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TEACHING COLLEGE STUDENTS HOW TO LEARN MATHEMATICS. WHAT IS LEARNED IN MATHEMATICAL DISCOVERY.

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THIS STUDY RELATED TO DISCOVERY METHODS OF TEACHING AND LEARNING WAS CONCERNED WITH TWO MAJOR QUESTIONS. FIRST, CAN "WHAT IS LEARNED" THROUGH MATHEMATICAL DISCOVERY BE IDENTIFIED AND TAUGHT BY EXPOSITION WITH EQUIVALENT RESULTS. SECOND, HOW DOES "WHAT IS LEARNED" DEPEND ON PRIOR LEARNING AND ON THE NATURE OF THE DISCOVERY TREATMENT ITSELF. IN A PREVIOUS STUDY, GAGNE AND BROWN FOUND THAT DISCOVERY GROUPS WERE BETTER ABLE TO DERIVE NEW FORMULAS THAN WERE RULE-GIVEN GROUPS. IN THE PRESENT STUDY IT WAS HYPOTHESIZED THAT (1) WHAT WAS LEARNED BY GUIDED DISCOVERY IN THE GAGNE AND BROWN STUDY CAN BE PRESENTED BY EXPOSITION WITH EQUIVALENT RESULTS, (2) PRESENTATION ORDER IS CRITICAL WHEN THE HINTS PROVIDED DURING DISCOVERY ARE SPECIFIC TO THE RESPECTIVE FORMULAS SOUGHT RATHER THAN RELEVANT TO A GENERAL STRATEGY, AND (3) PRESENTATION ORDER IS NOT CRITICAL WHEN THE PROGRAM EFFECTIVELY FORCES THE STUDENT TO LEARN THE GENERAL STRATEGY IRRESPECTIVE OF THE EXPOSITION OR DISCOVERY LEARNING METHOD. ONE OR TWO OF FOUR PROGRAMS--RULE GIVEN (R), DISCOVERY (D), GUIDED DISCOVERY (G), AND HIGHER-ORDER EXPOSITION (E)--WERE ADMINISTERED TO SEVEN GROUPS--R, RD, DR, RG, GR, RE, AND ER. ALL STUDENTS WERE REQUIRED TO DERIVE NEW RULES WITHIN THE SCOPE OF THE IDENTIFIED HIGHER-ORDER RULE. AS HYPOTHESIZED, GROUPS R AND RD PERFORMED AT ONE LEVEL WHICH WAS RELIABLY BELOW THE COMMON LEVEL OF THE OTHER FIVE GROUPS. TWO POINTS OF EMPHASIS IN THE CONCLUSION AND IMPLICATIONS WERE (1) "WHAT IS LEARNED" DURING GUIDED DISCOVERY CAN AT LEAST BE IDENTIFIED AND TAUGHT BY EXPOSITION WITH EQUIVALENT RESULTS, AND (2) IF A PERSON ALREADY KNOWS THE DESIRED RESPONSES, HE IS NOT LIKELY TO DISCOVER A HIGHER ORDER RULE BY WHICH SUCH RESPONSES MAY BE DERIVED. (RP)

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TEACHING COLLEGE STUDENTS HOW TO LEARN MATHEMATICS ("WHAT IS LEARNED" IN MATHEMATICAL DISCOVERY)

Project No. OEC 1-7-068798-0360

Joseph M. Scandura
University of Pennsylvania
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We are indebted to Robert M. Gagne and Gabriel Della Piana for making copies of their experimental materials available to us. We would also like to thank Joan Bracker and John Durnin for their general assistance and Joanna P. Williams for her comments on a draft of this paper. The participation of Bracker and Durnin was made possible by a U. S. Office of Education Graduate Training Grant to the University of Pennsylvania in Mathematics Education Research.

ABSTRACT

"What is Learned" in Mathematical Discovery

Joseph M. Scandura
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The concern was twofold: (1) can what is learned in mathematical discovery be identified and taught by exposition with equivalent results and (2) how does "what is learned" depend on prior learning and on the nature of discovery. It was hypothesized that discovery Ss may discover higher-order rules for deriving rules. Four programs, rule-given (R), discovery (D), guided discovery (G), and higher-order exposition (E) were administered to seven groups: R, RD, DR, RG, GR, RE, ER. All Ss were required to derive new rules within the scope of the identified higher-order rule. As hypothesized, groups R and RD performed at one level which was reliably ($p < .001$) below the common level of the other five groups. Theoretical and practical implications were discussed.

Teaching College Students How to Learn Mathematics

("What is Learned" in Mathematical Discovery[1])

Joseph M. Scandura — University of Pennsylvania
and
William G. Roughead — North Georgia College

One of the fundamental assumptions underlying many of the new mathematics curricula is that discovery methods of teaching and learning increase the students' ability to learn new content (e.g., Beberman, 1958; Davis, 1960; Peak, 1963). The last decade of research on discovery learning, however, has produced only partial and tentative support for this contention. Even where the experiments have been relatively free of methodological defects, the results have often been inconsistent (e.g., see Ausubel, 1961; Kersh & Wittrock, 1962). More particularly, the interpretation of research on discovery learning has been made difficult by differences in terminology, the tendency to compare identical groups on a variety of dependent measures, and vagueness as to what is being taught and discovered.

While most discrepancies due to differences in terminology can be reconciled by a careful analysis of what was actually

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[1]This paper is based on a Ph. D. dissertation submitted to the Florida State University by the second author under the chairmanship of the first author. The second author was primarily responsible for the conducting of the experiment and the analysis of the data. The first author was primarily responsible for formulating the problem and for the preparation of this report. This research was supported, in part, by a U. S. Office of Education grant to the first author.

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done in the experiments (e.g., Kersh & Wittrock, 1962) and thus present a relatively minor problem, the failure to equate original learning has often made it difficult to interpret transfer (and retention) results in an unambiguous manner. Thus, several studies (e.g., Craig, 1956; Wittrock, 1963) have shown that rule-given groups perform better on "near" transfer tests than do discovery groups. The obtained differences. however, may have been due to the fact that the discovery groups did not learn the originally presented materials as well as the rule-given groups.

When the degree of original learning was equated, Gagne and Brown (1961) found that their discovery groups were better able to derive new formulas than were their rule (i.e., formula)-given groups. They attributed this result to differences in "what was learned" but added that they were unable to specify precisely what these differences were. On the basis of an analysis of the experimental programs used by Gagne and Brown (1961), Eldredge (in Della-Piana, Eldredge, & Worthen, 1965) hypothesized that the differences found by Gagne and Brown (1961) were due to uncontrolled factors. Eldredge conjectured that if the treatment differences were limited to the order of presentation of the discovery hints and the to-be-learned formulas, no differences in transfer ability would result. However, Eldredge's results contradicted his hypothesis. In subsequent studies, Guthrie (1967) and Worthen (1967) obtained similar sequence effects.

Using the Set-Function Language (SFL)[2] as a guide, Scandura (1966) proposed an analysis of discovery learning that seems to

[2] In the SFL, the principle or rule, rather than the association, is viewed as the basic unit of behavior. In fact, it has

be in accord with experimental findings. The main point was that in order to succeed, discovery Ss must learn to derive solutions whereas solution-given Ss need not. In attaining criterion, discovery Ss may discover a derivation rule by which solutions to new, though related, problems may be derived. Under these circumstances, discovery Ss would be expected to perform better than expository Ss on tasks which are within the scope of such a derivation rule. If the new problems presented have solutions beyond the scope of a discovered derivation rule, however, there would be no reason to expect discovery Ss to have any special advantage.[3]

This study was concerned with two major questions. First, can "what is learned" in mathematical discovery be identified and, if so, can it be taught by exposition with equivalent results? Second, how does "what is learned" depend on prior learning and on the nature of the discovery treatment itself?

The SFL was used as an aid in analyzing the guided discovery programs used by Gagne and Brown (1961) and Eldredge (Della-Piana, Eldredge, and Worthen, 1965) to determine "what was learned." As a result of this analysis, we were able to devise an expository

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been shown that both associations and concepts may quite naturally be viewed as special cases of the rule. The denotation of a rule was defined as a set of functionally distinct stimulus-response pairs--the instances of the rule. The rule itself (i.e., the underlying unit of knowledge) was characterized as an ordered triple (D, O, R) where D refers to those stimulus properties which determine the corresponding responses, and O refers to the operation or transformation by which the derived stimulus properties or (internal) responses in the set R are derived from the properties in D. For more details, the reader is referred elsewhere (Scandura, 1966, 1967a, 1967b).

[3]In this discussion, the terms "solution" and "derivation rule" may be replaced by the more general terms, "response" and "rule," respectively.

statement of the derivation rule. In the manner described by Scandura, Woodward, and Lee (1967), we were also able to determine, on an *a priori* basis, which kinds of transfer item could be solved by using this derivation rule and which could not.

Assuming that transfer depends only on whether or not the derivation rule is learned, the order in which the formulas (i.e., the solutions) and the derivation rule are presented should have no effect on transfer provided *S* actually learns the derivation rule. If, on the other hand, a discovery program simply provides an *opportunity* to discover and does not guide the learner through the derivation procedure, sequence of presentation might have a large effect on transfer. That is, if a capable and motivated subject is given appropriate hints, he might well succeed in discovering the appropriate formulas and in the process discover the derivation rule. It is not likely, however, that he would exert much effort when given an opportunity to discover a formula he already knew. Something analogous may well have been involved in the studies by Eldredge (Della-Piana *et al*, 1965), Guthrie (1967), and Worthen(1967).

In particular, the following hypotheses were made. First, what was learned by guided discovery in the Gagne and Brown (1961) study can be presented by exposition with equivalent results. Second, presentation order is critical when the hints provided during discovery are *specific* to the respective formulas sought rather than relevant to a general strategy (i.e., a derivation rule). Third, presentation order is *not* critical when the program effectively forces *S* to learn the derivation rule, regardless of whether the learning takes place by exposition or by discovery.

METHOD

Materials.-[4] There were seven treatments. Each consisted of a common introductory program followed by various combinations of four basic instructional programs. The introductory program was designed to generally familiarize the Ss with number sequences and with the terminology used in the four basic programs. In particular, four concepts were clarified: sequence; term value, T_n; term number, n; and sum of the first n terms of a sequence, $\sum^n$.

Each of the four basic instructional programs was based on the same three arithmetic series and their respective summing formulas: $1 + 3 + 5 + \ldots + (2n-1) \longrightarrow n^2$; $2 + 6 + 10 + \ldots + (4n-2) \longrightarrow 2n^2$; $1 + 5 + 9 + \ldots + (4n-3) \longrightarrow (2n-1)n$. Following Gagne and Brown (1961), each series was presented as a three-row display--e.g.,

Term number	n :	1	2	3	4	...
Term Value	T_n:	2	6	10	14	...
Sum	$\sum^n$:	2	8	18	32	...

The rule and example (R) program consisted of the three series displays together with the respective summing formulas. The presentation of each summing formula was followed by three application problems--e.g., find the sum of 2 + 6 + 10 ($=2 \cdot 3^2 = 18$). S was also required to write out each formula in both words and symbols, but no rationale for the formula was provided.

The three other basic programs included differing kinds of directions and/or hints as to how the summing formulas might be determined. The expository (E) and highly guided discovery (G) programs were based on a simplified variant of that derivation

[4]Copies of the experimental materials used are included in Roughead's (1966) dissertation and in Scandura's (1967b) final report.

rule presumably learned by the guided discovery Ss in the Gagne and Brown (1961) study. The identified derivation rule can be stated,

> "... formulas for $\sum^n$ may be written as the product of an expression involving n (i.e., f(n)) and n itself. The required expression in n can be obtained by constructing a three columned table showing: (1) the first few sums $\sum^n$, (2) the corresponding values of n, and (3) a column of numbers $f(n) = \sum^n/n$ which when multiplied by n yields the corresponding values of $\sum^n$. Next, determine the expression $f(n) = \sum^n/n$ by comparing the numbers in the columns labeled n and $\sum^n/n$ and uncovering the (linear) relationship between them. The required formula is simply $\sum^n = n \cdot f(n)$."

As an example, consider the display,

Term number	n	:	1	2	3	4	...
Term value	T_n	:	2	6	10	14	...
Sum	$\sum^n$	:	2	8	18	32	...

The three-columned table would look like,

f(n)	n	$\sum^n$
2	1	2
4	2	8
6	3	18
8	4	32
.	.	.
.	.	.
.	.	.
2n	n	$2n^2$

The emerging pattern is f(n) = 2n; so, $\sum^n = 2n \cdot n = 2n^2$.

The E program consisted of a simplified statement of the derivation rule as it applied to each of the three training series. To insure that S learned how to use the derivation rule, a vanishing procedure was used which ultimately required *S* to apply the procedure without any instructions. The G program paralleled the E program in all respects. The only difference was that the G program consisted of questions whereas the E program consisted of yoked direct statements, each followed by a parallel question or completion statement to see whether *S* had read the original statement correctly. For example, the E statement, "When $n = 3$, you can multiply 6 times n to get $\sum^3 = 18$. What times n gives $\sum^3 = 18$?" corresponded to the question, "When $n = 3$, what times n gives $\sum^3 = 18$?" which appeared in the G program. Since the degree of overt responding was held constant, the only difference between the E and G programs was whether the information was acquired by reception or by reacting to a question (i.e., by discovery). The discovery (D) program, on the other hand, simply provided *S* with an opportunity to discover the respective summing formulas. *S* was guided by questions and hints which were specific to the formulas involved (e.g., "the formula has a 2 in it") rather than relevant to any general strategy or derivation rule. The questions and hints were interspersed with liberal amounts of encouragement (e.g., "Good try," "You can do it," etc.) to provide motivation.

There were two transfer tests. The within-scope transfer test consisted of two new series displays which could be solved by the identified derivation rule. These series and their respective summing formulas were $3 + 5 + 7 + \ldots + (2n + 1) \rightarrow (n + 2) \cdot n$ and $4 + 10 + 16 + \ldots + (6n - 2) \rightarrow (2n + 1) \cdot n$. The extra-scope transfer test involved the series, $2 + 4 + 8 + \ldots + 2^n \rightarrow (2T_n - 2) = (T_n + 1 - 2)$ and $1/2 + 1/6 + 1/12 + \ldots + [1/n(n + 1)] \rightarrow n/(n + 1) = n^2 T_n$, which, strictly speaking, were beyond the scope of the identified derivation rule.[5] A series of hints, paralleling those used in the D program, were constructed to accompany each test series.

The introductory and treatment programs were mimeographed and stapled together into separate 5 1/2" x 8 1/2" booklets. The four transfer series were presented on separate pages in a test booklet in the same three-row form used in the learning programs. The hints were put on 5" x 7" cards, bound by metal rings.

Subjects, Design, and Procedure.-- The naive Ss were 105 (103 females) junior and senior elementary education majors enrolled in required mathematics education courses at the Florida State University. Participation was a course requirement.

The Ss were randomly assigned to the seven treatment groups. In addition to the common introductory program, the rule-given treatment group (R) received only the R program. The other six treatment groups received the R program together with one of the other three basic instructional programs. The RE, RG, and RD groups, received the R program followed by the E (expository), G (guided discovery), and D (discovery) programs, respectively, while the ER, GR, and DR groups received these same respective programs in the reverse order.

The Ss were scheduled to come to the experimental room in groups of four or less and were arranged at the ends of two tables which were partitioned to provide separate study carrels. A brief quiz was used to screen out any Ss who were already familiar with

[5]The derivation rule, however, was potentially generalizable to the extra-scope series.

number series and/or formulas for summing them. Then, they were told,

"This is an experiment in learning mathematics. You will be given two programmed booklets to study. You are expected to try to learn. You should work at a good pace, but read everything for understanding.... If you have an error, don't change your answer, but write the correct answer under your original answer. If you cannot respond to a question within a minute or so, put an "X" in the blank and continue. You should, of course, look back at the question after finding the answer to be sure you understand...."

The Ss worked at their own rate. E recorded the times taken on the introductory and treatment booklets.

As soon as all of the Ss in the testing group had completed the treatment programs, they were told to review for a test. After two minutes, the booklets were collected and the tests and hint cards were presented. The Ss were instructed,

"On this test you will be timed. You also will be provided with hints to aid you when necessary. The less time it takes you and the fewer hints you need on a given problem, the better your score. You will be asked to find the formula for four new problems on this test. On each problem, you will have 5 minutes to find the correct summing formula. You should show any necessary work in your booklet. When you get an answer, raise your right hand immediately. Like this! Try it!... I'll tell you whether you are correct or incorrect. If incorrect, continue searching for the answer. Be sure to show me your answer

quickly so that you get the best possible time score....
When I tell you that the 5 minutes are up, if you have not found the formula, you may begin using the hints. You may use as many of the hints as you wish, and when you wish, after the 5 minute period. But remember, the fewer hints you use, the better your score."

Before continuing on to the second problem, each S read all of the hint cards pertaining to the first problem. The four Ss in each testing group began each problem at the same time. If an S solved a problem before the others, he was allowed to read the rest of the hints for that problem and, then, was required to wait for the others to finish. Before being released, the Ss were asked not to discuss particulars of the experiment with others who might participate.

Three indices of performance on the transfer tasks were obtained: (1) time to solution, (2) number of hints prior to solution, and (3) a weighted score similar to that used by Gagne and Brown (1961). The weighted score was equal to the time to solution in minutes plus a penalty of 4, 7, 9, or 10 depending on whether S used 1, 2, 3, or 4 hints, respectively. Theoretically, a range of scores from 0 to 20 was possible on this measure. Standard analysis of variance procedures were used to analyze the data after Cochran's C test failed to detect heterogeneity of variance.

RESULTS

Treatment Programs.-- All treatment groups performed at essentially the same level on the introductory program, both in terms of time to completion ($F(6, 98) = 1.74$, $p > .05$) and number of errors ($F(6, 98) = 1.35$, $p > .05$). Since the number of frames varied among the treatment programs, no overall comparisons were warranted.[6]

Performance on Learning and Transfer Tests.-- The results on the within-scope transfer test conformed to prediction. Irrespective of the transfer measure used, the group (R) given the formula program only and the group (RD) given the formula program followed by the opportunity to discover program performed at one level ($F(1, 28) < 1$) while the other five groups performed at a common ($F(4,70) < 1$) and significantly higher level ($F_{time}(1, 98) = 32.66$, $p < .001$; $F_{hints}(1, 98) = 54.52$, $p < .001$; $F_{weighted}(1, 98) = 57.99$, $p < .001$). In particular, only that sequence effect involving groups RD and DR was significant ($p < .01$).

While there were no overall treatment differences on the extra-scope transfer test (maximum $F(6, 98) = 1.31$, $p > .05$), the contrast between groups R and RD and groups DR, RG, GR, RE, and ER attained a borderline significance level ($F_{time}(1, 98) = 3.66$, $.05 < p < .10$; $F_{hints}(1, 98) = 4.02$, $p < .05$; $F_{weighted}(1, 98) = 4.61$, $p < .05$).[7] There were, however no reliable performance differences

[6] Still, it is interesting to note that those groups which received the R program first, in each case, spent less time on the learning program than did the corresponding groups who received the R program last. The differences, however, were not reliable at the .05 level (see table 1).

[7] In a study on rule generality, Scandura et al (1967) obtained a similar extra-scope transfer effect. While no extra-scope transfer was almost universally the case, one of the rules (i.e., "50 x 50") introduced was apparently generalized (to "n x n") and thereby provided an adequate basis for solving an extra-scope item.

While not sufficient as presented, potentially, the derivation rule introduced in this study could also be generalized. In fact, the first hint available on item 3 provided a basis for making appropriate modifications in the derivation rule so that transfer to this item was possible (but less likely than on the within-scope test). Similarly, although item 4 involved fractional term values,

Table 1

Summary of means and standard deviations of time and errors on introductory and treatment programs; items correct on the test of criterion learning; and time, hints and weighted scores on within and extra-scope transfer items.

Program	R $\bar{x}$	R s.d.	RD $\bar{x}$	RD s.d.	DR $\bar{x}$	DR s.d.	RG $\bar{x}$	RG s.d.	GR $\bar{x}$	GR s.d.	RE $\bar{x}$	RE s.d.	ER $\bar{x}$	ER s.d.
Introductory														
Time	8.93	2.20	9.00	2.71	7.13	1.59	7.73	2.38	8.73	2.93	7.27	1.00	8.13	2.09
Errors	1.20	.83	.80	1.04	.60	.80	.60	.61	.60	.71	.47	.88	.60	1.02
Treatment														
Time	23.07	7.42	29.33	8.70	33.00	7.70	36.40	10.29	41.47	8.30	35.87	8.70	40.20	8.60
Errors	1.47	1.86	1.20	1.04	1.53	2.22	2.80	2.51	2.67	2.87	1.73	1.65	2.20	1.48
Learning test														
Items correct	5.53	1.09	5.47	1.06	5.47	1.15	5.67	1.01	5.53	.81	6.00	0.00	5.73	.68
Within-Scope														
Time	6.48	1.37	5.98	1.26	4.36	1.78	4.00	1.37	4.21	2.00	3.97	1.63	3.61	1.68
Hints	2.00	.79	1.73	.76	.80	.91	.50	.69	.60	.86	.64	.65	.50	.66
Weighted	12.87	3.56	11.65	3.30	6.97	4.31	5.73	3.40	6.27	4.68	6.20	3.82	5.37	3.91
Extra-Scope														
Time	6.82	1.69	5.91	1.48	5.51	1.71	5.70	1.20	5.51	2.07	5.88	1.35	5.76	1.72
Hints	1.84	1.00	1.60	.88	1.20	.85	1.24	.73	1.37	1.11	1.44	.73	1.40	.71
Weighted	12.22	4.09	11.01	3.77	9.60	3.83	9.63	3.12	9.71	4.87	10.14	2.93	10.03	3.10

between: (1) the guided discovery (RG and GR) groups and the exposition (RE and ER) groups ($F<1$), (2) those guided discovery and exposition groups (RG and RE) given the formulas first and those groups (GR and ER) given the formulas last ($F<1$), (3) the opportunity to discover--formula-given (DR) group and the four guided discovery and exposition groups ($F<1$), or, most critically, (4) between groups DR and RD ($F<1$).

These transfer effects can not be attributed to differences in original learning. A learning test embedded within the common R program, indicated that the Ss had well-learned the appropriate summing formulas to the three training series before they took the transfer tests. The group means ranged from 5.5 to 6.0 with a possible maximum of 6.0 and minimum of 0.0.

DISCUSSION AND IMPLICATIONS

Two points need to be emphasized. First, "what is learned" during guided discovery can at least sometimes be identified and taught by exposition--with equivalent results. While this conclusion may appear somewhat surprising at first glance, further reflection indicates that we have always known it to be at least partially true. As has been documented in the laboratory (e.g., Kersh, 1958) as well as by innumerable classroom teachers of

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the summing formula could be obtained by a relatively simple extension of the derivation rule presented. Although educated guesses can be made as to the sources of this transfer, the underlying mechanisms are not well understood. John Durnin and the first author have a study underway which may provide some of the necessary information.

For these reasons and because the results on the extra-scope test were subject to possible transfer effects of testing on the within-scope test, caution is advised in interpreting the extra-scope results. We originally included the extra-scope test to obtain experimental hypotheses and not definitive information. It should be emphasized, however, that these comments in no way apply to the clear results on the within-scope test.

mathematics, it is equally possible to teach rules (e.g., n^2) by exposition and by discovery. No one to our knowledge, however, had ever seriously considered identifying "what is learned" in discovering rules in addition to the (discovered) rules themselves. In the present study, we were apparently successful in identifying a derivation rule--i.e., a rule for deriving first order rules. No differences in the ability to derive new (within-scope) formulas (i.e., first order rules) could be detected between those Ss who discovered a derivation rule and those who were explicitely given one.

What we did not do in this study was to consider the possibility that our discovery Ss may have acquired a still higher order ability--namely, and ability to derive derivation rules. A strictly logical argument would seem to indicate that an indeterminate number of higher order abilities might exist. As soon as one identifies "what is learned" by discovery in one situation, the question immediately arises as to whether there is some still higher order ability which makes it possible to derive the identified knowledge. In so far as behavior is concerned, of course, it is still an open question whether such higher order derivations rules do exist in fact. Whether they do or not, there are undoubtedly a large number of situations where, because of the complexity of the situation, "what is learned" by discovery may be difficult, if not impossible, to identify. In these situations, there may be no real alternative to learning by discovery.

Nonetheless, intuition-based claims that learning by self-discovery produces superior ability to solve new problems (as compared with learning by exposition) have not withstood experimental test. The value to transfer ability of learning by discovery does not appear to exceed the value of learning by some forms of exposition. Apparently, the discovery myth has come into being not so much because teaching by exposition is a poor technique as such but because what has typically been taught by exposition leaves much to be desired. Before definitive predictions can be made, careful consideration must be given to "what is learned," the nature of the transfer items, and the relationships between them. As we identify what it is that is learned by discovery in a greater variety of situations, we shall be in an increasingly better position to impart that same knowledge by exposition.

The second point to be emphasized concerns the sequence effect--if a person already knows the desired responses, then he is not likely to discover a higher order rule by which such responses may be derived. An extrapolation of this result suggests that if S knows a specific derivation rule, then he may not discover a still higher order derivation rule even if he has all of the prerequisites and is given the opportunity to do so. The reverse order of presentation may enhance discovery without making it more difficult to learn more specific[8] rules at a later time. In short, prior knowledge may actually interfere in a very substantial way with later opportunities for discovery.[9]

Why and how sequence affects "what is learned" is still open to speculation. In attempting to provide some clarification,

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[8] Whether the rather vague (non-specific) term "specific," used above, refers to lower order rules, less general rules (e.g., Scandura et al, 1967), or both is an open but extremely important question.

[9] In spite of this fact there may be some advantages inherent in learning more specific rules. Although data are practically nonexistent on this point, it is quite possible that specific rules may result in shorter response latencies.

Guthrie (1967, 48) has suggested that rules in verbal learning are analogous to the unconditioned stimuli in classical conditioning, while not giving rules results in behavior more closely approximate to that observed in operant conditioning. Unfortunately, the analogy is a poor one. Not only does it provide little in the way of explanation, but the analogy itself is incorrect. To insure learning, for example, unconditioned stimuli must appear contiguously or shortly after the to-be-conditioned stimuli; yet, in learning rules by exposition, the rules (i.e., the "unconditioned stimuli") are presented first and then the stimulus instances (i.e., the "conditioned stimuli"). Perhaps what Guthrie means is that <u>once learned</u>, rules may act in a manner similar to the <u>reflexes</u> of classical conditioning. Rules (and reflexes) "tell how to get from where to where"; eliciting stimuli only provide the occasion for such actions. Yonge (1966, 118) has offered a more reasonable explanation in terms of the total structure of prior experiences, but it was formulated in relatively imprecise cognitive terms.

Our own interpretation is as follows. When <u>S</u> is presented with a stimulus and is required to produce a response he does not already know, he <u>necessarily</u> must first turn his attention to selecting a rule by which he can generate the appropriate response. In effect, <u>S</u> must adopt a secondary goal (i.e., find a rule) before he can hope to obtain his primary one (i.e., find the response). To achieve this secondary goal, <u>S</u> is forced to come up with a derivation rule, which might well be adequate for deriving other rules in addition to the one needed. The kind and amount of guidance given would presumably help to determine the precise nature of the derivation rule so acquired. On the other hand, if S already knows the response, it is not likely that he will waste much time trying to find another way to determine that response. Under these conditions, the only way to get S to adopt a secondary goal is to change the context. Presumably, the expository and guided discovery Ss in this study learned the derivation rule because this appeared to be the desirable thing to do. Some such mechanism may prove crucial to any theory based on the rule construct and framed in the SFL (Scandura, 1966, 1967a, 1967b).

The obtained sequencing result may also have important practical implications, as will be attested to by any junior high school mathematics teacher who has attempted to teach the "meaning" underlying the various computational algorithms after the children have already learned to compute. The children must effectively say to themselves something like, "I already know how to get the answer. Why should I care why the prodedure works?" Similarly, drilling students in their multiplication facts before they know what it means to multiply, may interfere with their later learning what multiplication is. Let me make this point clear, because it is an important one. We are not saying that we should teach meaning first simply out of some sort of dislike for rote learning--for certain purposes rote learning may be quite adequate and the most efficient procedure to follow. What we are saying is that learning such things as how to multiply, without knowing what multiplication means, may actually make it more difficult to learn the underlying meaning later on.

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NAME ______________________________

CLASS______________________________

This test is to see how much you may already know about the topic discussed in the programs. Do not be disappointed if you can not provide answers. Place an x in blanks for which you are unable to reply to the question.

1. In order to add up the numbers as shown here, 1 + 3 + 5 + 7 + . . . + 199, you could actually add or you could use the rule or formula, ______________________________.

2. In order to add up the seven numbers shown here, 2 + 6 + 18 + 54 + 162 + 486 + 1458, you could actually add or instead you could use the rule or formula, ______________________________.

3. I have taken and passed the following courses:

I have taken ______ years of algebra in high school;

I have taken ______ years of geometry in high school;

I have taken courses numbered ______________________ in college or their equivalent; and I also have taken ______________________

____.

4. My mathematics programs have used the (traditional, new) approach. (circle one)

INTRODUCTORY PROGRAM FOR SEQUENCES

(PLEASE PRINT ON THIS PAGE)

NAME ______________________________

CLASS ______________________________

DATE ______________________________

PLEASE DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

CONGRATULATIONS ! ! ! You have completed the introduction. The time shown on the board is now __________. Turn in this booklet for another, please.

EXAMPLE

Here is a sequence of numbers. The ellipsis (. . .) means that the pattern shown is to continue on indefinitely.

1 4 7 10 13 . . .

The next two numbers in this sequence are ____ and ____.

TURN PAGE AFTER ANSWERING IN THE BLANKS.

EXAMPLE

Answer: 16 and 19.

DO NOT TURN THIS PAGE UNTIL DIRECTED TO DO SO.

I1.

Obviously a sequence may be extended indefinitely.

1 4 7 10 13 16 19 22 25 28 31 . . .

Each of the numbers in the sequence is called a term-value. Thus, the first term-value is 1, the third term-value is 7, and the fourth term-value is ________.

TURN PAGE AFTER ANSWERING IN THE BLANK.

I1.

Answer: 10.

GO TO NEXT PAGE WHEN READY.

I2.

The term-values of the sequence have a position in the sequence. In the sequence:

1 4 7 10 13 . . .

term-value 1 has the first position, term-value 4 has the second position, and so on. We call the position of a term-value its term number. Term-value 7 has the _______ position or its term number is _______.

TURN PAGE AFTER ANSWERING IN THE BLANKS.

I2.

Answer: third position, term number three.

GO TO NEXT PAGE WHEN READY.

13.

We can write the term number above each term-value, if we wish, as shown below.

Term number:	1	2	3	4	5	6	7	. . .
Term-value:	1	4	7	10	13	. . .		

This form shows that term-value 1 is term number 1, term-value 4 is term number 2 and term-value 13 is ___________ ___________ ___________.

TURN PAGE AFTER ANSWERING IN THE BLANKS.

13.

Answer: Term number five.

GO TO THE NEXT PAGE WHEN READY.

I4.

Just as we can use the letter n to represent the term number of any term-value, we can use T_n to mean the actual term-value associated with n. In the following sequence of numbers, if n = 3, then $T_n = 7$ and if n = 5, then T_n = ________.

Term number, n:	1	2	3	4	5	6	7	. . .
Term-value, T_n:	1	4	7	10	13	. . .		

TURN PAGE AFTER ANSWERING IN THE BLANK.

I4.

Answer: 13.

GO TO THE NEXT PAGE WHEN READY.

I5.

Term number, n:	1	2	3	4	5	6	7	. . .
Term-value, T_n:	1	4	7	10	13	16	. . .	

You are going to learn about summing the term-values of a sequence. $\sum^n$ means the <u>sum of the first n term-values</u>. For the sequence above, since the sum of the first three term-values is $1 + 4 + 7 = 12$, then $\sum^3 = 12$. The superscript 3 on $\sum^3$ tells us to add the first ____________ term-values of the sequence.

TURN PAGE AFTER ANSWERING IN THE BLANK.

I5.

Answer: <u>three.</u>

GO TO THE NEXT PAGE WHEN READY.

16.

Since $\sum^2$ means the sum of the first two term-values or $T_1 + T_2$ and $\sum^5$ means $T_1 + T_2 + \ldots + T_5$, then regardless of the value given to n, $\sum^n = T_1 + T_2 + \ldots + T\_\_$. (Give the last addend.)

TURN PAGE AFTER ANSWERING IN THE BLANK.

16.

Answer: T_n.

GO TO THE NEXT PAGE WHEN READY.

17.

Let us briefly review our terms. When we write the actual sequence down, we write its successive term-____________.

TURN PAGE AFTER ANSWERING IN THE BLANK.

17.

Answer: term-values.

GO TO THE NEXT PAGE WHEN READY.

18.

The position number of each term-value in a sequence may be called its __________ __________ .

TURN THE PAGE AFTER ANSWERING IN THE BLANK.

18.

Answer: term number.

GO TO THE NEXT PAGE WHEN READY.

I9.

The symbol used to represent the sum of the first n term-values of a sequence is written ___________.

TURN PAGE AFTER ANSWERING IN THE BLANK.

I9.

Answer: $\sum^n$.

GO TO THE NEXT PAGE WHEN READY.

110.

If $n = 8$, $\sum^{n}$ means that we wish to sum the first __________ term-values of a sequence.

TURN PAGE AFTER ANSWERING IN THE BLANK.

110.

Answer: <u>eight</u>.

GO TO THE NEXT PAGE WHEN READY.

111.

While working with any sequence, you may write out three rows of information as below.

Term number, n:	1	2	3	4	5	6	7	. . .
Term-value, T_n:	1	4	7	10	13	. . .		
Sum, Σ^n:	1	5	12	22	. . .			

This form shows the various sums directly below the values of n and T_n. In other words, $1 + 4 + 7 = 12 = \Sigma^3$ so 12 goes below $n = 3$ and $T_n = 7$. The correct sum to put under the term number 5 and the term-value 13 in this table is __________.

TURN PAGE AFTER ANSWERING IN THE BLANK.

111.

Answer: 35.

GO TO THE NEXT PAGE WHEN READY.

112.

Complete the three row form for the sequence shown below.

Provide answers for each of the underscored positions in the array.

Term number, n:	1	2	3	___	___	. . .
Term-value, T_n:	0	2	___	6	8	. . .
Sum, $\sum^n$:	0	2	6	___	___	. . .

TURN PAGE AFTER ANSWERING IN ALL THE BLANKS.

112.

Answer:

Term number, n:	1	2	3	$\underline{4}$	$\underline{5}$	. . .
Term-value, T_n:	0	2	$\underline{4}$	6	8	. . .
Sum, $\sum^n$:	0	2	6	$\underline{12}$	$\underline{20}$	. . .

GO TO THE NEXT PAGE WHEN READY.

SEQUENCES, PART TWO--STARTING TIME ___________.

(PLEASE PRINT ON THIS PAGE.)

NAME ______________________________

CLASS______________________________

AS SOON AS YOU HAVE FILLED IN THE ABOVE INFORMATION, PROCEED AS DIRECTED.

CONGRATULATIONS ! ! ! You have completed your entire lesson about sequence summing. The time shown on the board now is ___________. On the back cover of this booklet, would you make any comments you wish about these materials, good or bad, too hard, too easy, interesting or not, etc. After you have made your comments, would you please put down your pencil. You may look over the booklet if you wish. A test will be given to the class over these materials. Thank you.

In this program, you are to learn how to sum term-values of various sequences by use of a formula. In each case, it will be possible to express Σ^n as a formula involving n and/or T_n instead of actually adding up $T_1 + T_2 + T_3 + \ldots + T_n$. As an example, the sequence 4, 8, 12, . . . may be summed by use of $\Sigma^n = 2n^2 + 2n$. That is, if we wish to sum the first 10 term-values of this sequence, $4 + 8 + \ldots + 40$, we use the formula and get $2 \times 10 \times 10 + 2 \times 10$ or ______ is the sum of the first ten term-values. Different sequences will have different formulas.

TURN PAGE AFTER ANSWERING IN THE BLANK.

Answer: $2 \times 10 \times 10 + 2 \times 10 = 200 + 20 = \underline{220}$.

On the next page is a new sequence which we want you to be able to sum.

GO TO THE NEXT PAGE WHEN READY.

R1.

Term number,	n:	1	2	3	4	5	. . .
Term-value,	T_n:	1	3	5	7	. . .	
Sum,	Σ^n:	1	4	9	16	. . .	

The rule for summing this sequence is $\Sigma^n = n \times n = n^2$. That is, to find the sum of the first n term-values, we only need to multiply the number of term-values to be summed by ___________. Remember the rule.

TURN PAGE AFTER ANSWERING IN THE BLANK.

R1.

Answer: multiply the number of term-values to be summed by itself. (or an equivalent statement)

GO TO THE NEXT PAGE WHEN READY.

R2.

Term number,	n:	1	2	3	4	5	. . .
Term-value,	T_n:	1	3	5	7	. . .	
Sum,	Σ^n:	1	4	9	16	. . .	

As an example of the application of the rule $\Sigma^n = n \times n$ for summing this sequence, suppose we wanted to sum the first three term-values. Then $n = 3$ and the rule tells us that Σ^3 is ___ x ___ or $\Sigma^3 =$ ___. Compare this result with the sum $T_1 + T_2 + T_3$.

TURN PAGE AFTER ANSWERING IN THE BLANKS.

R2.

Answer: 3 x 3 or $\Sigma^3 = 9$.

GO TO THE NEXT PAGE WHEN READY.

R3.

Term number,	n:	1	2	3	4	5	. . .
Term-value,	T_n:	1	3	5	7	. . .	
Sum,	$\sum^n$:	1	4	9	16	. . .	

The rule for summing this sequence is $\sum^n =$ ____________ or in words, the sum of the first n term-values is found by ______________ ____.

When $n = 10$, $\sum^n =$ ____________.

TURN PAGE AFTER ANSWERING IN THE BLANKS.

R3.

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx n^2//

R3.

Answer: $\sum^n$ = n x n or n^2.

multiply the number of term-values, n, by itself.

(An equivalent statement is acceptable.)

$\sum^{10}$ = 100.

GO TO THE NEXT PAGE WHEN READY.

In this program, you are to learn how to sum term-values of various sequences by use of a rule or formula. In each case, it will be possible to express $\sum^n$ as a rule or formula involving n and/or T_n instead of actually adding up $T_1 + T_2 + \ldots + T_n$. On the next page is a sequence we want you to be able to sum.

GO TO THE NEXT PAGE WHEN READY.

R4.

Term number,	n:	1	2	3	4	5	. . .
Term-value,	T_n:	2	6	10	14	. . .	
Sum,	$\sum^n$:	2	8	18	32	. . .	

The rule or formula for summing this sequence is $\sum^n = 2n^2 = (2) \times (n) \times (n)$. That is, to find the sum of the first n term-values, we need only to multiply together n times n and ________. Remember this rule.

TURN PAGE AFTER ANSWERING IN THE BLANK.

R4.

Answer: 2.

GO TO THE NEXT PAGE WHEN READY.

R5.

Term number,	n:	1	2	3	4	5	. . .
Term-value,	T_n:	2	6	10	14	. . .	
Sum,	$\sum^n$:	2	8	18	32	. . .	

As an example of applying the rule $\sum^n = 2n^2$, suppose we want to find the sum of the first three term-values without actually adding $T_1 + T_2 + T_3$. Since we would have $n = 3$, then $\sum^3$ would be $(2) \times (\_\_) \times (\_\_) =$ ______. Compare this result with the sum shown in the table.

TURN PAGE AFTER ANSWERING IN THE BLANK.

r

In this program, you are to learn how to sum term-values of various sequences by use of a rule or formula. In each case, it will be possible to express $\sum^n$ as a rule or formula involving n and/or T_n instead of actually adding up $T_1 + T_2 + \ldots + T_n$. On the next page is a sequence we want you to be able to sum.

GO TO THE NEXT PAGE WHEN READY.

R5.

Answer: $2 \times \underline{3} \times \underline{3} = \underline{18}$.

GO TO THE NEXT PAGE WHEN READY.

R6.

Term number,	n:	1	2	3	4	5	. . .
Term-value,	T_n:	2	6	10	14	. . .	
Sum,	Σ^n:	2	8	18	32	. . .	

The rule for this sequence is Σ^n = ________ or in words, multiply two times ______________________________________.

If $n = 10$, then Σ^n = _______.

TURN PAGE AFTER ANSWERING IN THE BLANKS.

R6.

Answer: $\Sigma^n = \underline{2n^2}$ or $\underline{(2) \times (n) \times (n)}$. That is, two times <u>n squared</u> or two times <u>n times n again</u>. Thus, when n is 10, Σ^n is <u>200</u>.

GO TO THE NEXT PAGE WHEN READY.

R7.

Term number,	n:	1	2	3	4	5	. . .
Term-value,	T_n:	1	5	9	13	. . .	
Sum,	$\sum^n$:	1	6	15	28	. . .	

The rule for summing this sequence is $\sum^n = (2n-1) \times (n)$. That is, to find $\sum^n$ for a given value of n, we multiply the value of one less than twice n by ___________. Remember the rule.

TURN PAGE AFTER ANSWERING IN THE BLANKS.

R7.
Answer: n.

GO TO THE NEXT PAGE WHEN READY.

R8.

Term number,	n:	1	2	3	4	5 . . .
Term-value,	T_n:	1	5	9	13	. . .
Sum,	Σ^n:	1	6	15	28	. . .

Let us apply the rule $\Sigma^n = (2n-1) \times (n)$ to find Σ^3. Since $n = 3$, we get $\Sigma^3 = (5) \times (3) = 15$. Compare this result with the sum shown under $n = 3$ in the table. Notice that $2n-1$ is found before multiplying by n. For $n = 3$, $2n-1 = [(2) \times (\_\_)] - 1 =$ _____.

TURN PAGE AFTER ANSWERING IN THE BLANKS.

R8.

Answer: $[(2) \times (\underline{3})] - 1 = 6 - 1 = \underline{5}$.

GO TO THE NEXT PAGE WHEN READY.

R9.

Term number,	n:	1	2	3	4	5	. . .
Term-value,	T_n:	1	5	9	13	. . .	
Sum,	Σ^n:	1	6	15	28	. . .	

The rule for this sequence is Σ^n = __________ or in words, the sum is the quantity ________________________ times n.

For n = 10, we find Σ^n = _______.

TURN PAGE AFTER ANSWERING IN THE BLANKS

R9.

Answer: $\Sigma^n = \underline{(2n-1) \times (n)}$.

The sum is the quantity <u>two n minus one</u> times n.

(Equivalent statements are acceptable.)

$\Sigma^{10} = (19) \times (10) = \underline{190}$.

GO TO THE NEXT PAGE WHEN READY.

R 10.

In review, if asked to sum the sequence 1, 3, 5, 7, . . . you should use the rule or formula $\sum^n =$ ________________.

To sum the sequence 2, 6, 10, 14, . . . use $\sum^n =$ ____________.

To sum the sequence 1, 5, 9, 13, . . . use $\sum^n =$ ____________.

TURN PAGE AFTER ANSWERING IN THE BLANKS.

R 10.

Answer: $\underline{n \times n \text{ or } n^2}$.

$\underline{2 \times n \times n \text{ or } 2n^2}$.

$\underline{n \times (2n - 1)}$.

GO TO THE NEXT PAGE WHEN READY.

R11. DO NOT TURN BACK TO EARLIER PAGES FOR THIS QUESTION.

To sum	Term number,	n:	1	2	3	4	5	...	
	Term-value,	T_n:	1	5	9	13	...		use Σ^n = ______
	Sum,	Σ^n:	1	6	15	28	...		

To sum	Term number,	n:	1	2	3	4	5	...	
	Term-value,	T_n:	1	3	5	7	...		use Σ^n = ______
	Sum,	Σ^n:	1	4	9	16	...		

To sum	Term number,	n:	1	2	3	4	5	...	
	Term-value,	T_n:	2	6	10	14	...		use Σ^n = ______
	Sum,	Σ^n:	2	8	18	32	...		

NO ANSWERS ARE PROVIDED FOR THIS QUESTION.

GO TO THE NEXT QUESTION WHEN READY.

R12. DO NOT TURN BACK TO EARLIER PAGES FOR THIS QUESTION.

To sum	Term number,	n:	1	2	3	4	5	...	
	Term-value,	T_n:	2	6	10	14	...		use Σ^n = ______
	Sum,	Σ^n:	2	8	18	32	...		

To sum	Term number,	n:	1	2	3	4	5	...	
	Term-value,	T_n:	1	5	9	13	...		use Σ^n = ______
	Sum,	Σ^n:	1	6	15	28	...		

To sum	Term number,	n:	1	2	3	4	5	...	
	Term-value,	T_n:	1	3	5	7	...		use Σ^n = ______
	Sum,	Σ^n:	1	4	9	16	...		

NO ANSWERS ARE PROVIDED FOR THIS QUESTION.

GO TO THE NEXT PAGE WHEN READY.

E1. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 3 5 7 . . .

Sum, Σ^n: 1 4 9 16 . . .

Your job is to find a formula for Σ^n. The formula for Σ^n can be written as the product of an expression involving n and n itself. When n = 1, you can multiply 1 times n to get $\Sigma^1 = 1$. What times n gives $\Sigma^1 = 1$?___ When n = 3, you can multiply 3 times n to get $\Sigma^3 = 9$. What times n gives $\Sigma^3 = 9$?___ Let's make a table showing these facts. Fill in the blanks.

That number which when multiplied times n equals Σ^n.	n	Σ^n
1	1	1
___	2	4
3	3	9
___	4	___

TURN PAGE AFTER ANSWERING IN ALL BLANKS.

E1.

Answer: 1

3

That number which when multiplied times n equals Σ^n.	n	Σ^n
1	1	1
2	2	4
3	3	9
4	4	16

GO TO THE NEXT PAGE WHEN READY.

E2. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 3 5 7 . . .

Sum; Σ^n: 1 4 9 16 . . .

Complete the table below. Do it!

That ______ which when ______ times ___ equals ______.	n	$\underline{\quad}^n$
___	1	___
___	___	___
___	3	___
___	___	___

TURN PAGE AFTER ANSWERING IN ALL BLANKS.

E2.

Answer:

That <u>number</u> which when <u>multiplied</u> times <u>n</u> equals <u>Σ^n</u>.	n	Σ^n
1	1	1
2	2	4
3	3	9
4	4	16

NOTE: To shorten the writing, let's just replace the words, "that number which when multiplied times n equals Σ^n," with, "the multiplier of n."

GO TO THE NEXT PAGE WHEN READY.

E3. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 3 5 7 . . .

Sum, Σ^n: 1 4 9 16 . . .

The multiplier of n	n	Σ^n
1	1	1
2	2	4
3	3	9

There is a relationship between the first two columns. To find it, determine how the first column can be obtained from the second: If 5 is the second column number, the first column number is 5. If 5 is the second column number, what is the first column number?____ The first column numbers are the same as those in the second. How can you get the first column numbers from those in the second?______ If n stands for an arbitrary second column number (i.e., any second column number we have in mind), the corresponding first column number is n. If n stands for an arbitrary second column number, what is the corresponding first column number?_____

TURN PAGE AFTER ANSWERING IN THE BLANKS.

E3.

Answer: 5

They're the same (an equivalent answer is ok.).

n

GO TO THE NEXT PAGE WHEN READY.

E4. Term number, n: 1 2 3 4 5 . . .

Term-value, t_n: 1 3 5 7 . . .

Sum, Σ^n: 1 4 9 16 . . .

Produce the entire table we have been discussing. Show in the table you construct below that Σ^n is something times n. Do it for n = 1, 2, 3, and 4. When n is the second column value, the first column value is n. When n is the second column value, what is the first column value? ____ Write the formula for Σ^n.

Σ^n = ______ x n = ____.

TURN PAGE AFTER ANSWERING IN BLANKS AND MAKING TABLE.

24.

Answer: Your table should look something like:

(The multiplier of n)	x	(n)	=	Σ^n
1	x	1	=	1
2	x	2	=	4
3	x	3	=	9
4	x	4	=	16

If n is the second column value, the first column value is n.

Σ^n = n x n = n^2.

GO TO THE NEXT PAGE WHEN READY.

E5. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 2 6 10 14 . . .

Sum, Σ^n: 2 8 18 32 . . .

Your job is to find a formula for Σ^n. The formula for Σ^n can be written as the product of an expression involving n and n itself. When n = 1, you can multiply 2 times n to get $\Sigma^1 = 2$. What times n gives $\Sigma^1 = 2$? ______ When n = 3, you can multiply 6 times n to get $\Sigma^3 = 18$. What times n gives $\Sigma^3 = 18$? ______ Let's make a table showing these facts. Fill in the blanks.

The multiplier of n	n	Σ^n
2	1	2
___	2	8
6	3	18
___	4	___

TURN PAGE AFTER ANSWERING IN ALL BLANKS.

E5.

Answer: 2

6

The multiplier of n	n	Σ^n
2	1	2
4	2	8
6	3	18
8	4	32

GO TO THE NEXT PAGE WHEN READY.

E6. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 2 6 10 14 . . .

Sum, Σ^n: 2 8 18 32 . . .

Complete the table below. Do it!

The ________ of ____	n	____
____	1	____
____	____	____
____	3	____
____	____	____

TURN PAGE AFTER ANSWERING IN THE BLANKS.

E6.

Answer:

The multiplier of n	n	Σ^n
2	1	2
4	2	8
6	3	18
8	4	32

GO TO THE NEXT PAGE WHEN READY.

E7. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 2 6 10 14 . . .

Sum, Σ^n: 2 8 18 32 . . .

The multiplier of n	n	Σ^n
2	1	2
4	2	8
6	3	18

There is a relationship between the first two columns. To find it, determine how the first column can be obtained from the second. If 5 is the second column number, the first column number is 10. If 5 is the second column number, what is the first column number?___ The first column numbers are twice those in the second. How can you get the first column numbers from those in the second?______________ If n stands for an arbitrary second column number, the corresponding first column number is 2n. If n stands for an arbitrary second column number, what is the corresponding first column number?________

TURN PAGE AFTER ANSWERING IN THE BLANKS.

E7.

Answer: 10

double them or multiply by 2 or an equivalent

2n = 2 x n

GO TO THE NEXT PAGE WHEN READY.

E8. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 2 6 10 14 . . .

Sum, Σ^n: 2 8 18 32 . . .

Produce the entire table we have been discussing. Show in the table you construct below that Σ^n is something times n. Do it for n = 1, 2, 3, and 4. When n is the second column value, the first column value is 2n. When n is the second column value, what is the first column value?_______ Write the formula for Σ^n.

Σ^n = _____ x n = _______.

TURN PAGE AFTER ANSWERING IN BLANKS AND MAKING TABLE.

E8.

Answer: Your table should look something like:

(The multiplier of n)	x	(n)	=	Σ^n
2	x	1	=	2
4	x	2	=	8
6	x	3	=	18
8	x	4	=	32

If n is the second column value, the first column value is <u>2n</u>.

$\Sigma^n = \underline{2n} \times n = \underline{2n^2}$.

GO TO THE NEXT PAGE WHEN READY.

E9. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 5 9 13 . . .

Sum, Σ^n: 1 6 15 28 . . .

Your job is to find a formula for Σ^n. The formula for Σ^n can be written as the product of an expression involving n and n itself. When n = 1, you can multiply 1 times n to get $\Sigma^1 = 1$. What times n gives $\Sigma^1 = 1$? ____ When n = 3, you can multiply 5 times n to get $\Sigma^3 = 15$. What times n gives $\Sigma^3 = 15$? ____ Let's make a table showing these facts. Fill in the blanks.

The multiplier of n	n	Σ^n
1	1	1
___	2	6
5	3	15
___	4	___

TURN PAGE AFTER ANSWERING IN THE BLANKS.

E9.

Answer: 1

5

The multiplier of n	n	Σ^n
1	1	1
3	2	6
5	3	15
7	4	28

GO TO THE NEXT PAGE WHEN READY.

E10. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 5 9 13 . . .

Sum, Σ^n: 1 6 15 28 . . .

Complete the table below. Do it!

The ____ of __	n	____
____	1	____
____	____	____
____	3	____
____	____	____

TURN PAGE AFTER ANSWERING IN THE BLANKS.

E10.

Answer:

The multiplier of n	n	Σ^n
1	1	1
3	2	6
5	3	15
7	4	28

GO TO THE NEXT PAGE WHEN READY.

E11. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 5 9 13 . . .

Sum, Σ^n: 1 6 15 28 . . .

The multiplier of n	n	Σ^n
1	1	1
3	2	6
5	3	15

There is a relationship between the first two columns. To find it, determine how the first column can be obtained from the second. If 5 is the second column number, the first column number is 9. If 5 is the second column number, what is the first column number?____ The first column numbers are one less than two times those in the second. How can you get the first column numbers from those in the second? Take 1 away from what times n?____. If n stands for an arbitrary second column number, the corresponding first column number is 2n-1. If n stands for an arbitrary second column number, what is the corresponding first column number?______

TURN PAGE AFTER ANSWERING IN THE BLANKS.

E11.

Answer: 9

2

$2 \times n - 1 = 2n-1$

GO TO THE NEXT PAGE WHEN READY.

E12. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 5 9 13 . . .

Sum, Σ^n: 1 6 15 28 . . .

Produce the entire table we have been discussing. Show in the table you construct below that Σ^n is something times n. Do it for n = 1, 2, 3, and 4. When n is the second column value, the first column value is one less than 2n. When n is the second column value, the first column value is one less than what?_____ Write the formula for Σ^n. Σ^n = _____ x n.

TURN PAGE AFTER ANSWERING IN THE BLANKS AND MAKING TABLE.

E12.

Answer: Your table should look something like:

(The multiplier of n)	x	(n)	=	Σ^n
1	x	1	=	1
3	x	2	=	6
5	x	3	=	15
7	x	4	=	28

If n is the second column value, the first column value is one less than 2n or (2n-1).

$\Sigma^n = \underline{(2n-1)}$ x n.

GO TO THE NEXT PAGE WHEN READY.

D 1. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 3 5 7 . . .

Sum, Σ^n: 1 4 9 16 . . .

Can you write a formula for Σ^n for <u>this</u> sequence? If you can, write it here. Σ^n = _____. You are supposed to find a formula for _____. If you don't know one, try to find a way to tell what Σ^n is for each value of n. You may use the margins for any scribbling you wish. Do you see a way? _____(yes or no).

TURN PAGE AFTER ANSWERING IN THE BLANKS.

D 1.

Answer: No answer for Σ^n is provided for you on this page.

Σ^n.

Whatever you answered for the third blank is probably correct, but if you said NO, keep trying to find a way. YOU CAN DO IT!

GO TO THE NEXT PAGE WHEN READY.

D 2. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 3 5 7 . . .

Sum, Σ^n: 1 4 9 16 . . .

The desired formula for Σ^n involves n. Can you write a formula for Σ^n? If you can, write it here. Σ^n = ______. You are to find a formula for Σ^n which involves ___. If you don't know a formula, try to find a way to get Σ^n from knowledge of what n is. Do you see a way? ______(yes or no).

TURN PAGE AFTER ANSWERING IN THE BLANKS.

D 2.

Answer: No answer for Σ^n is provided for you on this page.

n.

Whatever you answered for the third blank is probably correct, but if you said NO, keep trying to find a way. YOU CAN DO IT!

GO TO THE NEXT PAGE WHEN READY.

3. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 3 5 7 . . .

Sum, Σ^n: 1 4 9 16 . . .

The desired formula for Σ^n involves n and multiplication. Can you now write a formula for Σ^n? If you can, write it here. Σ^n = ______. You are supposed to find a formula for Σ^n which involves ______________________________. If you don't know a formula, try to get one by multiplication of n. Do you see a way? ________(yes or no).

TURN PAGE AFTER ANSWERING IN THE BLANKS.

3.

Answer: $\Sigma^n = \underline{n \times n} = \underline{n^2}$.

n and multiplication.

Whatever you answered in the third blank is probably correct but if you said NO, better luck on the next sequence--don't stop trying. YOU CAN DO IT!

GO TO THE NEXT PAGE WHEN READY.

D 4. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 2 6 10 14 . . .

Sum, Σ^n: 2 8 18 32 . . .

Can you write a formula for Σ^n for <u>this</u> sequence? If you can, write it here. Σ^n = ______ You are supposed to find a formula for ______. If you don't know one, try to find a way to tell what Σ^n is for each value of n. You may use the margins for any scribbling you wish. Do you see a way? ______(yes or no).

TURN PAGE AFTER ANSWERING IN THE BLANKS.

D 4.

Answer: No answer for Σ^n is provided for you on this page.

Σ^n.

Whatever you answered for the third blank is probably correct, but if you said NO, keep trying to find a way. YOU CAN DO IT!

GO TO THE NEXT PAGE WHEN READY.

D 5. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 2 6 10 14 . . .

Sum, Σ^n: 2 8 18 32 . . .

The desired formula for Σ^n involves 2n. Can you write a formula for Σ^n? If you can, write it here. Σ^n = ______. You are to find a formula for Σ^n which involves ______. If you don't know a formula, try to find a way to get Σ^n from knowledge of what 2n is. Do you see a way?______(yes or no).

TURN PAGE AFTER ANSWERING IN THE BLANKS.

D 5.

Answer: No answer for Σ^n is provided for you on this page.

2n.

Whatever you answered for the third blank is probably correct, but if you said NO, keep trying to find a way. YOU CAN DO IT!

GO TO THE NEXT PAGE WHEN READY.

D 5.

Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 2 6 10 14 . . .

Sum, Σ^n: 2 8 18 32 . . .

The desired formula for Σ^n involves <u>multiplication of 2n</u>. Can you now write a formula for Σ^n? If you can, write it here. Σ^n = ______. You are supposed to find a formula for Σ^n which involves __________________________. If you don't know a formula, try to get one by multiplication of 2n. Do you see a way? ____________ (yes or no).

TURN PAGE AFTER ANSWERING IN THE BLANKS.

D 5.

Answer: $\Sigma^n = \underline{2 \times n \times n} = \underline{2n^2}$.

<u>multiplication of 2n</u>.

Whatever you answered in the third blank is probably correct, but if you said NO, better luck on the next sequence--don't stop trying. YOU CAN DO IT!

GO TO THE NEXT PAGE WHEN READY.

D 7.

Term number, n:	1	2	3	4	5 . . .
Term-value, T_n:	1	5	9	13 . . .	
Sum, Σ^n:	1	6	15	28 . . .	

Can you write a formula for Σ^n for <u>this</u> sequence? If you can, write it here. Σ^n = ________. You are supposed to find a formula for ______. If you don't know one, try to find a way to tell what Σ^n is for each value of n. You may use the margins for any scribbling you wish. Do you see a way? _____(yes or no).

TURN PAGE AFTER ANSWERING IN THE BLANKS.

D 7.

Answer: No answer for Σ^n is provided for you on this page.

Σ^n.

Whatever you answered for the third blank is probably correct, but if you said NO, keep trying to find a way. YOU CAN DO IT!

GO TO THE NEXT PAGE WHEN READY.

D 3.

Term number, n:	1	2	3	4	5 . . .
Term-value, T_n:	1	5	9	13 . . .	
Sum, Σ^n:	1	6	15	28 . . .	

The desired formula for Σ^n involves (<u>2n-1</u>). Can you write a formula for Σ^n? If you can, write it here. Σ^n = ________. You are to find a formula for Σ^n which involves ______. If you don't know a formula, try to find a way to get Σ^n from knowledge of what 2n-1 is. Do you see a way? _______ (yes or no).

TURN PAGE AFTER ANSWERING IN THE BLANKS.

D 3.

Answer: No answer for Σ^n is provided for you on this page.

<u>2n-1</u>

Whatever you answered for the third blank is probably correct, but if you said NO, keep trying to find a way. YOU CAN DO IT!

GO TO THE NEXT PAGE WHEN READY.

C 9.

Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 5 9 13 . . .

Sum, Σ^n: 1 6 15 28 . . .

The desired formula for Σ^n involves multiplication of 2n-1. Can you now write a formula for Σ^n? If you can, write it here. Σ^n = __________. You are supposed to find a formula for Σ^n which involves ______________________________. If you don't know a formula, try to get one by multiplication of 2n-1. Do you see a way?_____ (yes or no).

TURN PAGE AFTER ANSWERING IN THE BLANKS.

D 9.

Answer: $\Sigma^n = (\underline{2n-1}) \times (\underline{n})$ or an equivalent.

multiplication of 2n-1.

Whatever you answered in the third blank is probably correct, but if you said NO, better luck on the test.

GO TO THE NEXT PAGE WHEN READY.

1. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 3 5 7 . . .

Sum, Σ^n: 1 4 9 16 . . .

Your job is to find a formula for Σ^n. Can you write the formula for Σ^n as the product of an expression involving n and n itself? When n = 1, what times n gives $\Sigma^1 = 1$?____ When n = 3, what times n gives $\Sigma^3 = 9$?____ Let's make a table showing these facts. Can you fill in the blanks? Try it!

That number which when multiplied times n equals Σ^n	n	Σ^n
1	1	1
___	2	4
3	3	9
___	4	___

TURN PAGE AFTER ANSWERING IN ALL BLANKS.

1.

Answer: 1

3

That number which when multiplied times n equals Σ^n	n	Σ^n
1	1	1
2	2	4
3	3	9
4	4	16

GO TO THE NEXT PAGE WHEN READY.

2. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 3 5 7 . . .

Sum, Σ^n: 1 4 9 16 . . .

Can you complete the table below? Try it!

That ______ which when ______ times ____ equals ______.	n	$\overline{}^n$
___	1	___
___	___	___
___	3	___
___	___	___

TURN PAGE AFTER ANSWERING IN ALL BLANKS.

2.

Answer:

That number which when multiplied times n equals Σ^n.	n	Σ^n
1	1	1
2	2	4
3	3	9
4	4	16

NOTE: To shorten the writing, let's replace the words, "that number which when multiplied times n equals Σ^n," with, "the multiplier of n."

GO TO THE NEXT PAGE WHEN READY.

C 3. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 3 5 7 . . .

Sum, Σ^n: 1 4 9 16 . . .

The multiplier of n	n	Σ^n
1	1	1
2	2	4
3	3	9

Do you notice any relationship between the first two columns? If 5 is the second column number, what is the first column number?___ How can you get the first column numbers from those in the second?

____ If n stands for an arbitrary second column number (i.e., any second column number we have in mind), what is the corresponding first column number?_____

TURN TO THE NEXT PAGE AFTER ANSWERING IN BLANKS.

C 3.

Answer: 5

They're the same (An equivalent answer is ok.).

n

GO TO THE NEXT PAGE WHEN READY.

G 4

Term number, n: 1   2   3   4   5 . . .

Term-value, T_n: 1   3   5   7 . . .

Sum, Σ^n: 1   4   9   16 . . .

Can you produce the entire table we have been discussing? Can you show in the table you construct below that Σ^n is something times n? Try it for n = 1, 2, 3, and 4. When n is the second column value, what is the first column value?_____ Can you write the formula for Σ^n? Σ^n = _____ x n = _____.

TURN PAGE AFTER ANSWERING IN BLANKS AND MAKING TABLE.

G 4

Answer: Your table should look something like:

(The multiplier of n)	x	(n)	=	Σ^n
1	x	1	=	1
2	x	2	=	4
3	x	3	=	9
4	x	4	=	16

If n is the second column value, the first column value is <u>n</u>.

$\Sigma^n = \underline{n} \times n = \underline{n^2}$.

GO TO THE NEXT PAGE WHEN READY.

G 5. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 2 6 10 14 . . .

Sum, Σ^n: 2 8 18 32 . . .

Your job is to find a formula for Σ^n. Can you write the formula for Σ^n as the product of an expression involving n and n itself? When n = 1, what times n gives $\Sigma^1 = 2$? When n = 3, what times n gives $\Sigma^3 = 18$?_____ Let's make a table showing these facts. Can you fill in the blanks? Try it!

The multiplier of n	n	Σ^n
2	1	2
___	2	8
6	3	18
___	4	___

TURN PAGE AFTER ANSWERING IN ALL BLANKS.

3 5.

Answer: 2

6

The multiplier of n	n	Σ^n
2	1	2
4	2	8
6	3	18
8	4	32

GO TO THE NEXT PAGE WHEN READY.

5. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 2 6 10 14 . . .

Sum, Σ^n: 2 8 18 32 . . .

Can you complete the table below? Try it!

The ____________ of ___	n	____
___	1	___
___	___	___
___	3	___
___	___	___

TURN PAGE AFTER ANSWERING IN THE BLANKS.

5.

Answer:

The multiplier of n	n	Σ^n
2	1	2
4	2	8
6	3	18
8	4	32

GO TO THE NEXT PAGE WHEN READY.

G 7. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 2 6 10 14 . . .

Sum, Σ^n: 2 8 18 32 . . .

The multiplier of n	n	Σ^n
2	1	2
4	2	8
6	3	18

Do you notice any relationship between the first two columns? If 5 is the second column number, what is the first column number?___ How can you get the first column numbers from those in the second? ____________________ If n stands for an arbitrary second column number, what is the corresponding first column number? ________

TURN TO THE NEXT PAGE AFTER ANSWERING IN THE BLANKS.

C 7.

Answer: 10

double them or multiply by 2 or an equivalent

2n = 2 x n

GO TO THE NEXT PAGE WHEN READY.

6 9. Term number, n: 1 2 3 4 5 . . .

Term number, n:	1	2	3	4	5 . . .
Term-value, T_n:	2	6	10	14 . . .	
Sum, Σ^n:	2	8	18	32 . . .	

Can you produce the entire table we have been discussing? Can you show in the table you construct below that Σ^n is something times n? Try it for n = 1, 2, 3, and 4. When n is the second column value, what is the first column value?_____ Can you write the formula for Σ^n ? Σ^n = ______ x n = ______.

TURN PAGE AFTER ANSWERING IN BLANKS AND MAKING TABLE.

C 8.

Answer: Your table should look something like:

(The multiplier of n)	x	(n)	=	Σ^n
2	x	1	=	2
4	x	2	=	8
6	x	3	=	18
8	x	4	=	32

If n is the second column value, the first column value is <u>2n</u>.

$\Sigma^n = \underline{2n} \times n = \underline{2n^2}$.

GO TO THE NEXT PAGE WHEN READY.

6 9. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 5 9 13 . . .

Sum, Σ^n: 1 6 15 28 . . .

Your job is to find a formula for Σ^n. Can you write the formula for Σ^n as the product of an expression involving n and n itself? When n = 1, what times n gives $\Sigma^1 = 1$?____ When n = 3, what times n gives $\Sigma^3 = 15$?_____ Let's make a table showing these facts. Can you fill in the blanks? Try it!

The multiplier of n	n	Σ^n
1	1	1
___	2	6
5	3	15
___	4	___

TURN PAGE AFTER ANSWERING IN THE BLANKS.

6 9.

Answer: 1

5

The multiplier of n	n	Σ^n
1	1	1
3	2	6
5	3	15
7	4	28

GO TO THE NEXT PAGE WHEN READY.

5 10. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 5 9 13 . . .

Sum, Σ^n: 1 6 15 28 . . .

Can you complete the table below? Try it!

The ______ of ___	n	___
___	1	___
___	___	___
___	3	___
___	___	___

TURN PAGE AFTER ANSWERING IN THE BLANKS.

6 10

Answer:

The multiplier of n	n	Σ^n
1	1	1
3	2	6
5	3	15
7	4	28

GO TO THE NEXT PAGE WHEN READY.

G 11. Term number, n: 1 2 3 4 5 . . .

Term-value, T_n: 1 5 9 13 . . .

Sum, Σ^n: 1 6 15 28 . . .

The multiplier of n	n	Σ^n
1	1	1
3	2	6
5	3	15

Do you notice any relationship between the first two columns? If 5 is the second column number, what is the first column number?___ How can you get the first column numbers from those in the second? Take 1 away from what times n?______ If n stands for an arbitrary second column number, what is the corresponding first column number?

____

TURN PAGE AFTER ANSWERING IN THE BLANKS.

G 11.

Answer: 9

2

$2 \times n - 1 = 2n-1$

GO TO THE NEXT PAGE WHEN READY.

6 12.

Term number, n:	1	2	3	4	5 . . .
Term-value, T_n:	1	5	9	13 . . .	
Sum, Σ^n:	1	6	15	28 . . .	

Can you produce the entire table we have been discussing? Can you show in the table you construct below that Σ^n is something times n? Try it for n = 1, 2, 3, and 4. When n is the second column value, the first column value is one less than what?_____ Can you write the formula for Σ^n? Σ^n = _______ x n.

TURN PAGE AFTER ANSWERING IN THE BLANKS AND MAKING TABLE.

6 12.

Answer: Your table should look something like:

(The multiplier of n)	x	(n)	=	Σ^n
1	x	1	=	1
3	x	2	=	6
5	x	3	=	15
7	x	4	=	28

If n is the second column value, the first column value is one less than 2n or (2n-1).

$\Sigma^n = (2n-1) \times n$.

GO TO THE NEXT PAGE WHEN READY.

TEACHING COLLEGE STUDENTS HOW TO LEARN MATHEMATICS

("WHAT IS LEARNED"IN MATHEMATICAL DISCOVERY)

Joseph M. Scandura

Project No. OEC 1-7-068798-0360

University of Pennsylvania

1967

Background

One of the fundamental assumptions underlying many of the new mathematics curricula is that discovery methods of teaching and learning increase the student's ability to learn new content. The last decade of research on discovery learning, however, has produced only partial and tentative support for this contention. Even where the experiments have been relatively free of methodological defects, the results have often been inconsistent. More particularly, the interpretation of research on discovery learning has been made difficult by differences in terminology, the tendency to compare identical groups on a variety of dependent measures, and vagueness as to what is being taught and discovered.

While most discrepancies due to differences in terminology can be reconciled by a careful analysis of what was actually done in the experiments and thus present a relatively minor problem, the failure to equate original learning has often made it difficult to interpret transfer (and retention) results in an unambiguous manner. Thus, several studies have shown that rule-given groups perform better on "near" transfer tests than do discovery groups. The obtained differences, however, may have been due to the fact that the discovery groups did not learn the originally presented materials as well as the rule-given groups.

When the degree of original learning was equated, Gagne and Brown found that their discovery groups were better able to derive new formulas than their rule (i.e. formula)-given groups. They attributed this result to differences in "what was learned" but added that they were unable to specify precisely what these differences were.

Using the Set-Function Language (SFL) as a guide, Scandura proposed an analysis of discovery learning that seems to be in accord with experimental findings. The main point was that in order to succeed, discovery S̲s *must* learn to derive solutions whereas solution-given S̲s need not. In attaining criterion, discovery S̲s may discover a *derivation rule* by which solutions to new, though related, problems may be derived. Under these circumstances, discovery S̲s would be expected to perform better than expository S̲s on tasks which are within the scope of such a derivation rule. If the new problems presented have solutions beyond the scope of a discovered derivation rule, however, there would be no reason to expect discovery S̲s to have any special advantage.

OBJECTIVES

This study was concerned with two major questions. First, can "what is learned" in mathematical discovery be identified and, if so, can it be taught by exposition with equivalent results? Second, how does "what is learned" depend on prior learning and on the nature of the discovery treatment itself?

Assuming that transfer depends only on whether or not the derivation rule is learned, then the order in which the formulas (i.e. the solutions) and the derivation rule are presented should have no effect on transfer provided S̲ actually learns the derivation rule. If, on the other hand, a discovery program simply provides an opportunity to discover and does not guide the learner through the derivation procedure, sequence of presentation might have a large effect on transfer. That is, if a capable and motivated subject is given appropriate hints, he might well succeed in discovering the appropriate formulas and in the process discover the derivation rule. It is not likely, however, that he would exert much effort when given an opportunity to discover a formula he already knew.

In particular, the following hypotheses were made. First, what was learned by guided discovery in the Gagne and Brown study can be presented by exposition with equivalent results. Second, presentation order is critical when the hints provided during discovery are specific to the respective formulas sought rather than relevant to a general strategy (i.e. a derivation rule). Third, presentation order is not critical when the program effectively forces S to learn the derivation rule, regardless of whether the learning takes place by exposition or by discovery.

PROCEDURE

The SFL was used as an aid in analyzing the guided discovery programs used by Gagne and Brown and Eldredge to determine "what is learned." As a result of this analysis, we were able to devise an expository statement of the derivation rule. We were also able to determine, on an a priori basis, which kinds of transfer item could be solved by using this derivation rule and which could not.

There were seven treatments. Each consisted of a common introductory program followed by various combinations of four basic instructional programs. The introductory program was designed to generally familiarize the Ss with number sequences and with the terminology used in the four basic programs. In particular, four concepts were clarified: sequence; term value, T_n; term number, n; and sum of the first n terms of a sequence Σ^n.

Each of the four basic instructional programs was based on the same three arithmetic series and their respective summing formulas: $1 + 3 + 5 + \ldots + (2n-1) \longrightarrow n^2$; $2 + 6 + 10 + \ldots + (4n-2) \longrightarrow 2n^2$; $1 + 5 + 9 + \ldots + (4n-3) \longrightarrow (2n-1)n$. Each series was presented as a three-row display -- e g.,

Term number n :	1	2	3	4	. . .
Term value T_n:	2	6	10	14	. . . .
Sum n:	2	8	18	32	. . .

The rule and example (R) program consisted of the three series displays together with the respective summing formulas. The presentation of each summing formula was followed by three application problems --- e.g., find the sum of 2 + 6 + 10 ($=2\cdot3^2=18$). S was also required to write out each formula in both words and symbols, but no rationale for the formula was provided.

The other three basic programs included differing kinds of directions and/or hints as to how the summing formulas might be determined. The expository (E) and (highly) guided discovery (G) programs were based on a simplified variant of that derivation rule presumably learned by the guided discovery Ss in the Gagne and Brown study.

The E program consisted of a simplified statement of the derivation rule as it applied to each of the three training series. To insure that S learned how to use the derivation rule, a vanishing procedure was used which ultimately required S to apply the procedure without any instructions. The G program paralleled the E program in all respects. The only difference was that the G program consisted of questions whereas the E program consisted of yoked direct statements, each followed by a parallel question or completion statement to see whether S had read the original statement correctly. The only difference between the E and G programs was whether the information was acquired by reception or by reacting to a question (i.e., by discovery). The discovery (D) program, on the other hand, simply provided S with an opportunity to discover the respective summing formulas. S was guided b questions and hints which were specific to the formulas involved rather than relevant to any general strategy or derivation rule. The questions and hints were interspersed with liberal amounts of encouragement to provide motivation.

There were two transfer tests. The within-scope transfer test consisted of two new series displays which could be solved by the identified derivation rule. The extra-scope transfer test involved series which, strictly speaking, were beyond the scope of the identified derivation rule. A series of hints paralleling those used in the D program were constructed to accompany each test series.

The naive Ss were 105 (103 females) junior and senior elementary education majors enrolled in required mathematics education courses at the Florida State University. Participation was a course requirement.

The Ss were randomly assigned to the seven treatment groups. In addition to the common introductory program, the rule-given treatment group (R) received only the R program. The other six treatment groups received the R program together with one of the other three basic instructional programs. The RE, RG, and RD groups received the R program followed by the E (expository), G (guided discovery), and D (discovery) programs, respectively, while the ER, GR, and DR groups received these same respective programs in the reverse order.

The Ss were scheduled to come to the experimental room in groups of four or less and were arranged at the ends of two tables which were partitioned to provide separate study carrels. A brief quiz was used to screen out any Ss who were already familiar with number series and/or formulas for summing them. They were then given the introductory and treatment booklets and the necessary instructions for working through the two programs. The Ss worked at their own rate. E recorded the times taken in the introductory and treatment booklets.

As soon as all of the Ss in the testing group had completed the treatment programs, they were told to review for a test. After two minutes, the booklets were collected and the tests and hint cards were presented. The Ss were given instructions to the effect that they would be timed, that they would be told whether a particular answer was correct or not and that they could use the hints after 5 minutes. It was emphasized that the fewer hints used, the better the score. Before continuing on to the second problem, each S read all of the hint cards pertaining to the first problem, etc. If an S solved a problem before the others, he was allowed to read the rest of the hints for that problem and, then, was required to wait for the others to finish.

Three indices of performance on the transfer tasks were obtained: (1) time to solution, (2) number of hints prior to solution, and (3) a weighted score. The weighted score was equal to the time to solution in minutes plus a penalty of 4, 7, 9, or 10 depending on whether S used 1, 2, 3, or 4 hints, respectively.

RESULTS

All treatment groups performed at essentially the same level on the introductory program, both in terms of time to completion and number of errors.

The results on the within-scope transfer test conformed to prediction. Irrespective of the transfer measure used, the group (R) given the formula program only and the group (RD) given the formula program followed by the opportunity to discover program performed at one level, while the other five groups performed at a common and significantly higher level. In particular, only that sequence effect involving groups RD and DR was significant.

While there were no overall treatment differences on the extra-scope transfer test, the contrast between groups R and RD and groups DR, RG, GR, RE, and ER attained a borderline significance level. There were, however, no reliable performance differences between: (1) the guided discovery (RG and GR) groups and the exposition (RE and ER) groups, (2) those guided discovery and exposition groups (RG and RE) given the formulas first and those groups (GR and ER) given the formulas last, (3) the opportunity to discover---formula-given (DR) group and the four guided discovery and exposition groups, or, most critically, (4) between groups DR and RD.

These transfer effects can not be attributed to differences in original learning. A learning test embedded within the common R program indicated that the Ss had well-learned the appropriate summing formulas to the three training series before they took the transfer tests.

CONCLUSION AND IMPLICATIONS

Two points need to be emphasized. First, "what is learned" during guided discovery can at least sometimes be identified and taught by exposition--with equivalent results. While this conclusion may appear somewhat surprising at first glance, further reflection indicates that we have always known it to be at least partially true. As has been documented in the laboratory as well as by innumerable classroom teachers of mathematics, it is equally possible to teach rules (e.g., n^2) by exposition and by discovery. No one to our knowledge, however, had ever seriously considered identifying "what is learned" in discovering rules in addition to the (discovered) rules themselves. In the present study, we were apparently successful in identifying a derivation rule--i.e., a rule for deriving first order rules. No differences in the ability to derive new (within-scope) formulas (i.e., first order rules) could be detected between those Ss who discovered a derivation rule and those who were explicitly given one.

What we did not do in this study was to consider the possibility that our discovery Ss may have acquired a still higher order ability--namely, an ability to derive derivation rules. A strictly logical argument would seem to indicate that an indeterminate number of higher order abilities might exist. As soon as one identifies "what is learned" by discovery in one situation, the question immediately arises as to whether there is some still higher order ability which makes it possible to derive the identified knowledge. In so far as behavior is concerned, of course, it is still an open question whether such higher order derivation rules do exist in fact. Whether they do or not, there are undoubtedly a large number of situations where, because of the complexity of the situation, "what is learned" by discovery may be difficult, if not impossible, to identify. In these situations, there may be no real alternative to learning by discovery.

Nonetheless, intuition-based claims that learning by self-discovery produces superior ability to solve new problems (as compared with learning by exposition) have not withstood experimental test. The value to transfer ability of learning by discovery does not appear to exceed the value of learning by some forms of exposition. Apparently, the discovery myth has come into being not so much because teaching by exposition is a poor technique as such but because what has typically been taught by exposition leaves much to be desired. Before definitive predictions can be made, careful consideration must be given to "what is learned," the nature of the transfer items, and the relationships between them. As we identify just what it is that is learned by discovery in a greater variety of situations, we shall be in an increasingly better position to impart that same knowledge by exposition.

The second point to be emphasized concerns the sequence effect--if a person already knows the desired responses, then he is not likely to discover a higher order rule by which such responses may be derived. An extrapolation of this result suggests that if S knows a specific derivation rule, then he may not discover a still higher order derivation rule even if he has all of the prerequisites and is given the opportunity to do so. The reverse order of presentation may enhance discovery withoug making it more difficult to learn more specific rules at a later time. In short, prior knowledge may actually interfere in a very substantial way with later opportunities for discovery.

Why and how sequence affects "what is learned" is still open to speculation. In attempting to provide some clarification, Guthrie has suggested that rules in verbal learning are analogous to the unconditioned stimuli in classical conditioning, while not giving rules results in behavior more closely approximate to that observed in operant conditioning. Unfortunately, the analogy is a poor one. Not only does it provide little in the way of explanation, but the analogy itself is incorrect. To insure learning, for example, unconditioned stimuli must appear contiguously or shortly after the to-be-conditioned stimuli; yet, in learning rules by exposition, the rules (i.e., the "unconditioned stimuli") are presented first and then the stimulus instances (i.e., the "conditioned stimuli"). Perhaps what Guthrie means is that once learned, rules may act in a manner similar to the reflexes of classical conditioning. Rules (and reflexes) "tell how to get from where to where"; eliciting stimuli only provide the occasion for such actions. Yonge has offered a more reasonable explanation in terms of the total structure of prior experiences, but it was formulated in relatively imprecise cognitive terms.

Our own interpretation is as follows. When S̲ is presented with a stimulus and is required to produce a response he does not already know, he necessarily must first turn his attention to selecting a rule by which he can generate the appropriate response. In effect, S̲ must adopt a secondary goal (i.e., find a rule) before he can hope to obtain his primary one (i.e., find the response). To achieve this secondary goal, S̲ is forced to come up with a derivation rule, which might well be adequate for deriving other rules in addition to the one needed. The kind and amount of guidance given would presumably help to determine the precise nature of the derivation rule so acquired. On the other hand, if S̲ already knows the response, it is not likely that he will waste much time trying to find another way to determine that response. Under these conditions, the only way to get S̲ to adopt a secondary goal is to change the context. Presumably, the expository and guided discovery S̲s in this study learned the derivation rule because this appeared to be the desirable thing to do. Some such mechanism may prove crucial to any theory based on the rule construct and framed in the SFL

The obtained sequencing result may also have important practical implicatiins, as will be attested to by any junior high school mathematics teacher who has attempted to teach the "meaning" underlying the various computational algorithms after the children have already learned to compute. The children must effectively say to themselves something like, "I already know how to get the answer. Why should I care why the procedure works?" Similarly, drilling students in their multiplicati n facts before they know what it means to multiply, may interfere with their later learning what multiplication is. Let me make this point clear, because it is an important one. We are not saying that we should teach meaning first simply out of some sort of dislike for rote learning--for certain purposes rote learning may be quite adequate and the most efficient procedure to follow. What we are saying is that learning such things as how to multiply, without knowing what multiplication means, may actually make it more difficult to learn the underlying meaning later on.

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