. The solution set of an inequality of the type $|a| < 8$ consists of the intersection of the sets $\{a|a < 8\}$ and $\{a|a > -8\}$. This set may also be written $\{a|-8 < a < 8\}$.

<u>Teacher Notes</u>

UNIT 4: OPEN SENTENCES

PART 2. QUESTIONS AND ACTIVITIES FOR CLASSROOM USE

4.1 INTRODUCTION

The questions and activities in this unit are designed as a guide to show the teacher how a particular line of questioning can be used to present the various concepts of equations and inequalities through the discovery approach. However, this does not mean that these questions and only these questions must be used to elicit from the pupils the answer indicated. The teacher may actually have to ask several questions for every one furnished in this unit in order to receive the answer indicated, but the questions furnished will serve as a guide for the line of questioning to be used by the teacher. In teaching elementary algebra, the discovery approach has certain limitations but it will be a method of great value if the pupils are given a firm foundation in the basic fundamentals of the symbols used, definitions of terms, and basic properties of equations. If the pupils have a good mastery of the fundamental mathematical concepts of algebra, they can use the discovery approach in a good part of the course, and each new discovery can be a reward with satisfaction.

4.2 EQUATIONS

<u>Concept</u>: Classification of equations.

(1) <u>Answer the following questions</u>.

(a) What is an equation?

(b) What can you say about the truthfulness of the equation $x + 3 = x$? Is it always true, or never true?

(c) An equation that is never true is called an internally inconsistent equation. Describe the solution set of an internally inconsistent equation.

<u>Answers</u>:

(a) A statement of equality between two expressions

(b) It is never true.

(c) The solution set is the null set.

(2) <u>Answer the following questions</u>.

(a) Describe the truthfulness of the equation $x + 3 = 3 + x$ when any element in the replacement set of real numbers is substituted for x.

(b) An equation that is always true regardless of what element from the replacement set is substituted for the variable is called an identity. An identity is always true. Describe the solution set of an identity by comparing it to the replacement set of the variable contained in the identity.

Answers:

(a) The equation $x + 3 = 3 + x$ is always true regardless of what number is substituted for x. This is the commutative property under addition.

(b) The solution set of an identity contains all the elements of the replacement set of the variable.

(3) Answer the following questions.

(a) Describe the truthfulness of the equation $x + 2 = 3$ when an element from the replacement set is substituted for the variable.

(b) An equation such as $x + 2 = 3$, which is not always true and not always false but whose truthfulness depends on what number is substituted for the variable is called a conditional equation. Describe the solution set of a conditional equation by comparing it to the replacement set.

Answers:

(a) The equation may be true or it may be false, depending on which number is substituted for x. If the number 1 is substituted, the resulting equation is true. If any other number is substituted, the resulting equation is false.

(b) The solution set is a proper subset of the replacement set. The solution set consists of at least one, but not all, of the elements in the replacement set.

(4) Classify each of the following equations. The replacement set is the set of natural numbers. Indicate the solution set of each equation.

(a) $x = x$ (c) $3 - x = 5$

(b) $x + 3 = 7$ (d) $y^2 = 16$

Answers:

(a) Identity. The solution set is the entire replacement set.

(b) Conditional equation. The solution set is $\{4\}$.

(c) Internally inconsistent equation in natural numbers. The solution set is $\emptyset$ in this set.

(d) Conditional equation. The solution set is $\{4\}$ if the replacement set is the set of natural numbers.

(5) Classify each of the following equations. The replacement set is the set of integers. Indicate the solution set of each equation.

(a) $3 - x = 5$ (c) $3x = 1$

(b) $y^2 = 16$ (d) $8(\square + \triangle) = 8\square + 8\triangle$

Answers:

(a) Conditional equation. The solution set is $\{-2\}$.

(b) Conditional equation. The solution set is $\{4, -4\}$.

(c) Internally inconsistent equation in integers. The solution set is the null set if the replacement set is the set of integers.

(d) Identity. The solution set is the entire replacement set. This is the distributive property.

(6) Classify each of the following equations. The replacement set is the set of real numbers. Indicate the solution set of each equation.

(a) $|\triangle| = -3$

(b) $x^2 = 1$

(c) $3 - a = a - 3$

(d) $6 + (x + 4) = (6 + x) + 4$

(e) $|k + 9| = 9$

Answers:

(a) Internally inconsistent equation. The solution set is $\emptyset$.

(b) Conditional equation. The solution set is $\{1, -1\}$.

(c) Conditional equation. The solution set is $\{3\}$.

(d) Identity. The solution set is the entire replacement set. This is an application of the associative principle under addition.

(e) Conditional equation. The solution set is $\{-18, 0\}$.

Concept: Properties of equations.

(7) Three important properties of equations are the reflexive, symmetric, and transitive properties. Following is the definition of each of these terms as applied to equations.

Property	Definition (in terms of the relation, equals)
Reflexive	For any expression a, a = a.
Symmetric	For any two expressions a and b, if a = b then b = a.
Transitive	For any expressions a, b, and c if a = b and b = c, then a = c.

Make a table with four columns labeled respectively, Relation, Reflexive, Symmetric, and Transitive. Use "yes" or "no" as entries in the last 3 columns given the following relations: not equal to ($\neq$), less than ($<$), and perpendicular to ($\perp$).

Answer:

Relation	Reflexive	Symmetric	Transitive
$\neq$	No	Yes	No
$<$	No	No	Yes
$\perp$	No	Yes	No

There are other properties of equations that are discussed later.

Concept: Identities.

(8) Which of the following describe an identity?

(a) The solution set is a proper subset of the replacement set.

(b) The solution set is a subset of the replacement set.
(c) The solution set contains at least one element not in the replacement set.
(d) The replacement set contains at least one element not in the solution set.
(e) The solution set is the null set.
(f) The solution set and the replacement set are identical sets.

<u>Answers</u>: (b) and (f)

(9) <u>Answer the following questions</u>.

(a) One pupil claims the equation $\triangle \cdot \triangle + 6 = 5 \cdot \triangle$ is an identity because when the number 2 is substituted for $\triangle$, the equation is a true statement. Is the pupil correct? (The replacement set is the set of real numbers.)
(b) Another pupil asked whether it would be possible to prove that an equation is an identity by substituting numbers for the variable. How many substitutions would have to be made to prove an equation is an identity?
(c) How can we prove an equation is not an identity?
(d) Is the equation $x + 3 = 3 + x$ an identity. How do you know?
(e) How can you prove an equation is an identity?

<u>Answers</u>:
(a) No, the pupil is wrong. The solution set is $\{2, 3\}$. If for instance $\triangle$ is replaced by 5, the statement is false. The solution set and replacement set are not identical. It is not an identity; it is a conditional equation.
(b) Every element in the replacement set would have to be substituted for the variable. If the replacement set contains an infinite number of elements, an infinite number of substitutions would have to be made, which is impossible.
(c) By showing there is at least one element in the replacement set which is not in the solution set.
(d) It is an identity because it is a statement of the commutative principle under addition.
(e) An equation is an identity if it is a statement of or an application of one of the axioms, postulates, or definitions of our number system.

(10) <u>Which of the following are identities</u>? <u>State the reason for your conclusion</u>.

(a) $(x + 3) + 8 = x + (3 + 8)$
(b) $3 \cdot \triangle = \triangle \cdot 3$
(c) $x = x$
(d) $11 + y = y + 11$
(e) $6 (\triangle + \square) = 6\triangle + 6\square$
(f) $x \cdot 0 = 0$
(g) $3 \cdot (2x) = (3 \cdot 2)x$

(h) $x - 6 = 6 - x$
(i) $2x + 3x = x\cdot(2 + 3)$
(j) $(\triangle - 6) - 8 = \triangle - (6 - 8)$
(k) $y + 0 = y$
(l) $\triangle - \triangle = 0$
(m) $\frac{a}{0} = 0$

<u>Answers</u>:

(a) An identity. Associative principle under addition
(b) An identity. Commutative principle under multiplication
(c) An identity. Reflexive property of an equation
(d) An identity. Commutative principle under addition
(e) An identity. Distributive principle
(f) An identity. Multiplicative property of zero
(g) An identity. Associative principle under multiplication
(h) Not an identity. The solution set $\{6\}$ is not identical to the set of real numbers.
(i) An identity. Distributive and commutative principles
(j) Not an identity. Subtraction is not associative. The solution set is Ø not the set of real numbers.
(k) An identity. Definition of additive identity element
(l) An identity. Definition of subtraction
(m) Not an identity. Division by zero is undefined and not permitted. This is an internally inconsistent equation.

<u>Concept</u>: Equivalent equations.

(11) <u>What do you think is meant by "equivalent equations?"</u>

<u>Answer</u>: Equations with the same solution set

<u>Concept</u>: The additive property of equations.

(12) <u>Which of the following pairs of equations are equivalent equations?</u>
(a) $x = 3$ and $x + 9 = 3 + 9$
(b) $x = 51$ and $x + (-21) = 51 + (-21)$
(c) $x + 3 = 6$ and $x + 3 + 3 = 6 + (-3)$
(d) $x = 8$ and $x + 4 = 8 + 3$
(e) $x + 3 = 7$ and $x + 3 + 0 = 7 + 0$

<u>Answers</u>: (a), (b), and (e)

(13) <u>What does the equation a = b mean</u>?

Answer: a and b are different symbols for the same number.

(14) If a and b are symbols for the same number, compare the sum a + c with the sum b + c.

Answer: The sum a + c and the sum b + c must be the same.

(15) The additive property of equations is stated as follows: For every a, b, and c, a = b if and only if a + c = b + c.
Restate the above property in two sentences.

Answer: For every a, b, and c, if a = b, then a + b = b + c. For every a, b, and c, if a + c = b + c, then a = b.

(16) If the same number is added to or subtracted from both sides of an equation to form a new equation, how does the solution set of the new equation compare with that of the original equation?

Answer: The solution sets of the equations will be identical.

(17) If the same number is added to or subtracted from both sides of an equation to form a new equation, are the two equations equivalent?

Answer: Yes, because they have the same solution set.

(18) Do the following exercises.

(a) Write an equation equivalent to x + 3 = 9, formed by adding -3 to both sides of the equation.
(b) Write an equation equivalent to x + 3 + (-3) = 9 + (-3), formed by performing the indicated addition of like terms.

Answers:

(a) x + 3 + (-3) = 9 + (-3)
(b) x = 6

(19) The equation x = 6 indicates that the solution set is {6}. Using this additive property of equations, find the solution set to each of the following equations.

(a) y + 8 = 24	(d) 19 = x + 4
(b) 7 + y = 5	(e) -3.8 + y = -3.8
(c) k + (-6) = -9	(f) x + 3 = x + 3

Answers:

(a)

y + 8 = 24
y + 8 +(-8) = 24 + (-8)
y = 16

Solution set is {16}.

Check: 16 + 8 = 24 true

(b)

$$7 + y = 5$$
$$7 + y + (-7) = 5 + (-7)$$
$$y + 7 + (-7) = 5 + (-7)$$
$$y = -2$$

Solution set is $\{-2\}$.

Check: $7 + (-2) = 5$ true

(c)

$$k + (-6) = -9$$
$$k + (-6) + 6 = -9 + 6$$
$$k = -3$$

Solution set is $\{-3\}$.

Check: $(-3) + (-6) = -9$ true

(d)

$$19 = x + 4$$
$$19 + (-4) = x + 4 + (-4)$$
$$15 = x$$

Solution set is $\{15\}$.

Check: $19 = 15 + 4$ true

(e)

$$-3.8 + y = -3.8$$
$$-3.8 + y + 3.8 = -3.8 + 3.8$$
$$y + (-3.8) + 3.8 = -3.8 + 3.8$$
$$y = 0$$

Solution set is $\{0\}$.

Check: $-3.8 + 0 = -3.8$ true

(f)

$$x + 3 = x + 3$$
$$x + 3 + (-3) = x + 3 + (-3)$$
$$x = x$$

Solution set is {all real numbers}.

The equation is an identity.

(20) <u>Do the following exercises</u>.

(a) Write an equation equivalent to $x - 6 = 8$ which involves addition rather than subtraction.

(b) Find the solution set of $x + (-6) = 8$ by using the additive property for equations.

<u>Answers</u>:

(a) $x + (-6) = 8$

(b)

$$x + (-6) = 8$$
$$x + (-6) + 6 = 8 + 6$$
$$x = 14$$

Solution set is $\{14\}$.

Check: $14 + (-6) = 8$ true

(21) <u>Find the solution set of each of the following by using the additive property of equations</u>.

(a) $f - 3 = 9$

(b) $y - 0.97 = \frac{3}{4}$

(c) $9.4 = z - 8.7$

(d) $\triangle - (-7) = 13$

(e) $\square - \frac{1}{2} = -\frac{1}{2}$

Answers:

(a)

$$f - 3 = 9$$
$$f + (-3) = 9$$
$$f + (-3) + 3 = 9 + 3$$
$$f = 12$$

Solution set is {12}.

Check: $12 - 3 = 9$ true

(b)

$$y - 0.97 = \frac{3}{4}$$
$$y + (-0.97) = .75$$
$$y + (-0.97) + 0.97 = 0.75 + 0.97$$
$$y = 1.72$$

Solution set is {1.72}.

Check: $1.72 - .97 \stackrel{?}{=} \frac{3}{4}$

$.75 = \frac{3}{4}$ true

(c)

$$9.4 = z - 8.7$$
$$9.4 = z + (-8.7)$$
$$9.4 + 8.7 = z + (-8.7) + 8.7$$
$$18.1 = z$$

Solution set is {18.1}.

Check: $9.4 = 18.1 - 8.7$ true

(d)

$$\triangle - (-7) = 13$$
$$\triangle + 7 = 13$$
$$\triangle + 7 + (-7) = 13 + (-7)$$
$$\triangle = 6$$

Solution set is {6}.

Check: $6 - (-7) = 13$ true

(e)

$$\square - \frac{1}{2} = -\frac{1}{2}$$
$$\square + \left(-\frac{1}{2}\right) = -\frac{1}{2}$$
$$\square + \left(-\frac{1}{2}\right) + \frac{1}{2} = -\frac{1}{2} + \frac{1}{2}$$
$$\square = 0$$

Solution set is {0}.

Check: $0 - \frac{1}{2} \stackrel{?}{=} -\frac{1}{2}$

$0 + \left(-\frac{1}{2}\right) = -\frac{1}{2}$ true

Concept: Multiplicative property of equations.

(22) Write an equation containing one variable. Multiply both sides of this equation by any number except zero. How do the solution sets of the two equations compare?

Answer: The solution sets are the same.

(23) When both sides of an equation are multiplied by the same number, not zero, is the resulting equation equivalent to the original equation?

Answer: Yes

(24) The multiplicative property of equations may be stated as follows: For every a, b, and c ($c \neq 0$), $a = b$ if and only if $ac = bc$.
Restate the above principle in two sentences.

Answer: For every a, b, and c ($c \neq 0$) if $a = b$ then $ac = bc$. For every a, b, and c ($c \neq 0$), if $ac = bc$ then $a = b$.

(25) Multiplying by $\frac{1}{3}$ is equivalent to dividing by what number?

Answer: It is equivalent to dividing by 3.

(26) Dividing by 9 is equivalent to multiplying by what number?

Answer: It is equivalent to multiplying by $\frac{1}{9}$.

(27) State the multiplicative property of equations in a word sentence in terms of multiplying or dividing both sides of an equation by the same number.

Answer: Multiplying or dividing both sides of an equation by the same number (except zero) results in a new equation having the same solution set as that of the original equation.

(28) Write an equation equivalent to $16y = 48$, formed by multiplying both sides of the original equation by $\frac{1}{16}$.

Answer: $\left(\frac{1}{16}\right)(16y) = \left(\frac{1}{16}\right)(48)$

(29) Write an equation equivalent to $\left(\frac{1}{16}\right)(16y) = \left(\frac{1}{16}\right)(48)$ formed by performing the indicated multiplication and reducing all terms to lowest terms.

Answer: $y = 3$

(30) What is the solution set of the equation $16y = 48$?

Answer: $\{3\}$.

(31) <u>Find the solution set to each of the following equations by applying the multiplicative property of equations</u>.

(a) $21x = -84$ (d) $\frac{y}{8} = -2$

(b) $-13x = -65$

(c) $1.8k = -3.6$ (e) $\frac{7}{9}y = -\frac{2}{3}$

<u>Answers</u>:

(a)

$$21x = -84$$
$$\left(\frac{1}{21}\right)(21x) = \left(\frac{1}{21}\right)(-84)$$
$$x = -4$$

Solution set is $\{-4\}$.

Check: $21(-4) = -84$ true

(b)

$$-13x = -65$$
$$\left(\frac{1}{-13}\right)(-13x) = \left(\frac{1}{-13}\right)(-65)$$
$$x = 5$$

Solution set is $\{5\}$.

Check: $-13(5) = -65$ true

(c)

$$1.8k = -3.6$$
$$\left(\frac{1}{1.8}\right)(1.8k) = \left(\frac{1}{1.8}\right)(-3.6)$$
$$k = -2$$

Solution set is $\{-2\}$.

Check: $1.8(-2) = -3.6$ true

(d)

$$\frac{y}{8} = -2$$
$$8\left(\frac{y}{8}\right) = (8)(-2)$$
$$y = -16$$

Solution set is $\{-16\}$.

Check: $\frac{-16}{8} = -2$ true

(e)

$$-\frac{7y}{9} = -\frac{2}{3}$$
$$\left(-\frac{9}{7}\right)\left(-\frac{7}{9}\right)y = \left(-\frac{9}{7}\right)\left(-\frac{2}{3}\right)$$
$$y = \frac{6}{7}$$

Solution set is $\left\{\frac{6}{7}\right\}$.

Check: $\left(-\frac{7}{9}\right)\left(\frac{6}{7}\right) \overset{?}{=} -\frac{2}{3}$

$-\frac{6}{9} = -\frac{2}{3}$ true

(32) **Complete the following exercises.**

(a) Consider the equation $\frac{3}{x} = 2$. What is the restriction on the replacement set of the variable x?

(b) What does x represent in the equation $\frac{3}{x} = 2$?

(c) If both sides of the equation $\frac{3}{x} = 2$ are multiplied by the placeholder x, is this an application of the multiplicative property of equations?

(d) Multiply both sides of the equation $\frac{3}{x} = 2$ by x and reduce all terms. What is the resulting equation?

(e) Find the solution set of the resulting equation in (d). Is this the same as the solution set of the original equation?

Answers:

(a) $x \neq 0$

(b) x is a placeholder for a number in the set of real numbers.

(c) Yes. The placeholder x represents a number (other than zero) and both sides of this type of fractional equation may be multiplied by the placeholder without changing the solution set.

(d) $3 = 2x$

(e)
$$3 = 2x$$
$$\left(\frac{1}{2}\right)(3) = \left(\frac{1}{2}\right)(2x)$$
$$\frac{3}{2} = x$$

Solution set is $\left\{\frac{3}{2}\right\}$.

Check: $\frac{3}{\frac{3}{2}} \stackrel{?}{=} 2$

$3\left(\frac{2}{3}\right) = 2$ true

This is the same as the solution set of the original equation.

(33) **Find the solution set of each of the following equations.**

(a) $\frac{-7}{x} = 5$ (b) $\frac{1.5}{k} = -6$ (c) $-\frac{5}{2} = \frac{12}{a}$

Answers:

(a)

$$\frac{-7}{x} = 5$$

$$x\left(\frac{-7}{x}\right) = x \cdot 5$$

$$-7 = 5x$$

$$\left(\frac{1}{5}\right)(-7) = \left(\frac{1}{5}\right)(5x)$$

$$-\frac{7}{5} = x$$

Solution set is $\left\{-\frac{7}{5}\right\}$.

Check: $\frac{-7}{-\frac{7}{5}} \stackrel{?}{=} 5$

$$-7\left(-\frac{5}{7}\right) = 5 \text{ true}$$

(b)

$$\frac{1.5}{k} = -6$$

$$k\left(\frac{1.5}{k}\right) = k \cdot (-6)$$

$$1.5 = -6k$$

$$\left(-\frac{1}{6}\right)(1.5) = \left(-\frac{1}{6}\right)(-6k)$$

$$-\frac{1}{4} = k$$

Solution set is $\left\{-\frac{1}{4}\right\}$.

Check: $\frac{1.5}{-\frac{1}{4}} \stackrel{?}{=} -6$

$$(1.5)(-4) \stackrel{?}{=} -6$$

$$-6 = -6 \text{ true}$$

(c)

$$-\frac{5}{2} = \frac{12}{a}$$

$$a\left(-\frac{5}{2}\right) = a\left(\frac{12}{a}\right)$$

$$\left(-\frac{5}{2}\right)a = 12$$

$$-\frac{2}{5}\left(-\frac{5}{2}a\right) = -\frac{2}{5}(12)$$

$$a = -\frac{24}{5}$$

Solution set is $\left\{-\frac{24}{5}\right\}$.

Check: $-\frac{5}{2} \stackrel{?}{=} \frac{12}{-\frac{24}{5}}$ $\quad -\frac{5}{2} = (12)\left(-\frac{5}{24}\right)$ true.

Concept: Adding and subtracting variables from each side of an equation.

(34) What is the solution set of the equation $4 \triangle = 20$?

Answer: $\{5\}$

(35) Add the variable $\triangle$ to each side of the equation. What is the solution set of the new equation?

Answer: $\{5\}$

(36) Add $7 \triangle$ to both sides of the equation $4 \triangle = 20$. What is the solution set of the new equation?

Answer: $\{5\}$

(37) Subtract $2 \triangle$ from each side of the equation $4 \triangle = 20$. What is the solution set of the new equation?

Answer: $\{5\}$

(38) When the same variable or the same multiple of the same variable is added to or subtracted from both sides of the equation $4 \triangle = 20$, does the solution set of the equation change?

Answer: No, the solution set remains the same.

(39) Is adding or subtracting the same multiple of the same variable from both sides of an equation an application of the addition property of equations?

Answer: Yes. The variable represents a number and any number may be added to or subtracted from both sides of an equation without changing the solution set.

(40) How can the principle in (39) be applied in solving the equation $3x = 2x + 6$?

Answer: By adding $-2x$ to both sides of the equation (or subtracting $2x$)

$$3x = 2x + 6$$
$$3x - 2x = 2x + 6 + (-2x)$$
$$x = 6$$

Solution set is $\{6\}$.

Check: $3 \cdot 6 \stackrel{?}{=} 2 \cdot 6 + 6$
$18 = 12 + 6$ true

(41) Find the solution sets of the following equations.

(a) $9x = 7x + 10$
(b) $3y - 2 = y + 4$
(c) $4b + 9 = 25 - 4b$
(d) $3(m + 4) = m - 7$

Answers:

(a)

$$9x = 7x + 10$$
$$9x + (-7x) = 7x + 10 + (-7x)$$
$$2x = 10$$
$$x = 5$$

Solution set is $\{5\}$.

Check: $9 \cdot 5 \stackrel{?}{=} 7 \cdot 5 + 10$

$45 = 35 + 10$ true

(b)

$$3y - 2 = y + 4$$
$$3y + (-2) + 2 + (-y) = y + 4 + 2 + (-y)$$
$$2y = 6$$
$$y = 3$$

Solution set is $\{3\}$.

Check: $3 \cdot 3 - 2 \stackrel{?}{=} 3 + 4$

$9 - 2 = 7$ true

(c)

$$4b + 9 = 25 - 4b$$
$$4b + 9 + (-9) + 4b = 25 + (-4b) + (-9) + 4b$$
$$8b = 16$$
$$b = 2$$

Solution set is $\{2\}$.

Check: $4 \cdot 2 + 9 \stackrel{?}{=} 25 - 4 \cdot 2$

$8 + 9 \stackrel{?}{=} 25 - 8$

$17 = 17$ true

(d)

$$3\ (m + 4) = m - 7$$
$$3m + 12 = m + (-7)$$
$$3m + 12 + (-12) + (-m) = m + (-7) + (-12) + (-m)$$
$$2m = -19$$
$$m = -\frac{19}{2}$$

Solution set is $\left\{-\frac{19}{2}\right\}$.

Check: $3\left(-\frac{19}{2} + 4\right) \stackrel{?}{=} -\frac{19}{2} - 7$

$3\left(-\frac{19}{2} + \frac{8}{2}\right) \stackrel{?}{=} -\frac{19}{2} - \frac{14}{2}$

$3\left(-\frac{11}{2}\right) \stackrel{?}{=} -\frac{33}{2}$

$-\frac{33}{2} = -\frac{33}{2}$ true

Concept: Solution to equations involving absolute value.

(42) Applying the definition of absolute value, what is the solution set of the equation $|x| = 5$?

Answer: $\{5, -5\}$.

(43) Consider the equation $|3x + 4| = 10$. The expression $3x + 4$ is equivalent to what two numbers? Why?

Answer: The expression $3x + 4$ is equivalent to 10 and to -10 because $|10| = 10$ and $|-10| = 10$.

(44) Write two equations that do not contain absolute values and which together are equivalent to the equation $|3x + 4| = 10$.

Answer: $3x + 4 = 10$ or $3x + 4 = -10$

(45) Find the solution set of the equation $3x + 4 = 10$ and of the equation $3x + 4 = -10$.

Answer:

$$3x + 4 = 10$$
$$3x + 4 + (-4) = 10 + (-4)$$
$$3x = 6$$
$$\frac{3x}{3} = \frac{6}{3}$$
$$x = 2$$

Solution set is $\{2\}$.

Check: $3 \cdot 2 + 4 = 10$ true

$$3x + 4 = -10$$
$$3x + 4 + (-4) = -10 + (-4)$$
$$3x = -14$$
$$\frac{3x}{3} = -\frac{14}{3}$$
$$x = -\frac{14}{3}$$

Solution set is $\left\{-\frac{14}{3}\right\}$.

Check: $3\left(-\frac{14}{3}\right) + 4 = -10$ true

(46) For each of the following equations, write an equivalent pair of equations that do not contain absolute value signs. Find the solution set of each of these equations and indicate the solution set of the original equation.

(a) $|5 + x| = 3$ (b) $\left|\frac{6x + 1}{5}\right| = 11$ (c) $|2y| = \frac{1}{2}$

Answers:

(a)

$$5 + x = 3$$
$$5 + x + (-5) = 3 + (-5)$$
$$x = -2$$

Solution set is $\{-2\}$.

$$5 + x = -3$$
$$5 + x + (-5) = -3 + (-5)$$
$$x = -8$$

Solution set is $\{-8\}$.

Solution set of $|5 + x| = 3$ is $\{-2, -8\}$.

check first

$$|5 + (-2)| \stackrel{?}{=} 3$$
$$|3| = 3 \text{ true}$$

check second

$$|5 + (-8)| \stackrel{?}{=} 3$$
$$|-3| = 3 \text{ true}$$

(b)

$$\frac{6x + 1}{5} = 11$$
$$5\left(\frac{6x + 1}{5}\right) = 5(11)$$
$$6x + 1 = 55$$
$$6x + 1 + (-1) = 55 + (-1)$$
$$6x = 54$$
$$\frac{1}{6}(6x) = \frac{1}{6}(54)$$
$$x = 9$$

Solution set is $\{9\}$.

$$\frac{6x + 1}{5} = -11$$
$$5\left(\frac{6x + 1}{5}\right) = 5(-11)$$
$$6x + 1 = -55$$
$$6x + 1 + (-1) = -55 + (-1)$$
$$6x = -56$$
$$\frac{1}{6}(6x) = \frac{1}{6}(-56)$$
$$x = -\frac{28}{3}$$

Solution set is $\left\{-\frac{28}{3}\right\}$

Solution set of $\left|\frac{6x + 1}{5}\right| = 11$ is $\left\{9, -\frac{28}{3}\right\}$

check first

$$\left|\frac{6(9) + 1}{5}\right| \stackrel{?}{=} 11$$
$$|11| = 11 \text{ true}$$

check second

$$\left|\frac{6\left(-\frac{28}{3}\right) + 1}{5}\right| \stackrel{?}{=} 11$$
$$\left|\frac{-56 + 1}{5}\right| \stackrel{?}{=} 11$$
$$|-11| = 11 \text{ true}$$

(c)

$$2y = \frac{1}{2}$$
$$\frac{1}{2}(2y) = \frac{1}{2}\left(\frac{1}{2}\right)$$
$$y = \frac{1}{4}$$

Solution set is $\left\{\frac{1}{4}\right\}$.

$$2y = -\frac{1}{2}$$

$$\frac{1}{2}(2y) = \frac{1}{2}\left(-\frac{1}{2}\right)$$

$$y = -\frac{1}{4}$$

Solution set is $\left\{-\frac{1}{4}\right\}$.

Solution set of $|2y| = \frac{1}{2}$ is $\left\{\frac{1}{4}, -\frac{1}{4}\right\}$.

check first

$$\left|2\left(\frac{1}{4}\right)\right| = \frac{1}{2} \text{ true}$$

check second

$$\left|2\left(-\frac{1}{4}\right)\right| = \frac{1}{2} \text{ true}$$

<u>Concept</u>: Writing equivalent expressions.

(47) <u>Applying the distributive principle, determine an expression equivalent to each of the following that does not contain parentheses.</u>
(a) $-1(2x + 5)$ (b) $12(y - 6)$ (c) $-3(k - m + 11)$

<u>Answers</u>:
(a) $-1(2x + 5) = -2x + (-5)$
$= -2x - 5$
(b) $12(y - 6) = 12y - 72$
(c) $-3(k - m + 11) = -3(k + (-m) + 11)$
$= -3k + (-3)(-m) + (-3)(11)$
$= -3k + 3m + (-33)$
$= -3k + 3m - 33$

(48) <u>Find the solution set of each of the following equations.</u>
(a) $30 + 2(b - 6) = 4 + 2(3b + 5)$
(b) $(8x - 12) + 2(-15 - x) = 0$
(c) $3(k + 5) + 4(k - 5) = 23$

<u>Answers</u>:
(a)
$$30 + 2(b - 6) = 4 + 2(3b + 5)$$
$$30 + 2[b + (-6)] = 4 + 6b + 10$$
$$30 + 2b - 12 = 14 + 6b$$
$$18 + 2b = 14 + 6b$$
$$18 + 2b + (-14) + (-2b) = 14 + 6b + (-14) + (-2b)$$
$$4 = 4b$$
$$1 = b$$

Solution set is $\{1\}$.

Check: $30 + 2(1 - 6) \stackrel{?}{=} 4 + 2(3\cdot 1 + 5)$
$30 + 2(-5) \stackrel{?}{=} 4 + 2(8)$
$30 - 10 \stackrel{?}{=} 4 + 16$
$20 = 20$ true

(b) $(8x - 12) + 2(-15 - x) = 0$

$$\begin{aligned}
[8x + (-12)] + 2[-15 + (-x)] &= 0 \\
8x + (-12) + (-30) + (-2x) &= 0 \\
6x + (-42) &= 0 \\
6x + (-42) + 42 &= 0 + 42 \\
6x &= 42 \\
x &= 7
\end{aligned}$$

Solution set is $\{7\}$.

Check: $(8\cdot 7 - 12) + 2(-15 - 7) \stackrel{?}{=} 0$

$(56 - 12) + 2(-22) \stackrel{?}{=} 0$

$44 + (-44) = 0$ true

(c) $3(k + 5) + 4(k - 5) = 23$

$$\begin{aligned}
3(k + 5) + 4[k + (-5)] &= 23 \\
3k + 15 + 4k + (-20) &= 23 \\
7k + (-5) &= 23 \\
7k + (-5) + 5 &= 23 + 5 \\
7k &= 28 \\
k &= 4
\end{aligned}$$

Solution set is $\{4\}$.

Check: $3(4 + 5) + 4(4 - 5) \stackrel{?}{=} 23$

$3(9) + 4(-1) \stackrel{?}{=} 23$

$27 + (-4) = 23$ true

<u>Concept</u>: To prove that $(-1)(x) = -x$ for all x.

(49) <u>Answer the following questions.</u>

(a) What does the symbol $-x$ represent?
(b) If x represents the number 3, what number does $-x$ represent?
(c) If x represents -6, what does $-x$ represent?
(d) If x represents 0, what does $-x$ represent?
(e) Can $-x$ represent a positive number?
(f) Should the symbol "$-x$" be read "minus x?" Should it be read "negative x?"
(g) How should the symbol "$-x$" be read?
(h) What is the sum of $x + (-x)$?
(i) Is the equation $x + (-1)(x) = (1)(x) + (-1)(x)$ an identity?
(j) Write an equation equivalent to the equation in (i), applying the distributive principle to the right hand side of the equation.
(k) Write an equation equivalent to the equation in (j), performing the addition and multiplication indicated on the right hand side of the equation.
(l) If $x + (-1)(x) = 0$, to what must $(-1)(x)$ be equal?

<u>Answers</u>:
(a) $-x$ represents the additive inverse of the number represented by x.
(b) -3
(c) The inverse of -6, or 6
(d) The inverse of 0, or 0

(e) Yes, it can represent a positive number, a negative number, or zero.
(f) No. Reading -x as "minus x" or as "negative x" may tend to imply that it is the symbol for a negative number, while it may represent a positive number, a negative number, or zero.
(g) It should be read "the inverse of x."
(h) x + (-x) = 0 because the sum of any number and its additive inverse is zero.
(i) Yes, because (1)(x) = x.
(j) x + (-1)(x) = x [1 + (-1)]
(k) x + (-1)(x) = 0
(l) (-1)(x) must be equal to -x, because if x + (-1)(x) = 0, then (-1)(x) meets the definition of the additive inverse of x. Therefore, the product of any algebraic expression and -1 is equal to the inverse of the expression. This is a fundamental principle of algebra.

(50) Applying the principle that -x = (-1)(x), prove that 3x - x = 2x.

Answer: $3x - x = 3x + (-x)$
$= 3x + (-1)(x)$
$= [3 + (-1)]\,x$
$= 2x$

(51) Prove that ax - bx = (a - b)x.

Answer: $ax - bx = ax + (-b)(x)$
$= [a + (-b)]\,x$
$= (a - b)\,x$

(52) To what is -1(-x) equal? Prove your answer.

Answer: $(-1)(-x) = (-1)(-1)(x)$
$= 1x$
$= x$

(53) Complete the following exercises.
(a) Write an equation equivalent to $\frac{x}{-1} = \square$ which involves multiplication.
(b) To what must $\square$ be equal?

Answers:
(a) $-1(\square) = x$
(b) $\square = -x$ because $(-1)(-x) = x$; therefore, $\frac{x}{-1} = -x$.

(54) Applying the principle that the inverse of any algebraic expression is equivalent to the product of the expression and -1, determine expressions not containing parentheses which are equivalent to each of following.
(a) -(x + y)
(b) -(x - y)
(c) -(x + y - z)
(d) -(-6x + 7y)
(e) -(-5x - 4y)
(f) -(-2x - 7y + 3z)

Answers:

(a) $-(x + y) = (-1)(x + y)$
$= -x + (-y)$
$= -x - y$

(b) $-(x - y) = (-1)[x + (-y)]$
$= -x + y$

(c) $-(x + y - z) = (-1)[x + y + (-z)]$
$= -x + (-y) + z$
$= -x - y + z$

(d) $-(-6x + 7y) = (-1)(-6x + 7y)$
$= 6x + (-7)y$
$= 6x - 7y$

(e) $-(-5x - 4y) = (-1)[-5x + (-4y)]$
$= 5x + 4y$

(f) $-(-2x - 7y + 3z = (-1)[-2x + (-7y) + 3z]$
$= 2x + 7y + (-3z)$
$= 2x + 7y - 3z$

(55) Prove or disprove: $-(x - y) = y - x$.

Answer: $-(x - y) = (-1)[x + (-y)]$
$= -x + y$
$= y + (-x)$
$= y - x$

Therefore, the statement is true.

(56) Prove or disprove: $(-x)(y) = -(xy)$

Answer: $(-x)(y) = (-1)(x)(y)$
$= (-1)(xy)$
$= -(xy)$

Therefore, the statement is true.

(57) Prove or disprove: $(-x)(-y) = xy$.

Answer: $(-x)(-y) = (-1)(x)(-1)(y)$
$= (-1)(-1)(x)(y)$
$= xy$

Therefore, the statement is true.

(58) Match each expression in Column A with its equivalent expression in Column B by inserting in each blank in Column A the letter from Column B that precedes the matching expression.

Column A	Column B
____ 1. $3a + 7k$	A. $-(3a - 7k)$
____ 2. $-3a - 7k$	B. $-(3a + 7k)$
____ 3. $-3a + 7k$	C. $-(-3a - 7k)$
____ 4. $-7k - 3a$	
____ 5. $7k + 3a$	
____ 6. $7k - 3a$	

Answers: 1. C 2. B 3. A 4. B 5. C 6. A

(59) Prove or disprove: $-(3x) = -3x$.

Answer: $-(3x) = (-1)(3)(x)$
$= -3x$
Therefore, the statement is true.

Concept: Combining like terms.

(60) Complete the following exercises.
(a) Writing expressions such as $4x + 7y - 3x - 5y$ in simplest form is sometimes called combining like terms or collecting like terms. Write an expression equivalent to $4x + 7y - 3x - 5y$ which contains only the operation addition.
(b) Applying the commutative law of addition rearrange the terms in the expression on the right side of the equation so that like terms are adjacent to one another.
(c) Perform the addition indicated on the right side of the equation in (b).

Answers:
(a) $4x + 7y - 3x - 5y = 4x + 7y + [-(3x)] + [-(5z)]$
$= 4x + 7y + (-3x) + (-5y)$
(b) $4x + 7y - 3x - 5y = 4x + (-3x) + 7y + (-5y)$
(c) $4x + 7y - 3x - 5y = x + 2y$

(61) Find simpler equivalent expressions for each of the following by combining like terms.
(a) $17k - 15d - 9k - 4d$
(b) $a - 3b - a + 5$
(c) $4xy + 5ab - 7xy - 3ab + 9$

Answers:
(a) $8k - 19d$
(b) $-3b + 5$
(c) $-3xy + 2ab + 9$

(62) For each of the following, indicate whether or not each equation is an identity.
(a) $a - b -(a + b) = 0$
(b) $7x - (3x + z) - (2x - 2y) = 2x$
(c) $5(3x - 2y) - 2(2x + y) = 11x - 12y$

Answers:
(a) Not an identity
(b) Not an identity
(c) An identity

(63) Find the solution set of each of the following equations.
(a) $4 - (3x - 2) = 8 + (-4x + 2)$
(b) $6a - (-2a - 4) = 2a - 2(a + 6)$
(c) $-(3\triangle - 8) + 6\triangle = \triangle - (\triangle - 4) - (-2\triangle) - 7$

Answers:

(a)
$$4 - (3x - 2) = 8 + (-4x + 2)$$
$$4 + (-1)[3x + (-2)] = 8 + (-4x + 2)$$
$$4 + (-3x) + 2 = 8 + (-4x) + 2$$
$$6 + (-3x) = 10 + (-4x)$$
$$6 + (-3x) + 4x + (-6) = 10 + (-4x) + 4x + (-6)$$
$$x = 4$$

Solution set is $\{4\}$.

Check: $4 - (3\cdot4 - 2) \stackrel{?}{=} 8 + (-4\cdot4 + 2)$
$$4 - 10 \stackrel{?}{=} 8 + (-14)$$
$$-6 = -6 \text{ true}$$

(b)
$$6a - (-2a - 4) = 2a - 2(a + 6)$$
$$6a + (-1)[-2a + (-4)] = 2a + (-2)(a + 6)$$
$$6a + 2a + 4 = 2a + (-2a) + (-12)$$
$$8a + 4 = -12$$
$$8a + 4 + (-4) = -12 + (-4)$$
$$8a = -16$$
$$\frac{1}{8}(8a) = \frac{1}{8}(-16)$$
$$a = -2$$

Solution set is $\{-2\}$.

Check: $6(-2) - [-2(-2)-4] \stackrel{?}{=} 2(-2) - 2(-2 + 6)$
$$-12 - 0 \stackrel{?}{=} -4 + (-2)(4)$$
$$-12 = -4 + (-8) \text{ true}$$

(c)
$$-(3\triangle - 8) + 6\triangle = \triangle - (\triangle - 4) - (-2\triangle) - 7$$
$$(-1)[3\triangle + (-8)] + 6\triangle = \triangle + (-1)(\triangle - 4) + (-1)(-2\triangle) - 7$$
$$-3\triangle + 8 + 6\triangle = \triangle + (-\triangle) + 4 + (2\triangle) + (-7)$$
$$3\triangle + 8 = 2\triangle + (-3)$$
$$3\triangle + 8 + (-2\triangle) + (-8) = 2\triangle + (-3) + (-2\triangle) + (-8)$$
$$\triangle = -11$$

Solution set is $\{-11\}$.

Check: $-[3(-11) - 8] + 6(-11) \stackrel{?}{=} -11 - (-11 - 4) - [-2(-11)] - 7$
$$-(-41) - 66 \stackrel{?}{=} -11 - (-15) - 22 - 7$$
$$-25 = -25 \text{ true}$$

4.3 INEQUALITIES

<u>Concept</u>: Classification of inequalities.

(1) <u>What is an inequality</u>?

<u>Answer</u>: An inequality is a statement that relates one quantity to another quantity as indicated by any of the following symbols or their equivalents: $<$, $>$, $\leq$, $\geq$, $\neq$.

(2) Indicate how each of the following algebraic sentences is read.

(a) $a < b$ (c) $g \leq p$ (e) $a \neq b$
(b) $x > y$ (d) $k \geq m$

Answers:

(a) $a < b$ is read "a is less than b."
(b) $x > y$ is read "x is greater than y."
(c) $g \leq p$ is read "g is less than or equal to p."
(d) $k \geq m$ is read "k is greater than or equal to m."
(e) $a \neq b$ is read "a is not equal to b."

(3) What can you say about the truthfulness of the inequality $x + 3 < x$? Is it always true, sometimes true, or never true as x is replaced in turn by elements from the replacement set?

Answer: It is never true.

(4) An inequality that is never true is called an internally inconsistent inequality. Describe the solution set of such an inequality.

Answer: The solution set is the null set.

(5) Describe the truthfulness of the inequality $y - 3 < y$ when elements from the replacement set of real numbers are substituted in turn for y.

Answer: The inequality $y - 3 < y$ is always true regardless of what number is substituted for y.

(6) An inequality that is always true regardless of which element from the replacement set is substituted for the variable is called an absolute inequality. Describe the solution set of an absolute inequality by comparing it to the replacement set.

Answer: The solution set is identical to the replacement set of the variable.

(7) Describe the truthfulness of the inequality $k - 2 < 5$ when elements from the replacement set are substituted in turn for the variable.

Answer: If any number less than 7 is substituted for the variable k, the inequality is true. If the number 7 and any number greater than 7 is substituted for k, the inequality is false.

(8) An inequality such as $k - 2 < 5$ whose truthfulness depends upon what number is substituted for the variable is called a conditional inequality. Describe the solution set of a conditional inequality by comparing it to the replacement set.

Answer: The solution set is a proper subset of the replacement set. The solution set consists of at least one, but not all, of the elements in the replacement set.

(9) Classify each of the following as an absolute inequality, an internally inconsistent inequality, or a conditional inequality.

(a) $y - |y| \leq y$ (d) $x - 4 > -6$

(b) $g - 1 \neq 1 - g$ (e) $y + 3 > y$

(c) $x < x$

Answers:

(a) Absolute inequality

(b) Conditional inequality

(c) Internally inconsistent inequality

(d) Conditional inequality

(e) Absolute inequality

Concept: Properties of inequality.

(10) If $x < y$, is the difference $y - x$ positive, negative, or zero?

Answer: The difference $y - x$ is positive. $y - x > 0$.

(11) The definition of the inequality, less than, is that $x < y$ if and only if $y - x > 0$. Restate this definition in two sentences.

Answer: If $x < y$, then $y - x > 0$.
If $y - x > 0$, then $x < y$.

(12) Indicate whether each of the relations $\neq$, $<$, and $\leq$ is reflexive, symmetric, and/or transitive.

Answer:

Relation	Reflexive	Symmetric	Transitive
$\neq$	No	Yes	No
$<$	No	No	Yes
$\leq$	Yes	No	Yes

(13) Which of the following describes an absolute inequality?

(a) The solution set is the null set.

(b) The solution set is a proper subset of the replacement set.

(c) The solution is identical to the replacement set.

Answer: (c)

(14) If a pupil claims that the inequality $x + 16 > 5$ is an absolute inequality because when he substituted the numbers -8, -2, 0, 6, 9, and 12 each in turn for the variable, the inequality was a true statement each time. Was the pupil correct if the replacement set is the set of real numbers?

Answer: No, the pupil was not correct. When any number less than -11 is substituted for the variable, the inequality is false. $x + 16 > 5$ is not an absolute inequality because its solution set is not identical to the replacement set.

(15) <u>How can we show an inequality is not an absolute inequality</u>?

<u>Answer</u>: By showing there is at least one element in the replacement set which is not in the solution set.

(16) <u>What is the definition of the inequality</u> $a < b$?

<u>Answer</u>: $a < b$ if and only if $b - a > 0$.

(17) <u>If we can prove that</u> $k - m > 0$, <u>have we proven that</u> $m < k$?

<u>Answer</u>: Yes, according to the definition of this inequality

(18) <u>How can we prove that</u> $(x + z) < (y + z)$?

<u>Answer</u>: By proving $(y + z) - (x + z) > 0$.

(19) <u>To any inequality involving two numbers, such as</u> $10 < 15$, <u>add any number to both sides of the inequality. Is the resulting inequality true</u>?

<u>Answer</u>: Yes

(20) <u>To any inequality involving two numbers such as</u> $10 < 15$, <u>subtract any number from both sides of the inequality. Is the resulting inequality true</u>?

<u>Answer</u>: Yes

(21) <u>Does it appear that inequalities may have an additive property like the additive property for equations</u>?

<u>Answer</u>: It appears that this may be true.

(22) <u>If</u> $x < y$, <u>what can be said about</u> $y - x$?

<u>Answer</u>: If $x < y$, then $y - x > 0$.

(23) <u>Write an equivalent expression for</u> $(y + z) - (x + z)$ <u>that contains no parentheses</u>.

<u>Answer</u>: $(y + z) - (x + z) = y + z + (-x) + (-z)$
$= y - x$

(24) <u>If</u> $x < y$, <u>then</u> $y - x > 0$. <u>Substitute</u> $(y + z) - (x + z)$ <u>for the equivalent expression</u> $y - x$ <u>in the inequality</u> $y - x > 0$. <u>If</u> $x < y$, <u>what can be said about</u> $(y + z) - (x + z)$?

<u>Answer</u>: If $x < y$, then $(y + z) - (x + z) > 0$.

(25) <u>If</u> $x < y$, <u>then</u> $(y + z) - (x + z) > 0$. <u>If</u> $(y + z) - (x + z) > 0$, <u>what is the relation between</u> $(y + z)$ <u>and</u> $(x + z)$?

<u>Answer</u>: $x + z < y + z$.

(26) <u>Therefore, if</u> $x < y$, <u>what is the relation between</u> $(x + z)$ <u>and</u> $(y + z)$?

<u>Answer</u>: If $x < y$, then $x + z < y + z$.

(27) <u>Summarize the proof that if</u> $x < y$ <u>then</u> $x + z < y + z$.

<u>Answer</u>: If $x < y$, then $y - x > 0$
$(y + z) - (x + z) = y - x$
If $y - x > 0$ then $(y + z) - (x + z) > 0$
If $(y + z) - (x + z) > 0$ then $x + z < y + z$
Therefore, if $x < y$, then $x + z < y + z$.

(28) <u>Prove that if</u> $x + z < y + z$, <u>then</u> $x < y$.

<u>Answer</u>: If $x + z < y + z$ then $(y + z) - (x + z) > 0$.
$(y + z) - (x + z) = y - x$
If $(y + z) - (x + z) > 0$, then $y - x > 0$
If $y - x > 0$, then $x < y$.

(29) <u>The statement "</u>$x < y$ <u>if and only if</u> $x + z < y + z$<u>" is the additive property of inequalities. Restate the principle in two sentences.</u>

<u>Answer</u>: If $x < y$ then $x + z < y + z$.
If $x + z < y + z$, then $x < y$.

(30) <u>Restate the additive property of inequalities in a word sentence.</u>

<u>Answer</u>: If any number is added to or subtracted from both sides of an inequality the solution set of the resulting inequality is the same as that of the original inequality.

(31) <u>Applying the additive property of inequalities, find the solution sets of each of the following. The solution set may be expressed in set builder notation such as</u> $\{x|x < 5\}$.

(a) $x + 3 < -11$ (b) $y - 9 > -6$ (c) $2b - 5 \leq b + 3$

<u>Answers</u>:

(a)
$x + 3 < -11$
$x + 3 + (-3) < -11 + (-3)$
$x < -14$

Solution set is $\{x|x < -14\}$.

(b)
$y - 9 > -6$
$y + (-9) + 9 > -6 + 9$
$y > 3$

Solution set is $\{y|y > 3\}$.

(c) $2b - 5 \leq b + 3$
$2b + (-5) + 5 + (-b) \leq b + 3 + (-b) + 5$
$b \leq 8$
Solution set is $\{b | b \leq 8\}$.

Concept: If $x < y$, and if $z > 0$, then $xz < yz$.

(32) Multiply both sides of the inequality $5 < 7$ by 6. Is the new inequality true?

Answer: $5 \cdot 6 < 7 \cdot 6$
$30 < 42$
The new inequality is true.

(33) Multiply both sides of the inequality $5 < 7$ by zero. Is the new inequality true?

Answer: $5 \cdot 0 < 7 \cdot 0$
$0 < 0$
The new inequality is not true.

(34) Answer the following questions.
(a) Can both sides of an inequality involving the relation, less than, be multiplied by zero without creating a false statement of inequality.
(b) What other types of inequalities have the same property?

Answers:
(a) No. When both sides of any inequality involving the relation, less than, are multiplied by zero, the new inequality formed becomes $0 < 0$, which is false.
(b) When both sides of any inequality involving the relations, "is greater than" or "is not equal to" are multiplied by zero, the resulting relation is false.

(35) Multiply both sides of the inequality $5 < 7$ by -3. Is the new inequality true?

Answer: $(5)(-3) < (7)(-3)$
$-15 < -21$
The new inequality is false.

(36) When both sides of an inequality involving "is less than" or "is greater than" are multiplied by a negative number, is the new inequality true?

Answer: No

(37) If $x < y$, then $y - x > 0$. If $z > 0$, then must $(y - x)z$ be positive or negative? Why?

Answer: If $z > 0$ and $y - x > 0$, then $(y - x)z$ must be positive because the product of two positive numbers is always a positive number.

(38) If $x < y$ what can be said about $(y - x)z$?

Answer: If $x < y$, then $(y - x)z > 0$ for all $z > 0$.

(39) **Write an expression equivalent to $(y - x)z$.**

Answer: $yz - xz$

(40) **If $x < y$, what can be said about $yz - xz$?**

Answer: If $x < y$, then $yz - xz > 0$ for all $z > 0$.

(41) **If $yz - xz > 0$, what is the relation between xz and yz?**

Answer: $xz < yz$.

(42) **If $x < y$ and $z > 0$, what is the relation between xz and yz?**

Answer: If $x < y$ and $z > 0$, then $xz < yz$.

(43) **Write a summary of the proof that if $x < y$ and $z > 0$, then $xz < yz$.**

Answer: If $x < y$ then $y - x > 0$
If $z > 0$ and $y - x > 0$ then $(y - x)z > 0$
If $x < y$ and $z > 0$, then $yz - xz > 0$
If $yz - xz > 0$, then $xz < yz$
Therefore, if $x < y$ and $z > 0$, then $xz < yz$.

(44) **Prove that if $xz < yz$ and $z > 0$, then $x < y$.**

Answer: If $xz < yz$, then $yz - xz > 0$.
If $yz - xz > 0$ then $(y - x)z > 0$
If $(y - x)z > 0$ and $z > 0$, then $y - x > 0$
If $y - x > 0$, then $x < y$.
Therefore, if $xz < yz$ and $z > 0$, then $x < y$.

(45) **If $z > 0$, then $x < y$ if and only if $xz < yz$. This is one part of the multiplicative property of inequalities. Restate this property in two sentences.**

Answer: If $z > 0$ and $x < y$, then $xz < yz$.
If $z > 0$ and $xz < yz$, then $x < y$.

(46) **Restate this part of the multiplicative property of inequalities in a word sentence.**

Answer: If both sides of an inequality are multiplied by or divided by any positive number, the solution set of the resulting inequality is the same as the solution set of the original inequality.

(47) **Using this part of the multiplicative property of inequalities, find the solution set of each of the following.**

(a) $6x > -12$ (b) $\frac{y}{5} < y - 9$ (c) $7b - 5 \leq 3b + 8$

Answers:
(a) $\{x|x > -2\}$ (b) $\{y|y > \frac{45}{4}\}$ (c) $\{b|b \leq \frac{13}{4}\}$

Concept: If $x < y$ and $z < 0$, then $xz > yz$.

(48) Multiply both sides of the inequality $-4 < 0$ by -5. Multiply both sides of the inequality $10 > 9$ by -3. Are the new inequalities true?

Answer: No, the new inequalities are false.

(49) Answer the following:
(a) If we multiply both sides of any true inequality involving "is greater than" or "is less than," such as $5 < 6$, by a negative number, is the resulting inequality true?
(b) What change must be made in the resulting inequality to make it a true statement?

Answers:
(a) No, the resulting inequality is false.
(b) The inequality sign must be reversed. The " < " sign must be changed to " > " and the " > " sign changed to " < ".

(50) If $x < y$, then $y - x > 0$. If $z < 0$, must $(y - x)z$ be positive or negative? Why?

Answer: If $z < 0$ and $y - x > 0$ then $(y - x)z$ must be negative because the product of a positive number and a negative number is always a negative number.

(51) If $x < y$ and $z < 0$ then $(y - x)z < 0$. Write an equivalent statement that does not contain parentheses.

Answer: If $x < y$ and $z < 0$ then $yz - xz < 0$.

(52) If $yz - xz < 0$ is the expression $-(yz - xz)$ positive or negative?

Answer: If $yz - xz$ is negative, its opposite or inverse $-(yz - xz)$ is positive. $-(yz - xz) > 0$.

(53) If $yz - xz < 0$ then $-(yz - xz) > 0$. Write an equivalent and simpler expression for $-(yz - xz) > 0$.

Answer: $-(yz - xz) = xz - yz$.
Therefore, if $yz - xz < 0$ then $xz - yz > 0$.

(54) If $yz - xz < 0$ then $xz - yz > 0$. What is the relation between xz and yz?

Answer: If $xz - yz > 0$, then $yz < xz$ or $xz > yz$.

(55) If $x < y$ and $z < 0$, what is the relation between xz and yz?

Answer: If $x < y$ and $z < 0$, then $xz > yz$.

(56) <u>Write a summary of the proof that if x < y and z < 0 then xz > yz.</u>

<u>Answer</u>: If $x < y$ then $y - x > 0$.
If $z < 0$ and $y - x > 0$ then $(y - x)z < 0$
$(y - x)z = yz - xz$
If $x < y$ and $z < 0$ then $yz - xz < 0$.
If $yz - xz < 0$ then $-(yz - xz) > 0$
$-(yz - xz) = xz - yz$
If $x < y$ and $z < 0$ then $xz - yz > 0$
If $xz - yz > 0$ then $yz < xz$
If $yz < xz$ then $xz > yz$
If $x < y$ and $z < 0$ then $xz > yz$.

(57) <u>Prove that if xz > yz and z < 0 then x < y.</u>

<u>Answer</u>: If $xz > yz$ then $yz < xz$
If $yz < xz$ then $xz - yz > 0$
If $xz - yz > 0$ then $-(xz - yz) < 0$
$-(xz - yz) = yz - xz$
$yz - xz = (y - x)z$
If $xz - yz > 0$ then $(y - x)z < 0$
If $z < 0$ and $(y - x)z < 0$ then $(y - x) > 0$
If $(y - x) > 0$ then $x < y$
Therefore, if $xz > yz$ and $z < 0$ then $x < y$.

(58) <u>If z < 0 then x < y if and only if xz > yz. This is the second part of the multiplicative property of inequalities. Restate this property in two sentences.</u>

<u>Answer</u>: If $z < 0$ and $x < y$ then $xz > yz$.
If $z < 0$ and $xz > yz$ then $x < y$.

(59) <u>Restate the above part of the multiplication property of inequalities in a word sentence.</u>

<u>Answer</u>: If both sides of an inequality involving "is greater than" or "is less than" are multiplied or divided by any negative number and if the inequality relation is reversed in the resulting inequality, then the solution set of the resulting inequality will be the same as the solution set of the original inequality.

(60) <u>Find the solution set of each of the following inequalities.</u>
(a) $\frac{x}{-3} < -4$ (b) $-7w > 0$ (c) $\frac{3c - 1}{-3} \leq 2c + 3$

<u>Answers</u>:
(a) $\{x | x > 12\}$
(b) $\{w | w < 0\}$
(c) $\{c | c \geq -\frac{8}{9}\}$

<u>Concept</u>: Solution of inequalities containing absolute value.

(61) Write two equations not containing absolute value that are equivalent to the equation $|a| = 8$.

Answer: $a = 8$ and $a = -8$

(62) If in the equation $|x| = c$, x is a variable and c is a given number, what will be the solution set of the equation if c is positive? If c is zero? If c is negative?

Answer: If c is a positive number, the variable x will equal c and the inverse of c, and the solution set is $\{c, -c\}$. If c is zero, the solution set is $\{0\}$. If c is negative, the solution set is $\emptyset$.

(63) Consider the inequality $|t| > 1$. Which elements in the replacement set $\{5,4,3,2,1,0,-1,-2,-3,-4,-5\}$ are elements in the solution set of this inequality?

Answer: The solution set is $\{5,4,3,2,-2,-3,-4,-5\}$.

(64) Write a statement not containing absolute value that is equivalent to $|y| > 7$.

Answer: $y > 7$ or $y < -7$. The solution set consists of any number greater than seven or less than -7.

(65) Describe the solution set of $|y| > k$ in terms of intersection or union of two sets if y represents a variable and k is a given positive number.

Answer: The solution set of $|y| > k$ is the union of the set $\{y|y > k\}$ and the set $\{y|y < -k\}$, where y is the variable and k is a given positive number.

(66) Find the solution set of each of the following.
(a) $|\triangle| > 4$ (b) $|v + 2| > 5$ (c) $|7g| > 10$

Answers:
(a) $\{\triangle|\triangle > 4 \text{ or } \triangle < -4\}$
(b) $v + 2 > 5$ or $v + 2 < -5$
$v > 3$ $\quad v < -7$
Solution set is $\{v|v > 3 \text{ or } v < -7\}$.
(c) $7g > 10$ or $7g < -10$
$g > \frac{10}{7}$ $\quad g < -\frac{10}{7}$
Solution set is $\left\{g|g > \frac{10}{7} \text{ or } g < -\frac{10}{7}\right\}$.

(67) Consider the inequality $|t| < 4$. Which elements in the replacement set $\{5,4,3,2,1,0,-1,-2,-3,-4,-5\}$ are elements in the solution set of this inequality?

Answer: $\{3,2,1,0,-1,-2,-3\}$

(68) **Write a statement containing two inequalities with no absolute value symbols that is equivalent to $|x| < 4$.**

Answer: $x < 4$ and $x > -4$. The solution set consists of any number less than 4 and greater than -4.

(69) **Describe the solution set of $|x| < k$ in terms of intersection or union of two sets if x represents a variable and k represents a positive number.**

Answer: The solution set of $|x| < k$ where k is a positive number is the intersection of the set $\{x|x < k\}$ and the set $\{x|x > -k\}$. The solution set is $\{x|-k < x < k\}$.

(70) **Find the solution set of each of the following.**

(a) $|\triangle| < 4$ (c) $|7n| < 10$

(b) $|v + 2| < 5$ (d) $|19w| < 0$

Answers:

(a) $\{\triangle|-4 < \triangle < 4\}$

(b) $v + 2 < 5$ and $v + 2 > -5$

$v < 3$ $v > -7$

Solution set is $\{v|-7 < v < 3\}$.

(c) $7n < 10$ and $7n > -10$

$n < \frac{10}{7}$ $n > -\frac{10}{7}$

Solution set is $\left\{n \middle| -\frac{10}{7} < n < \frac{10}{7}\right\}$.

(d) $\emptyset$. There is no expression whose absolute value is a negative number.

Teacher Notes