REPORT RESUMES

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AN EXPERIMENTAL COURSE IN MATHEMATICS FOR THE NINTH YEAR. UNITS 10 AND 11, OPEN SENTENCES IN TWO VARIABLES AND RELATIONS AND FUNCTIONS.

N.Y. STATE EDUC. DEPT., ALBANY, BUR. OF SEC. CURR. DEV.

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THIS TEACHING GUIDE IS THE FOURTH OF FIVE EXPERIMENTAL EDITIONS CONTAINING MATERIALS AND METHODS FOR TEACHING A REVISED MATHEMATICS PROGRAM IN GRADE 9. BACKGROUND MATERIAL FOR TEACHERS AS WELL AS QUESTIONS AND ACTIVITIES FOR CLASSROOM PRESENTATIONS ARE PROVIDED IN THE CONTENT AREAS OF (1) OPEN SENTENCES IN TWO VARIABLES (UNIT 10) AND (2) RELATIONS AND FUNCTIONS (UNIT 11). UNIT 10 INCLUDES SECTIONS ON ALGEBRAIC SOLUTIONS, SOLUTION BY GRAPHING, AND SOLUTION OF INEQUALITIES. UNIT 11 INCLUDES SECTIONS ON RELATIONS, FUNCTIONS (ALGEBRAIC AND TRIGONOMETRIC), RANGE AND DOMAIN, GRAPHING RELATIONS AND FUNCTIONS, AND SLOPE AND INTERCEPT. (RP)

Units 10 and 11

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an experimental course in MATHEMATICS FOR THE NINTH YEAR

SE 003 803

THE UNIVERSITY OF THE STATE OF NEW YORK / THE STATE EDUCATION DEPARTMENT
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AN EXPERIMENTAL COURSE

IN

M A T H E M A T I C S

FOR THE

NINTH YEAR

Unit 10. Open Sentences in Two Variables

Unit 11. Relations and Functions

The University of the State of New York
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AN EXPERIMENTAL COURSE IN MATHEMATICS FOR THE NINTH YEAR

Mathematics 9X

CONTENTS

Page

Syllabus Outline - Mathematics 9X iv

Foreword vi

Unit 10. Open Sentences in Two Variables 270

Part 1. Background Material for Teachers 270
10.1 Introduction 270
10.2 Solving a System of Linear Equations by Substitution 270
10.3 Solving a System of Linear Equations by the Addition Method 273
10.4 Solving a System of Linear Equations by Graphing 274
10.5 Solving a System of Linear Inequalities by Graphing 276
10.6 Verbal Problems Involving Linear Equations in Two Variables 277

Part 2. Questions and Activities for Classroom Use 278
10.1 Introduction 278
10.2 Solving a System of Linear Equations by Substitution 278
10.3 Solving a System of Linear Equations by the Addition Method 284
10.4 Solving a System of Linear Equations by Graphing 288
10.5 Solving a System of Inequalities by Graphing . . . 294
10.6 Verbal Problems Involving Linear Equations in Two Variables 301
Unit Test 306

Unit 11. Relations and Functions 310

Part 1. Background Material for Teachers 310
11.1 Introduction 310
11.2 Relations 310
11.3 Functions (algebraic and trigonometric) 312

Part 2. Questions and Activities for Classroom Use 316
11.1 Introduction 316
11.2 Relations 316
11.3 Functions (algebraic and trigonometric) 324
Unit Test 342

SYLLABUS OUTLINE

Mathematics 9X

Unit	Topics	Time Allotment (days)
	Optional topics are indicated by an asterisk (*).	
1.	Sets Sets (finite and infinite) Universe, subsets, null set Union and intersection of sets Disjoint sets Complement of a set Matching sets and one-to-one correspondence Euler circles and Venn diagrams Cartesian product of two sets Solution sets	5 - 6
2.	Algebraic Expressions Algebraic symbols Addition, subtraction, multiplication, and division of algebraic expressions Value of an expression	9 - 11
3.	The Set of Integers Properties of the natural numbers Operations in the set of integers Properties of the integers Absolute value	5 - 6
4.	Open Sentences Equations Identities Equations with no solution Inequalities Solution of equations Solving problems by use of equations Solution of inequalities Solving problems by use of inequalities Solution of equations and inequalities involving absolute value	30 - 35
5.	Algebraic Problems Formula problems Motion problems Value problems Mixture problems Business problems Work problems Geometric problems	25 - 30

6. The Set of Real Numbers — 9 - 11
The set of rational numbers
Irrational numbers
Properties of the real numbers
The real number line

7. Exponents and Radicals — 15 - 17
Non-negative exponents
Negative exponents
Operating with expressions containing exponents
Factoring and prime factorization
Equations in fractional form
Radicals
Simplification of radicals
Operating with expressions containing radicals
Fractional exponents

8. Polynomial Expressions — 10 - 12
Addition, subtraction, multiplication, and division of polynomial expressions
Factoring polynomial expressions

9. Quadratic Equations — 10 - 12
Solution by factoring
*Solution by completing the square
*Solution by quadratic formula
Graphing quadratic equations
Simple proofs

10. Open Sentences in Two Variables — 9 - 10
Algebraic solutions
(addition and subtraction of equations)
(substitution)
Solution by graphing
Solution of inequalities

11. Relations and Functions — 7 - 9
Relations
Functions (algebraic and trigonometric)
Range and domain
Graphing relations and functions
Slope and intercept

*12. Further Study of Trigonometric Functions
The unit circle and the line x = 1
Sine, cosine, and tangent defined in terms of unit circle and line x = 1

FOREWORD

In April 1961, an advisory committee on secondary school mathematics convened at the Department to discuss the direction that secondary mathematics curriculum revision should take. This committee consisted of college and secondary school teachers, supervisors, administrators, and a consultant from one of the national curriculum programs. As a result of this meeting, the recommendation was made that a revision of the mathematics 7-8-9 program be undertaken immediately.

This publication represents the fourth of a series of experimental units for a course in mathematics for the ninth grade. The other three consist of units 1-4, 5-7, and 8-9, respectively. The final publication, consisting of material for optional unit 12 and a glossary, is still in preparation and will be distributed as soon as it is printed.

The materials in the 9X experimental syllabus are based upon the foundations laid in the 7X and 8X experimental syllabuses. Therefore, it is to be understood that the 7X and 8X experimental courses are a prerequisite to the 9X experimental course. As in the 7X and 8X syllabuses, the chief emphasis is placed upon the understanding of basic mathematical concepts as contrasted with the all-too-frequently used program in which the mechanics of mathematics receives the greatest stress. The general approach and content used is that agreed upon by leading mathematical authorities as the most desirable. In the actual teaching of the program major emphasis is placed upon the "discovery process." The principal function of the teacher is to carefully set the stage for learning in an organized fashion such that the pupils will "discover" for themselves the fundamental concepts involved.

The materials in the mathematics 7X, 8X, and 9X experimental syllabuses include much of what today are called the basic ideas and concepts of mathematics. These concepts are those which the pupils will use throughout their study in mathematics. With this material the teacher should be able to aid the pupils to see the beauty of mathematics in terms of the fundamental structure found in mathematical systems. The important unifying concepts included in the new course of study for the ninth grade are:

- Algebraic Expressions and Open Sentences
- Analysis of Algebraic Problems
- The Set of Real Numbers
- Properties of Exponents and Radicals
- Operations with Polynomial Expressions
- Quadratic Equations
- Open Sentences in Two Variables
- Relations and Functions
- Trigonometric Functions

A new mathematical curriculum is not the sole answer to the improvement of mathematics instruction. Most important perhaps is the method of presenting the material. If the teacher develops lesson plans that will allow the pupils to discover concepts for themselves, the teaching and learning of mathematics will become excitingly different and no longer remain the dissemination of rules and tricks.

A special committee was formed to review the 9X syllabus and to make recommendations for the writing of materials. This committee consisted of the following: David Adams, Liverpool High School; Benjamin Bold, Coordinator of Mathematics, High School Division, New York City Board of Education; Mary Challis, Plattsburgh High School; Francis Foran, Garden City Junior High School; Eleanor Maderer, Coordinator of Mathematics, Board of Education, Utica; William Mooar, Benjamin Franklin Junior High School, Kenmore; Verna Rhodes, Corning Free Academy; Leonard Simon, Curriculum Center, New York City; Joan Vodek, Chestnut Hill Junior High School, Liverpool; Frank Wohlfort, Coordinator of Mathematics, Junior High School Division, New York City Board of Education.

The materials for the 9X syllabus were written by Charles Burdick, coordinator and teacher of mathematics, Oneida Junior High School, Schenectady. The project has been developed under the joint supervision of this Bureau and the office of Frank Hawthorne, Chief, Bureau of Mathematics Education, who guided the planning. Aaron Buchman, associate in mathematics education, reviewed and revised the original manuscript. Mrs. Evelyn Gutekunst, temporary associate in mathematics, assisted in checking the material and preparing original diagrams for the artist. Herbert Bothamley, acting as temporary curriculum associate, edited and prepared the final manuscript for publication.

Gordon E. Van Hooft
Chief, Bureau of Secondary Curriculum Development

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Director, Curriculum Development Center

UNIT 10: OPEN SENTENCES IN TWO VARIABLES

Part 1. Background Material for Teachers

10.1 INTRODUCTION

This unit is entitled "Open Sentences in Two Variables" rather than "Equations in Two Variables" because it includes the study of concepts pertaining to both equations and inequalities. Included among the topics pertaining to equations are the solution of a system of two linear equations in two variables using the substitution method, the addition method, the graphing method, and the application of these methods to solving verbal problems. Topics pertaining to inequalities have been limited to the solution of two inequalities involving two variables by the graphing method. In presenting each of the three methods for solving a system of two linear equations, emphasis has been placed upon the principle that the solution set may consist of one pair of elements, an infinite number of pairs of elements, or no elements, depending upon whether or not the system of linear equations consists of consistent, dependent, or inconsistent equations. A further discussion of linear functions and graphing linear functions is contained in unit 11.

10.2 SOLVING A SYSTEM OF LINEAR EQUATIONS BY SUBSTITUTION

A linear equation in two variables such as $2x + 3y = 12$ has a solution set consisting of an infinite number of elements each of which is an ordered number pair. A few such elements in the solution set are the pairs $(0, 4)$, $(6, 0)$, $(1, \frac{10}{3})$, and $(\frac{9}{2}, 1)$, where the first number in the pair is the value of x, and the second is the corresponding value of y. Substituting any real number for x in the given equation will result in an equation the solution to which will give a value for y. Substituting any real number for y will result in an equation the solution to which will give a value of x. The solution set will therefore be an infinite set of ordered number pairs.

However, if a second restriction is placed upon the replacement set of the variables, the solution set of the equation may consist of a single pair of numbers. For example, if the second restriction that $x = y + 1$ is placed upon the replacement set of the variables in the equation $2x + 3y = 12$, then there is only one pair of numbers that will satisfy the second restriction and also make the equation $2x + 3y = 12$ a true statement. One method of determining this pair of numbers is to substitute $y + 1$ for x in the first equation. The resulting equation is $2(y + 1) + 3y = 12$. The solution set of this equation is $\{2\}$. If y equals 2, the value of x can be determined by substituting 2 for y in either the equation $2x + 3y = 12$ or in the equation $x = y + 1$. Regardless of which equation is used for the substitution, x equals 3. Therefore, the solution set of the system of equations $2x + 3y = 12$ and $x = y + 1$ is $\{(x, y) \mid x = 3, y = 2\}$, which can be written as a set containing only one ordered number pair as a member, thus $\{(3, 2)\}$.

The standard form of a linear equation in two variables is $Ax + By = C$, where A, B, and C represent any real numbers. If the system of given equations is

$$\begin{aligned} 2x + 3y &= 12 \\ x - y &= 1 \end{aligned}$$

there are four possible procedures that may be used to solve this system of equations by the substitution method.

(1) The first equation is solved for x and this expression is substituted for x in the second equation to obtain the value of y, and finally $\{(x, y)\}$.
(2) The first equation is solved for y and this expression is substituted for y in the second equation to obtain the value of x, and finally $\{(x, y)\}$.
(3) The second equation is solved for x and this expression is substituted for x in the first equation to obtain the value of y, and finally $\{(x, y)\}$.
(4) The second equation is solved for y and this expression is substituted for y in the first equation to obtain the value of x, and finally $\{(x, y)\}$.

Many times one of these possible procedures is obviously easier to carry out than any of the other three. In the example above, it probably is easiest to solve the second equation for x and substitute this for x in the first equation in order to obtain the value of y, and finally the solution set $\{(x, y)\}$. The objective of the various procedures is to replace the given pair of simultaneous equations by an equivalent pair of simultaneous equations, the first containing only one letter and the second only the other letter, from which the solution set is at once evident. As was done in solving equations in one unknown, the axioms of algebra are used to derive the successive equivalent equations. The process is called solving the system of equations.

In solving a system of two linear equations in two variables, the solution set does not always consist of a single pair of numbers. Solving the system of equations $2x + 4y = 12$ and $x = 6 - 2y$ by the substitution method results in the equation $2(6-2y)+4y = 12$ if $6 - 2y$ is substituted for x in the first equation. This equation is equivalent to $12 - 4y + 4y = 12$, or $12 = 12$. This last equation is an identity. This means that the given system of equations is true for any value of y and therefore the solution set contains an infinite number of pairs of elements. The second equation in the pair of given equations did not place an additional restriction upon the replacement set of the variables. In fact, the second equation is simply a restatement of the first equation; the two equations are equivalent. The second equation is equivalent to $x + 2y = 6$. Multiplying both sides of this equation by 2 results in the first equation.

Two linear equations in two variables which are simply equivalent equations and which therefore have an infinite number of elements in the solution set, are called dependent equations. Such a system of equations is often very easy to recognize if both equations are given in standard form. One equation will be a multiple of the other equation. However, if a system of equations is expressed in a form such as $4 + 3(x + 2y) = 5x + y$ and $6x - 15y = 12$, it may not be easily recognized as a dependent system until the first equation is replaced by an equivalent equation in standard form. When the new set of equations $2x - 5y = 4$ and $6x - 15y = 12$ is examined, it is easily determined that the two equations are equivalent. The second equation is obtained by multiplying the first equation by 3; therefore, they are equivalent.

We see that the solution set of a system of two linear equations in two variables may also be the null set. If one equation places such an additional restriction upon the replacement set of the variables of the other equation that no pair of numbers can possible satisfy both equations, the solution set is the null set. An example of such a system of equations is $2x + 4y = 18$ and $x + 2y = 10$. When the second equation is solved for x and the equal expression $10 - 2y$ is substituted for x in the first equation, the result is $2(10 - 2y) + 4y = 18$ so that $20 - 4y + 4y = 18$ and $20 = 18$. This is an internally inconsistent equation. Therefore there is no pair of numbers in the solution set of the original system of equations. The solution set is the null set. This can also be seen by an inspection of the equations without even solving the system. If the first equation is divided by 2, the equivalent system of equations is $x + 2y = 9$ and $x + 2y = 10$. If 10 and 9 are both equal to $x + 2y$, then 10 and 9 must be equal to each other. This is impossible. The given system of equations places restrictions which are impossible to satisfy simultaneously and so the solution set is the null set. Such a system of equations is called inconsistent. With practice, pupils can learn to recognize many such inconsistent systems of equations on sight.

A variation of the substitution method of solving systems of equations is the comparison method. In this method, both equations are solved for the same variable. For example, $x + 7y = 13$ and $2x + 5y = 8$ are replaced by the equivalent equations $x = 13 - 7y$ and $x = \frac{8 - 5y}{2}$. Expressions equal to the same expression are equal to each other so $13 - 7y = \frac{8 - 5y}{2}$. The solution of this equation is $y = 2$, and if this value of y is substituted in either of the given equations, it follows that x = -1. The pair of equations, y = 2 and x = -1 is equivalent to the given pair of equations so the required solution set is {(-1, 2)}.

10.3 SOLVING A SYSTEM OF LINEAR EQUATIONS BY THE ADDITION METHOD

A second procedure for solving a system of two linear equations in two variables involves adding the two equations. The system 2x + 3y = 35 and -2x + 5y = 13 may be solved by adding -2x + 5y to the left side of the first equation and adding 13 to the right side of the first equation. Since it is indicated that for certain replacement -2x + 5y = 13 are different expressions for the same number, then the same number, in effect, is being added to each side of the first equation. The addition results in the equation 8y = 48, which yields y = 6 as one of a pair of equivalent equations. The value of y obtained is then substituted for y in either of the given equations to obtain $x = 8\frac{1}{2}$ as the second of the pair of equivalent equations. The solution set is then $\{(8\frac{1}{2}, 6)\}$.

Before the addition of the equations is performed, it may be necessary to replace the system of equations by an equivalent system since the coefficient of one of the variables in one of the equations must be the additive inverse of the coefficient of that same variable in the other equation. This condition is necessary so that the equation resulting from the addition will contain only one variable.

This condition is not met in the system 2x + 2y = 12 and 2x + 5y = 27. However, if either equation is multiplied by -1, the coefficient of the x term in one equation will be the additive inverse of the coefficient of the x term in the other equation. The system can then be solved by the addition method.

Sometimes it is necessary to multiply each equation by a different integer to yield equivalent equations in which the coefficients of one of the variables are additive inverses of each other. For example, the system 3x + 7y = 76 and 2x + 5y = 54 can be solved by multiplying the first equation by 2 and the second equation by -3. The resulting equivalent equations are 6x + 14y = 152 and -6x - 15y = -162 which can be solved by the addition method.

Another possible procedure for solving this system is to multiply the first equation by 5 and the second equation by -7. The resulting equivalent equations are 15x + 35y = 380 and -14x - 35y = -378 so that x = 2. From this, 4 + 5y = 54 and y = 10. The solution set is {(2, 10)}.

10.4 SOLVING A SYSTEM OF LINEAR EQUATION BY GRAPHING

The arrangement of the questions and activities in this unit is based upon the assumption that the pupils have mastered the basic concepts of coordinate geometry presented in unit 3 of the Mathematics 8X course. The first questions and activities have as their purpose a brief review of the coordinate geometry concepts. Included among these concepts are the following:

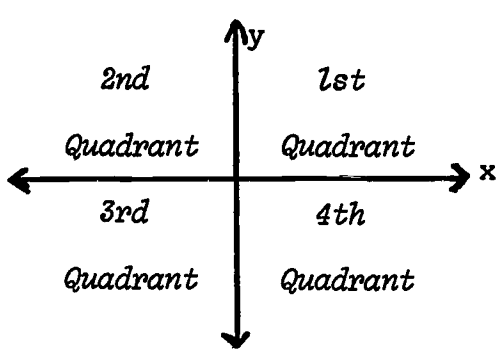
(1) The equation $x = 1$ identifies the set of all the points on the number plane that are one unit to the right of the y-axis. This set of points forms a straight line parallel to the y-axis. The equation $x = n$, where n is any positive real number, identifies a straight line parallel to the y-axis and n number of units to the right of it. If n is negative, the line is $|n|$ number of units to the left of the y-axis. If n is zero, the line is coincident with the y-axis.

(2) The equation $y = 1$ identifies the set of all points on the number plane 1 unit above the x-axis and parallel to it. The equation $y = n$, where n is any positive real number, is a straight line above the x-axis, parallel to it and n number of units from it. If n is negative, the line is $|n|$ number of units below the x-axis. If n is zero, the line is coincident with the x-axis.

(3) Every point on the number plane can be identified by a pair of lines, the line parallel to the y-axis and the line parallel to the x-axis, such that each contains the given point. The point is the intersection of the two lines.

(4) If a point is the intersection of the graph of the equations $x = a$, $y = b$, the point may be identified by the ordered number pair (a, b). For example, a point at the intersection of the graphs of $x = 6$, $y = -3$ may be identified as (6, -3). The two numbers that form the ordered number pair are called coordinates of the point, the first coordinate being called the *abscissa* and the second being called the *ordinate*.

The x-axis and the y-axis together divide the number plane into four parts or quadrants. The quadrants are identified as follows:



Just as zero is neither in the positive part nor in the negative part of the number line, so each axis is not in any quadrant. The two axes form the boundaries and dividing lines between the quadrants but are not contained in any quadrant.

The variables x and y in the linear equation Ax + By = C may represent the abscissa and ordinate of a set of ordered number pairs which identify points on a number plane. Such an equation may therefore be associated with a set of points on a number plane. If A and B are not zero, the equation Ax + By = C represents an infinite set of points, and these points form a straight line. Every linear equation in two variables represents a straight line. Therefore, it is necessary to determine only two points which are identified by the equation to determine the straight line which the equation represents. If two points are determined from an equation and plotted, the straight line drawn through these two points is the straight line represented by the equation.

The two equations of a system of linear equations in two variables may represent lines whose intersection consists of one point, an infinite number of points, or no point. If the two straight lines are not coincident but they intersect, they can intersect at only one point. Such lines are represented by consistent simultaneous equations. The solution set of the equations consists of a single ordered number pair.

If the two lines are parallel and therefore do not intersect, their intersection is the null set. Equations representing such lines must be inconsistent equations.

If the two lines are coincident, the lines must be represented by equivalent equations. The intersection of two such lines consists of an infinite number of points and therefore the solution set of such a system of equations consists of an infinite number of ordered number pairs. Such a system consists of dependent equations.

The graphing method of solving systems of equations has one disadvantage: it is sometimes difficult to determine the coordinates of the point of intersection of the lines with a sufficient degree of accuracy. One exercise in the questions and activities requires that a system of simultaneous equations be graphed and the coordinates of the point of intersection be determined. The lines actually intersect at the point $\left(\frac{23}{24}, \frac{25}{24}\right)$. However, when a pupil graphs these equations with a regular pencil on ordinary graph paper, it is logical for him to read the point of intersection as being (1, 1). When he finds that the coordinates of this point do not satisfy the given equations, he soon realizes that he must solve the equations by some other method if he is to determine the exact solution set of the system of equations. Even though the graphing method does have this disadvantage, it is a very valuable method to master as it has important applications in higher mathematics and science courses, particularly in determining a first approximation to the solution set rapidly.

10.5 SOLVING A SYSTEM OF LINEAR INEQUALITIES BY GRAPHING

In this section, some of the concepts of graphing a system of equations are applied to graphing a system of inequalities. The questions and activities begin with a brief review of the basic concepts of graphing inequalities in one variable on a number line as presented in unit 3 of the Mathematics 8X course. The topic of graphing inequalities on a number plane is then introduced and this is extended to graphing a system of inequalities in two variables. Such a graph consists of the graph of both inequalities on the same axes. The set of points in the intersection of the two regions represents the solution set of the system of inequalities. The pupil is not required to show the solution set of a system of two inequalities in any way other than the graphical representation.

In attempting to plot an inequality, it is sometimes advisable to first plot the related equality. For example, the inequality $x > y$ may be graphed by first graphing the equality $x = y$. This line separates the number plane into two half planes, one of which is the graph of the inequality $x > y$. The half plane that forms the solution set can be determined by selecting any point in either half plane and determining whether or not its coordinates satisfy the inequality. If they do, that point is in the solution set of the given inequality. If they do not, that point is in the half plane which does not represent the solution set of the inequality and so the other half plane is chosen. To solve the inequality $x > y$, the line $x = y$ is graphed. Any point such as (2, 3) is selected and its coordinates substituted for the variables in $x > y$. This results in the inequality $2 > 3$, which is a false statement. Therefore, the point (2, 3) is not in the solution set of the inequality. It can then be concluded that the half plane below the line $x = y$ represents the solution set.

In the finished graph the line representing $x = y$ is drawn as a dashed line, indicating that all the points below and right up to the line are in the solution set but that the points on the line are not in the solution set. The graph of $x \geq y$ would have a solid line representing $x = y$ as the boundary of the solution set to indicate that all the points on the line are also included in the region representing the solution set.

The graph of an inequality containing the absolute value of a variable may consist of the intersection of two sets or it may consist of the union of two sets. The solution set of the inequality $|x| > 3$ consists of the union of the two sets $x > 3$ and $x < -3$. The solution set of the inequality $|x| < 2$ consists of the intersection of the two sets x < 2 and x > -2. This intersection may be expressed as -2 < x < 2. If k is non-negative, the solution set of the inequality |x| > k is the union of the sets x > k and x < -k. The solution set of |x| < k is the set -k < x < k. If k is negative, the solution set of the inequality |x| > k is the set of real numbers, and the solution set of the inequality |x| < k is the null set.

10.6 VERBAL PROBLEMS INVOLVING LINEAR EQUATIONS IN TWO VARIABLES

Many of the verbal problems which the pupils have learned to solve by the use of one equation in one variable may also be solved by the use of two equations in two variables. After the pupils have mastered the concepts of solving a system of two linear equations in two variables they then have a greater freedom in selecting a procedure to use in solving many verbal problems.

There are many problems that are best solved by the use of two equations in two variables. The verbal problems included in this unit are those that are best solved by this method. Although one method of solution is indicated for each problem, this should not be construed as indicating that this is the only acceptable method. There is a variety of ways of expressing unknowns and relationships in solving problems, and any method that is correct mathematically should be accepted.

Teacher Notes

UNIT 10: OPEN SENTENCES IN TWO VARIABLES

Part 2. Questions and Activites for Classroom Use

10.1 INTRODUCTION

The concepts presented in this unit apply to the solution of any system of two linear equations in two variables. There is no restriction requiring the coefficients or constant terms to be rational numbers. The coefficients and constant terms may be any real number. However, the questions and activities have been limited to exercises involving only rational numbers. The teacher may, if he desires, supplement these exercises with problems involving linear equations containing irrational numbers as coefficients and constant terms.

10.2 SOLVING A SYSTEM OF LINEAR EQUATIONS BY SUBSTITUTION

Concept: Standard form of a linear equation in two variables.

(1) A first degree equation in the form Ax + By = C, where A, B, and C are any real numbers, is called a linear equation in two variables in standard form.

Express each of the following in standard form.

(a) $6x = 3(y + 9)$ (c) $5(y - 5x) + 17 = 12 - 3(x + 9y)$

(b) $\frac{2x - 5}{2y + 9} = -7$

Answers: (a) $6x + (-3)y = 27$ or $6x - 3y = 27$

(b) $2x + 14y = -58$

(c) $-22x + 32y = -5$

Concept: Substitution method of solving a system of equations.

(2) (a) *Solve the equation $2x + 3y = 12$ for x.*

Answer: $x = \frac{12 - 3y}{2}$

(b) *As real numbers are substituted for y in the equation $x = \frac{12 - 3y}{2}$ does each determine a numerical value to replace x?*

Answer: Yes

(c) *Solve the equation $2x + 3y = 12$ for y.*

Answer: $y = \frac{12 - 2x}{3}$

(d) *As real numbers are substituted for x in the answer (c) does each determine a numerical value to replace y?*

Answer: Yes

(e) *Describe the solution set of the equation $2x + 3y = 12$.*

Answer: For every real number y there is a real number x that will satisfy the equation and for every real number x there is a real number y that will satisfy the equation. The solution set is an infinite set of ordered pairs of numbers.

(3) (a) *If in the equation $2x + 3y = 12$, an additional restriction is placed upon the variables such that $y = 2x$, how can the value of x be determined?*

Answer: If 2x is substituted for y in the equation 2x + 3y = 12, this resulting equation can be solved for x.

(b) *Perform this substitution and determine x.*

Answer: $2x + 3(2x) = 12$

$$x = \frac{3}{2}$$

(c) *Once the value of x has been determined, how can the value of y be determined?*

Answer: $\frac{3}{2}$ can be substituted for x in either of the given equations and the resulting equation solved for y.

(d) *Perform this substitution and solve for y.*

Answer: $y = 2x$

$$y = 2\left(\frac{3}{2}\right)$$

$$y = 3$$

(e) *What simpler system of equations has the same solution set as the system of equations $y = 2x$ and $2x + 3y = 12$?*

Answer: $x = \frac{3}{2}$ and $y = 3$

(f) *Express the solution set as an ordered number pair.*

Answer: $\{(\frac{3}{2}, 3)\}$

(4) *Solve each of the following by performing the substitution as indicated in the previous question.*

(a) $2x + y = 4$, $y = 1 - x$ (b) $4x + y = -1$, $y = 4x - 7$ (c) $-3x + y = -2$, $x = 4y - 3$

Answers:

(a) $x = 3, y = -2$ (b) $x = \frac{3}{4}, y = -4$ (c) $x = 1, y = 1$

(5) (a) *Solve the system of equations $3x + 2y = 17$ and $x + 2y = 5$ by the substitution method, first solving the second equation for x and then substituting the result in the first equation.*

Answer:

$x = 5 - 2y$ $\qquad 3(5 - 2y) + 2y = 17$ $\qquad x + 2(-\frac{1}{2}) = 5$

$y = -\frac{1}{2}$ $\qquad x = 6$

The solution set is $\{(6, -\frac{1}{2})\}$.

(b) *Solve this same system of equations by the substitution method by first solving the second equation for y and then substituting the result in the first equation.*

Answer:

$y = \frac{5 - x}{2}$ $\qquad 3x + 2(\frac{5 - x}{2}) = 17$ $\qquad y = \frac{5 - 6}{2}$

$x = 6$ $\qquad y = -\frac{1}{2}$

The solution set is $\{(6, -\frac{1}{2})\}$.

(6) *Solve the system of equations $3x + 4y = 22$ and $2x - 2y = -4$ by use of the substitution method, using in turn, each of the following procedures.*

(1) First solve the second equation for x and substitute this in the first equation.

(2) First solve the second equation for y and substitute this in the first equation.

(3) First solve the first equation for x and substitute this in the second equation.

(4) First solve the first equation for y and substitute this in the second equation.

Is the solution set the same regardless of which of the above procedures is followed?

Answer: Yes. Regardless of which procedure is followed, the equivalent system $x = 2$, $y = 4$ is obtained and the solution set is $\{(2, 4)\}$.

(7) *Solve each of the following systems of equations by use of the substitution method, using any one of the four possible procedures described in the previous question.*

(a) $2x + 5y = -11$
$4x - 2y = -10$

(b) $3x + 5y = 5$
$2x + 3y = 3$

(c) $3k - 5m = -34$
$2k + m = 12$

(d) $2a - 5b - 33 = 0$
$b = a - 9$

Answers:

(a) $x = -3$, $y = -1$

(b) $x = 0$, $y = 1$

(c) $k = 2$, $m = 8$

(d) $a = 4$, $b = -5$

Concept: Inconsistent and consistent systems of equations.

(8) *Solve the system of equations $4x + 8y = -4$ and $x + 2y = 3$ by the substitution method, and describe the solution set of this system.*

Answer: $x = 3 - 2y$

$$4(3 - 2y) + 8y = -4$$
$$12 - 8y + 8y = -4$$
$$12 = -4$$

All values of x and y which make $x + 2y = 3$ true, make $4x + 8y = -4$ a false or internally inconsistent statement. The solution set of the system is the null set.

(9) A system of equations whose solution set is the null set is called an inconsistent system of equations. If the solution set consists of exactly one ordered pair of numbers, the system is consistent and the equations are called independent.

Solve each of the following systems of equations and indicate whether or not each system is inconsistent or consistent.

(a) $3a - 4b = 12$
$6a - 8b = 11$

(b) $3x - 4y = -14$
$6x - 7y = -28$

(c) $-8c - 12d = 33$
$6c + 9d = -24$

Answers:
(a) Inconsistent (b) {(-2, 2)}, consistent
(c) Inconsistent

Concept: Dependent systems of equations.

(10) (a) *What is the result when the first equation of the system $x + 2y = -3$ and $4y = -6 - 2x$ is solved for x and this expression for x is substituted for x in the second equation?*

Answer: $x = -3 - 2y$

$$4y = -6 - 2(-3 - 2y)$$
$$4y = -6 + 6 + 4y$$
$$4y = 4y$$

An identity results.

(b) *In the previous system of equations if a set of values for x and y is selected so that the first equation is a true statement, and these values are substituted in the second equation, what conclusion can be reached concerning the second equation?*

Answer: Any set of ordered pairs of values that will make the first equation true will also make the second equation true.

(c) *How many different ordered pairs of numbers are there that will make the first equation true?*

Answer: An infinite set of such ordered pairs of numbers.

(d) *Describe the solution set of the system of equations.*

Answer: The solution set is an infinite set.

(e) *Write both of the given equations in standard form.*

Answer:
$$x + 2y = -3$$
$$2x + 4y = -6$$

(f) *What operation can be performed on the first equation which will result in the second equation?*

Answer: If both sides of the first equation are multiplied by 2, the resulting equation is identical with the second given equation.

(g) *Are the equations $x + 2y = -3$ and $2x + 4y = -6$ really different relations between x and y?*

Answer: No. They are different forms which can be derived from the same equation.

(h) *Compare the solution set of the first equation with the solution set of the second equation.*

Answer: The solution set of the first equation must be identical with the solution set of the second equation as the equations are equivalent equations.

(11) *A system of two equations consisting of equivalent equations is called a dependent system. How can a dependent system of equations be recognized?*

Answer: Two equations are dependent if one is a multiple of the other or if an identity results when one equation is solved for one variable and the expression for this variable is substituted for it in the second equation.

(12) *Which of the following are systems of dependent equations?*

(a) $2x + y = 3$
$y - 5 = -4x$

(b) $3x = 7 - 5y$
$20y = -12x + 28$

(c) $3x + 2y = 4$
$9x + 8y = 14$

(d) $12a - 36 = -24b$
$-27 + 18b = -9a$

Answer: (b) and (d) are systems of dependent equations.

Concept: Comparison method of solving a system of equations.

(13) (a) *Solve each of the equations $x + 2y = 11$ and $3x + 4y = 23$ for x.*

Answer: $x = 11 - 2y$ and $x = \frac{23 - 4y}{3}$

(b) *Two expressions equal to the same expression are equal to each other. How can this principle be applied to the equation in (a)?*

Answer: If $x = 11 - 2y$ and $x = \frac{23 - 4y}{3}$, then

$11 - 2y = \frac{23 - 4y}{3}$.

(c) *Solve the equation in (b) for y and substitute its value into either of the given equations to determine the value of x.*

Answer: $y = 5$, $x = 1$ or the solution set is $\{(1, 5)\}$.

(14) The method just described for solving a system of equations is called the comparison method.

Solve each of the following by the use of the comparison method.

(a) $2a = b - 24$
$b = 18 - a$

(b) $w + t = 3w - t + 8$
$7 - t = 13 + w$

(c) $\frac{5x + y}{4} = 4$
$2x = 7y - 38$

Answers:

(a) $a = -2$, $b = 20$ (b) $w = -5$, $t = -1$
(c) $x = 2$, $y = 6$

10.3 SOLVING A SYSTEM OF LINEAR EQUATIONS BY THE ADDITION METHOD

Concept: Basis for the addition method.

(1) (a) *Consider the system of equations $x + y = 12$ and $x - y = 4$. If $x - y$ is added to the left side of the first equation and 4 is added to the right side of the first equation, is such an operation correct mathematically?*

Answer: For the correct values of x and y, the quantity $x - y$ and 4 are different symbols for the same number, so the same number, in effect, is being added to each side of the first equation.

(b) *What equation results when this addition is performed?*

Answer: $2x = 16$

(c) *Solve this equation for x and substitute this value of x in either of the given equations to determine the value of y.*

Answer: $x = 8$, $y = 4$ or the solution set is $\{(8, 4)\}$.

(2) The method of solving a system of equations by adding in this manner is called the addition method.

Solve each of the following systems of equations by the addition method.

(a) $2x + 4y = 48$
$-2x - 8y = -92$

(b) $-a + 3b = 8$
$a - b = -4$

(c) $p + q = 8$
$p - q = -6$

(d) $2x - 3y = 1$
$2x + 3y = 5$

Answers:

(a) $x = 2, y = 11$
(b) $a = -2, b = 2$
(c) $p = 1, q = 7$
(d) $x = \frac{3}{2}, y = \frac{2}{3}$

Concept: Subtracting or multiplying one equation by -1.

(3) (a) *What equation results when the addition method is applied to the system of equation $3h - 4p = 2$ and $3h - 2p = 4$?*

Answer: The equation $6h - 6p = 6$ results.

(b) *Can this equation be solved to give a value of one of the variables that is in the solution set of the given system of equations?*

Answer: No

(c) *The coefficients of one of the variables must meet what condition before a system of two equations can be solved by the addition method?*

Answer: The coefficient of one variable in one equation must be the additive inverse of the coefficient of that same variable in the other equation.

(d) *What operation can be performed on one of the equations of the above given pair that will enable the system to be solved by the addition method?*

Answer: If either of the given equations is multiplied by -1, the result is an equivalent equation, which forms with the other given equation, a system which can be solved by the addition method.

(e) *Solve the given system of equations in the manner just described.*

Answer:
$3h - 4p = 2$
$3h - 2p = 4$

$$\begin{array}{r} -3h + 4p = -2 \\ \underline{3h - 2p = 4} \\ 2p = 2 \\ p = 1 \end{array}$$

$3h - 2 = 4$
$h = 2$

The solution set is $\{(h, p) \mid h = 2, p = 1\}$.

(4) *Solve each of the following systems of equations, multiplying in each pair each side of one of the equations by -1 where necessary so that the system may be solved by the addition method.*

(a) $x + 3y = 18$ (b) $3a - b = 1$ (c) $2k - 3n = -14$
$x + y = 10$ $3a - 5b = -19$ $2k + 9n = 34$

Answers:

(a) $x = 6, y = 4$ (b) $a = 2, b = 5$ (c) $k = -1, n = 4$

Concept: Multiplying one of the equations by a constant other than -1.

(5) (a) *What operation may be performed on one of the equations in the system $6x + 5y = 52$ and $-3x + 2y = 10$ so that the coefficients of one of the variables in the pair of equivalent equations will be additive inverses of each other?*

Answer: If the first equation is left unaltered or multiplied by one and the second equation is multiplied by 2, the coefficients of the x terms in the equivalent equations will be additive inverses of each other.

(b) *Solve this system of equations using this procedure.*

Answer: $x = 2, y = 8$ or the solution set is $\{(2, 8)\}$.

(6) *For each of the following systems of equations, indicate what operation performed on one of the equations of the given pair will enable the system to be solved by the addition method.*

(a) $x + 6y = 7$
$3x - 5y = -2$

(b) $7c - 3h = 10$
$5c + 6h = 56$

(c) $17r + 12s = 122$
$13r - 3s = 142$

(d) $k + 2q = -6$
$10k + 3q = -26$

Answers:

(a) Multiply the first equation by -3. Then $x = 1, y = 1$.
(b) Multiply the first equation by 2. Then $c = 4, h = 6$.
(c) Multiply the second equation by 4. Then $r = 10, s = -4$.
(d) Multiply the first equation by -10. Then $k = -2, q = -2$.

Concept: Multiplying each equation by a different constant.

(7) (a) *Is there any integer such that multiplying one of the equations in the system $3x + 2y = 24$ and $2x + 3y = 26$ by that integer will result in a system of equations that can be solved by the addition method?*

Answer: There is no such integer.

(b) *Is there a set of two integers such that when the first equation is multiplied by one of the integers and the second equation is multiplied by the other integer, the resulting equivalent equations can be solved by the addition method?*

Answer: If the first equation is multiplied by 2 and the second equation by -3, the resulting equivalent equations can be solved by the addition method. One of the other possibilities is to multiply the first equation by -3 and the second equation by 2.

(c) *Solve this system of equations using this procedure.*

Answer: $x = 4$, $y = 6$ or the solution set is $\{(4, 6)\}$.

(8) *Solve each of the following systems of equations, first multiplying each of the equations by an appropriate integer so that the system can be solved by the addition method.*

(a) $3a + 5b = -21$
$2a - 3b = 24$

(b) $5x - 7y = -7$
$3x - 9y = -33$

(c) $-4w + 5x = -31$
$5w - 7x = 41$

(d) $.2x - .5y = -3.4$
$.7x + .3y = 8.6$

Answers:

(a) $a = 3$, $b = -6$
(b) $x = 7$, $y = 6$
(c) $w = 4$, $x = -3$
(d) $x = 8$, $y = 10$

(9) *Solve each of the following systems of equations using any one of the methods described previously.*

(a) $2x = -7 + 3y$
$4x = 5y - 9$

(b) $$\frac{8 + x}{3} - \frac{2x - y}{4} = \frac{5y - 2x - 1}{6}$$
$$\frac{3y - 4}{2} + \frac{5x - 2}{6} = \frac{x + 9y + 2}{6}$$

(c) $x + 3y + 17 = 3(4x - y - 1)$
$5(5x + 3y - 2) = 8x + 32y - 10$

Answers:

(a) $x = 4$, $y = 5$ (b) $x = 4$, $y = 6$ (c) $x = 4$, $y = 4$

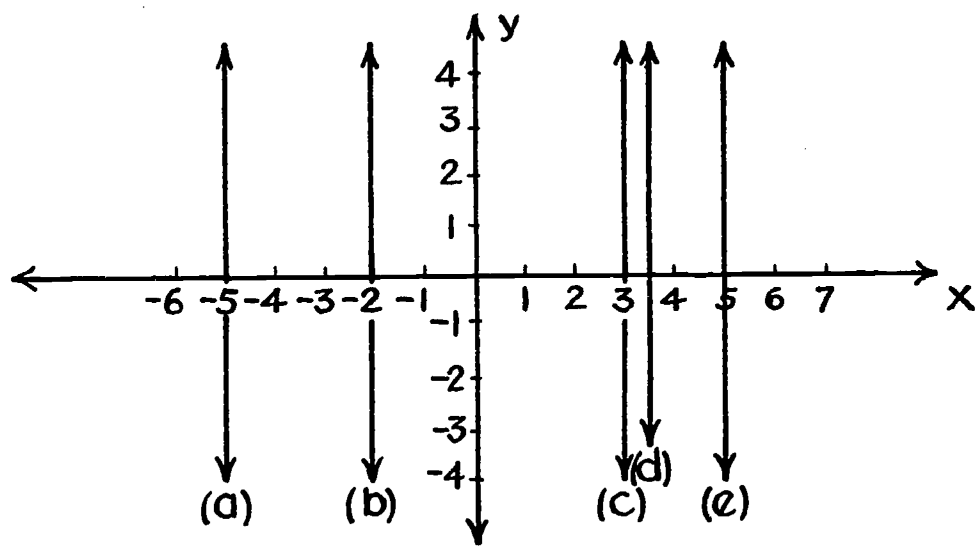
10.4 SOLVING A SYSTEM OF LINEAR EQUATIONS BY GRAPHING

Concept: Ordered number pairs.

(1) *On a number plane in which the coordinates of points are given with reference to an x-axis and a y-axis intersecting at right angles, describe the geometric configuration formed by the set of all points satisfying the equation $x = 1$.*

Answer: The set of all points satisfying the equation x = 1 is a straight line parallel to and one unit to the right of the y-axis.

(2) *Identify each of the lines drawn on the number plane below.*



Answers: (a) x = -5 (b) x = -2 (c) x = 3 (d) $x = 3\frac{1}{2}$ (e) x = 5

(3) *Describe the line formed by the set of all points on the number plane that satisfy the equation $y = 5$.*

Answer: The set of all points satisfying the equation y = 5 is a straight line parallel to and 5 units above the x-axis.

(4) *Identify each of the lines drawn on the number plane on the following page.*

Answers:

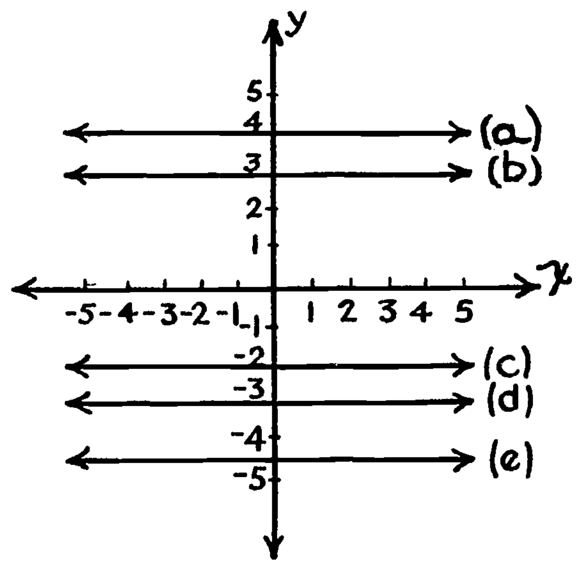
(a) y = 4

(b) y = 3

(c) y = -2

(d) y = -3

(e) $y = -4\frac{1}{2}$



(5) (a) *Is every point on the number plane contained on a line parallel to the y-axis?*

Answer: Yes

(b) *Is every point on the number plane contained on a line parallel to the x-axis?*

Answer: Yes

(c) *Of what does the intersection of one line parallel to the y-axis and one line parallel to the x-axis consist?*

Answer: A single point

(d) *How can any point on a number plane be identified?*

Answer: By identifying the line parallel to the y-axis and the line parallel to the x-axis that intersect at the given point.

(e) *How can the point at the intersection of the lines $x = 3$ and $y = 5$ be denoted?*

Answer: This point can be denoted as (3, 5).

(f) *What concept may be associated with the ordered number pair (a, b) where a and b are any real numbers?*

Answer: The ordered number pair (a, b) is the point of intersection of the lines x = a, and y = b.

(g) *What is meant by the terms coordinates, abscissa, and ordinate?*

Answer: The numbers a and b of the ordered number pair (a, b) are called the coordinates of the point; a is the abscissa and b is the ordinate.

(6) (a) *The intersecting x-axis and y-axis divide the number plane into how many parts?*

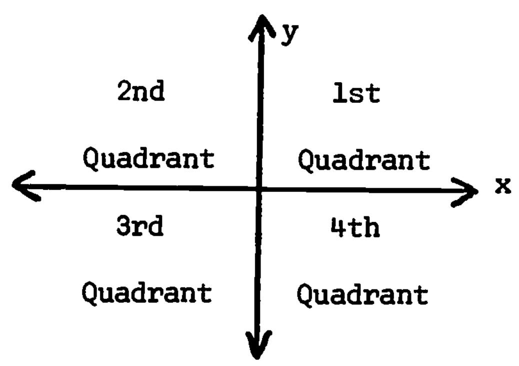
Answer: Four

(b) *What name is given to one such part?*

Answer: Quadrant

(c) *On a diagram show how each of the four quadrants of the number plane is identified.*

Answer:



(d) *Is the point on a number line which represents the number zero in the positive part of the number line or in the negative part?*

Answer: The point which represents zero is neither in the positive part nor in the negative part.

(e) *The points on the x-axis and on the y-axis are in which, if any, quadrant?*

Answer: The points on the axes are not in any quadrant.

(7) *Identify which quadrant, if any, each of the following points is located:*

(a) (3, -4) (c) (-4, 3) (e) (0, -2)
(b) (10, 12) (d) (-8, -6)

Answers: (a) 4th (b) 1st (c) 2nd (d) 3rd
(e) No quadrant

Concept: Graphing a linear equation.

(8) (a) Pairs of corresponding values of the variables x and y in the linear equation Ax + By = C can represent the abscissa and ordinate respectively of ordered number pairs that represent points on a plane.

How many different ordered number pairs exist that satisfy the equation $x = y$?

Answer: An unlimited number of ordered number pairs

(b) *Determine several ordered number pairs that satisfy the equation $x = y$ and graph the points on a number plane. What geometric configuration would the set of all points whose coordinates satisfy this equation form?*

Answer: A straight line

(c) *Graph several points whose coordinates satisfy the equation $x + y = 12$. What geometric configuration would be formed by the set of all points whose coordinates satisfy this equation?*

Answer: A straight line

(d) *Write a linear equation in the form $Ax + By = C$, and graph several points whose coordinates satisfy the equation. Are all the graphed points on the same straight line?*

Answer: Yes

(9) (a) *Given two points on a number plane, how many straight lines exist that contain both points?*

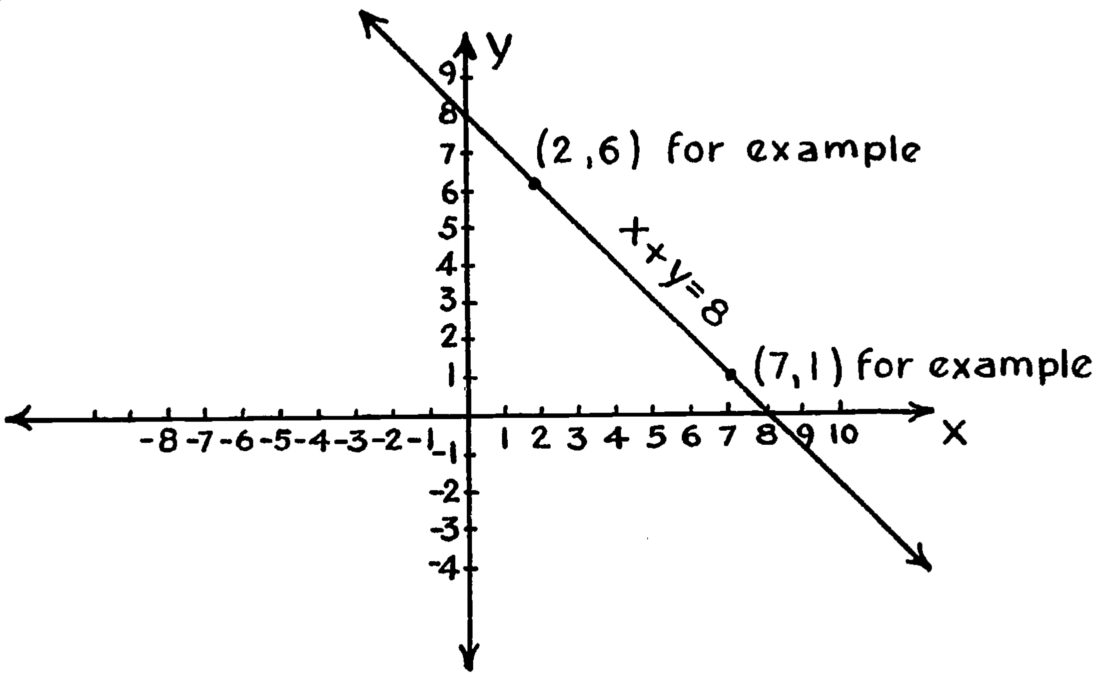
Answer: One and only one straight line

(b) *Assuming that the set of all points whose coordinates satisfy the equation $Ax + By = C$ form a straight line, how many ordered number pairs must be determined in order to determine the straight line?*

Answer: Two ordered number pairs

(c) *Determine two points whose coordinates satisfy the equation $x + y = 8$, graph the points, and draw a line through the points to represent the set of all points whose coordinates satisfy this equation.*

Answer:

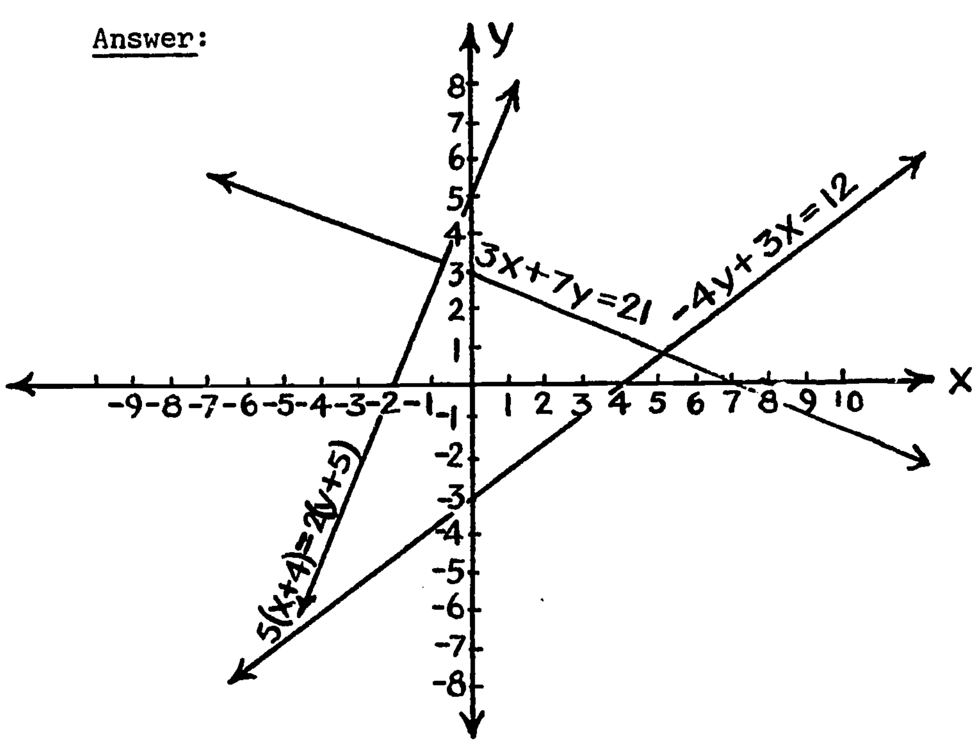


(10) *Plot on one graph the three lines representing respectively the three equations below, and label each line with the proper equation.*

(a) 3x + 7y = 21 (c) 5(x + 4) = 2(y + 5)

(b) -4y + 3x = 12

Answer:



(11) (a) *The coordinates of which of the following points satisfy the equation* $2x + 3y = 6$: *(-3, 4), (3, 0), (5, 6), (3, 2), (0, 2)?*

Answer: (-3, 4), (3, 0), (0, 2)

(b) *Which of the following sets of coordinates satisfy both equations* $x + 4y = 16$ *and* $2x + 3y = 17$: *(0, 4), (3, 4), (4, 3), (7, 1), (16, 0)?*

Answer: (4, 3)

(c) *If its coordinates satisfy both equations, what does the point (4, 3) represent?*

Answer: The point (4, 3) is the intersection of the two lines represented by the given equations.

(12) *If the equations* $Ax + By = C$ *and* $Dx + Ey = F$, *where A, B, C, D, E, and F are any real numbers, represent two straight lines, describe the three different types of sets which could represent the intersection of two such lines.*

Answer: If the two straight lines are parallel, their intersection will be the null set. If the two straight lines are coincident, their intersection will be an infinite set of points. If the two straight lines are not coincident, but intersect, their intersection will consist of a single point.

(13) (a) *If a system of equations is graphed and the two lines intersect at one point, what is the name given to such a system of equations?*

Answer: A system of consistent equations

(b) *If a system of equations is graphed and the equations are parallel so that the solution set is the null set, what is such a system of equations called?*

Answer: A system of inconsistent equations

(c) *If a system of equations is graphed and the two lines are coincident, what is the name given to such a system?*

Answer: A system of dependent equations

(14) *Graph each of the following system of equations, and determine the solution set of the system by deter-*

mining the intersection of the two equations. Label each system as consisting of dependent, inconsistent, or consistent equations.

(a) $5x + 4y = 20$
$10x = -8y + 80$

(b) $2x - 7y = -46$
$6x = 21y - 138$

(c) $5x - y = 10$
$4x + y = -1$

(d) $-2x + 3y = 16$
$3x - 5y = -26$

Answers:

(a) The null set. The equations are inconsistent.
(b) An infinite set. The equations are dependent.
(c) $\{(1, -5)\}$. The equations are consistent.
(d) $\{(-2, 4)\}$. The equations are consistent.

Concept: Disadvantage of the graphing method.

(15) (a) *Graph the system of equations $24x + 48y = 73$ and $x + y = 2$. At what point do the two lines appear to intersect?*

Answer: The two straight lines appear to intersect at the point (1, 1).

(b) *Do the values $x = 1$, $y = 1$ satisfy both equations?*

Answer: No. The values do not satisfy the first equation.

(c) *Solve this system of equations by the substitution or by the addition method. What is the solution set of this system?*

Answer: $\left\{\left(\frac{23}{24}, \frac{25}{24}\right)\right\}$

(d) *What limitation has the graphing method of solving two linear equations in two variables?*

Answer: The graphing method has the limitation that all measurements have only a certain degree of precision which may not be fine enough to yield the exact solutions.

10.5 SOLVING A SYSTEM OF INEQUALITIES BY GRAPHING

Concept: Graphing an inequality on a number line.

(1) A real number greater than 1 may be expressed algebraically by the open sentence $x > 1$. The solution set of this open sentence is $\{x \mid x > 1\}$.

Express each of the following as an algebraic open sentence. Also indicate the solution set of the open sentence.

(a) X is greater than 8.
(b) X is less than 4.
(c) Y is greater than or equal to 2.
(d) K is less than or equal to -3.
(e) K is greater than -3 and less than -1.
(f) W is less than 0 and greater than -5.
(g) W is greater than or equal to -3 and less than or equal to 8.
(h) H is less than or equal to 10 and greater than 0.
(i) B is less than 5 and greater than or equal to -5.

<u>Answers</u>:

(a) $X > 8$, $\{X \mid X > 8\}$

(b) $X < 4$, $\{X \mid X < 4\}$

(c) $Y \geq 2$, $\{Y \mid Y \geq 2\}$

(d) $K \leq -3$, $\{K \mid K \leq -3\}$

(e) $-3 < K < -1$, $\{K \mid -3 < K < -1\}$

(f) $-5 < W < 0$, $\{W \mid -5 < W < 0\}$

(g) $-3 \leq W \leq 8$, $\{W \mid -3 \leq W \leq 8\}$

(h) $0 < H \leq 10$, $\{H \mid 0 < H \leq 10\}$

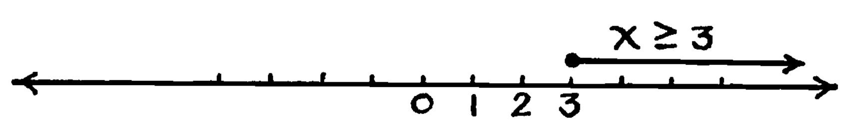
(i) $-5 \leq B < 5$, $\{B \mid -5 \leq B < 5\}$

(2) *Graph each of the following inequalities by drawing a line, segment, or a ray above the points on a number line included in the solution set of the inequality.*

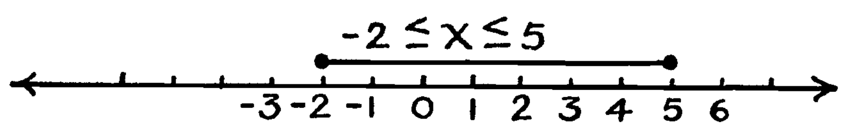
(a) $x \geq 3$ (b) $-2 \leq x \leq 5$ (c) $x \leq -1$
(d) $-2 < x < 1$ (e) $-6 \leq x < -3$

<u>Answers</u>:

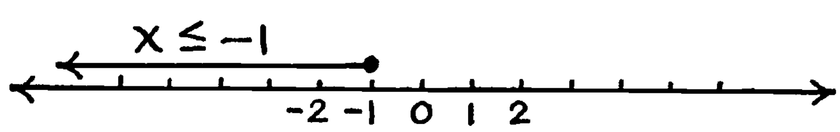
(a)



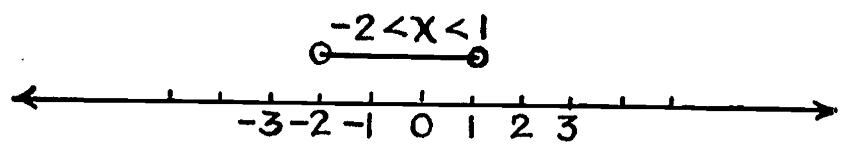
(b)



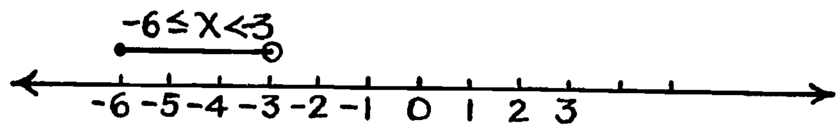
(c)



(d)



(e)

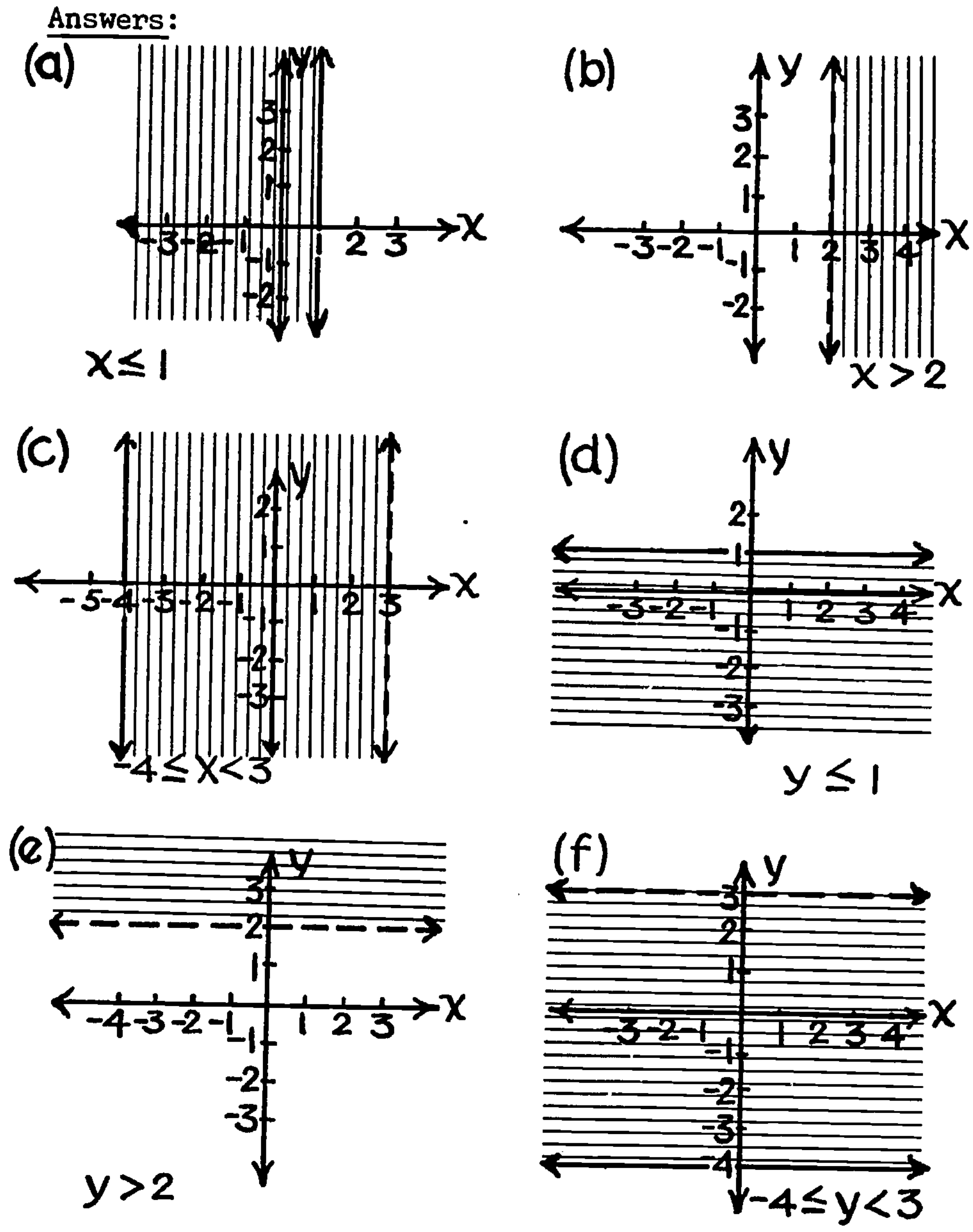


(3) *Sketch on a number plane the solution set of each of the following inequalities.*

(a) $x \le 1$ (b) $x > 2$ (c) $-4 \le x < 3$

(d) $y \le 1$ (e) $y > 2$ (f) $-4 \le y < 3$

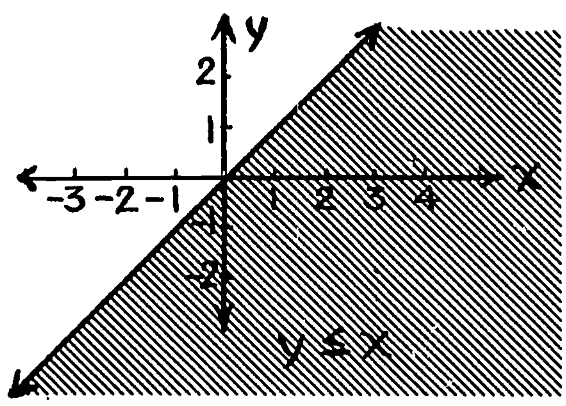
Answers:



(4) (a) *Graph the inequality $y \leq x$. The set of all points whose coordinates satisfy this inequality forms what geometric configuration?*

Answer:

The set of points forms a half plane.



(b) *What line forms the boundary of this half plane?*

Answer: The line y = x

(c) *Is the line $y = x$ included in the solution set of the inequality?*

Answer: Yes

(d) *Does the line $y = x$ form the boundary of the half plane represented by $y \leq x$?*

Answer: Yes

(e) *Is the set of points on the line $y = x$ in the solution set of the inequality $y < x$?*

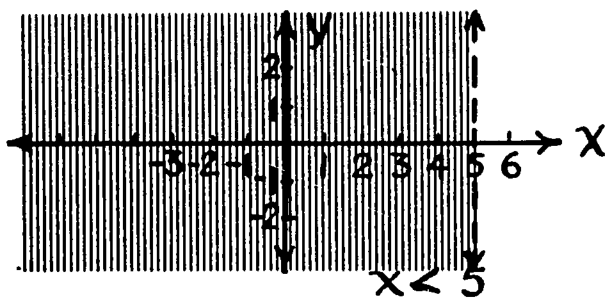
Answer: No

(f) *How is the boundary of the inequality $y < x$ indicated?*

Answer: The boundary, which is the line y = x, is indicated by a dashed line.

(5) (a) *Solve the inequality $x - 5 < 0$ for x and graph the resulting inequality.*

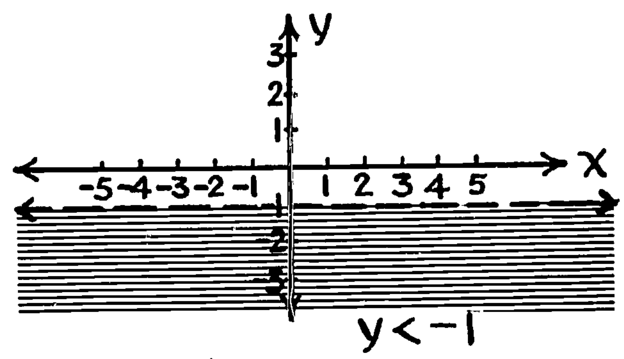
Answer: x < 5



(b) *Solve the inequality y + 1 < 0 and graph the resulting inequality.*

Answer:

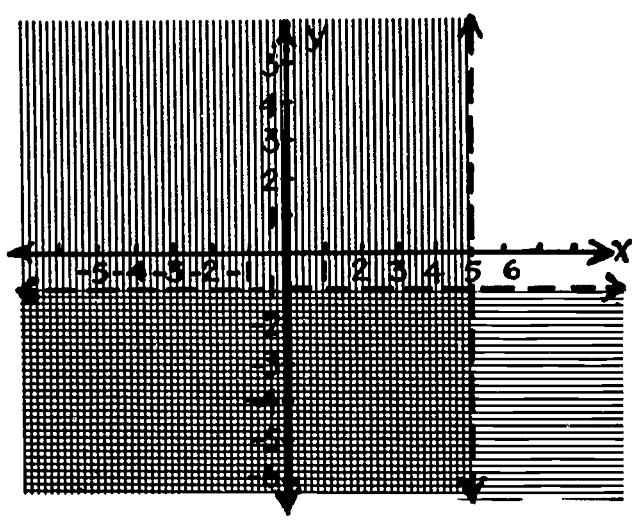
y < -1



(c) *Graph the system of inequalities x - 5 < 0 and y + 1 < 0 on the same number plane. How is the solution set of this system of inequalities represented on this graph?*

Answer:

The intersection of the graphs of the set of points satisfying the inequality x - 5 < 0 and the set of points satisfying the inequality y + 1 < 0 represents the solution set of the system of inequalities.



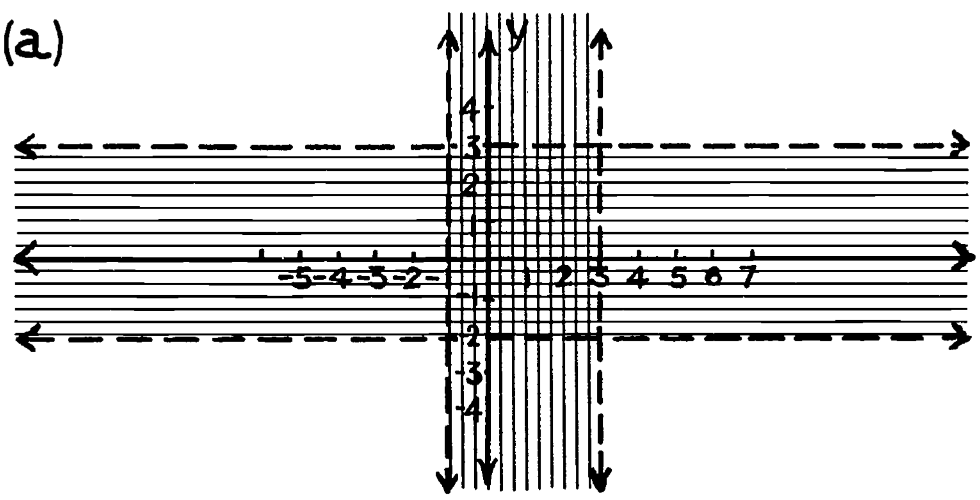
(6) *Graph the system of inequalities below indicating the portion of the graph representing the solution set of the system of inequalities.*

(a) $-1 < x < 3$ (b) $x - y > 0$ (c) $x \leq -y$

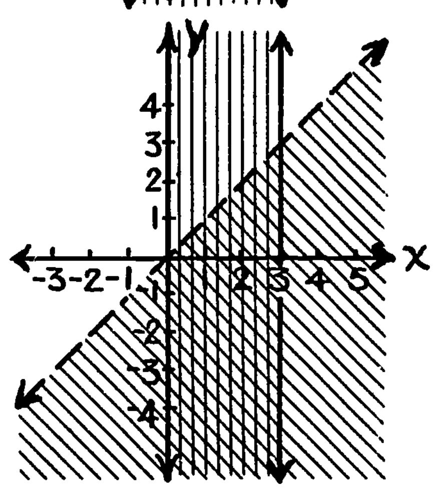
$-2 < y < 3$ $0 \leq x \leq 3$ $-6 < x \leq -2$

<u>Answers:</u>

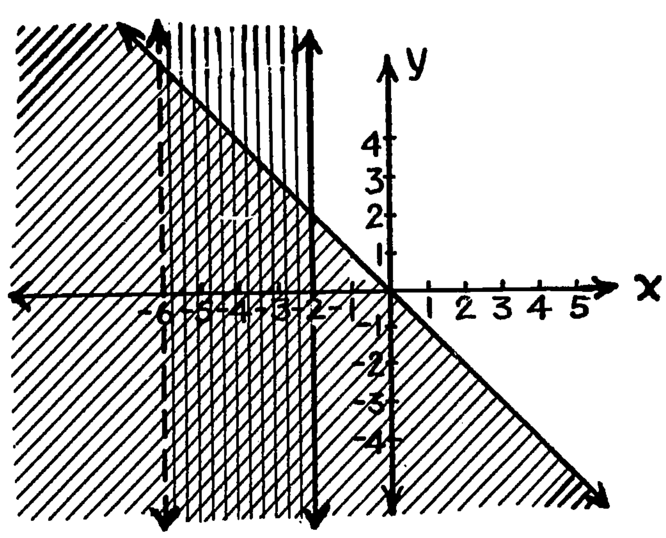
(a)



(b)



(c)

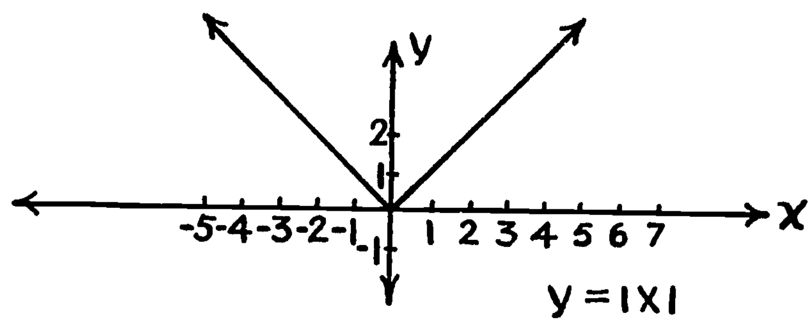


(7) (a) *The equations $y = | x |$ has what restriction on the variables?*

Answer: y cannot be a negative number as the absolute value of every real number is non-negative.

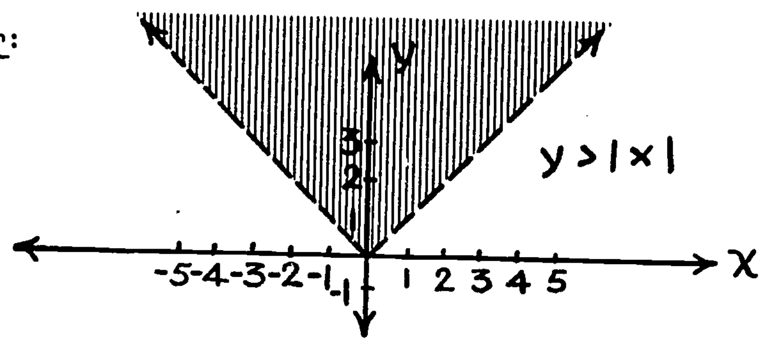
(b) *Graph the equation $y = | x |$.*

Answer:



(c) *Graph the inequality $y > | x |$.*

Answer:

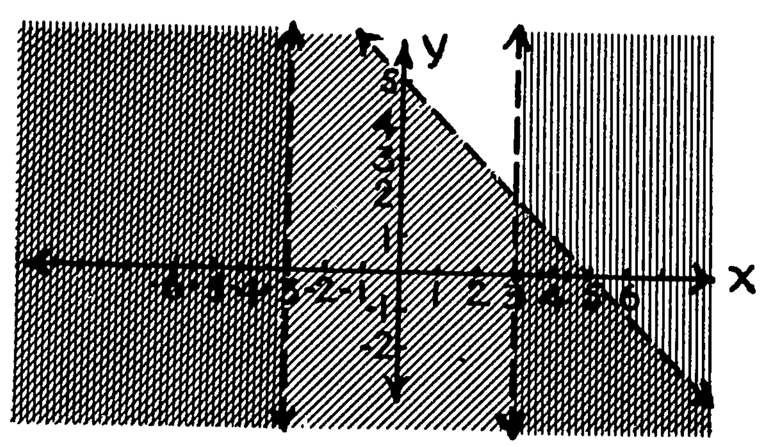


(8) *Graph each of the following systems of inequalities and indicate the portion of the graph that represents the solution set of the system.*

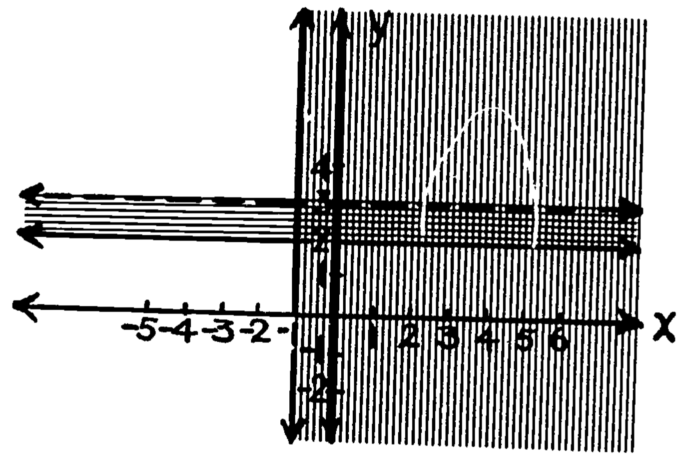
(a) $| x | > 3$, $x + y < 5$ (b) $2 \le y < 3$, $-2x \le 2$ (c) $| x | > -y$, $4 < y < 5$

Answers:

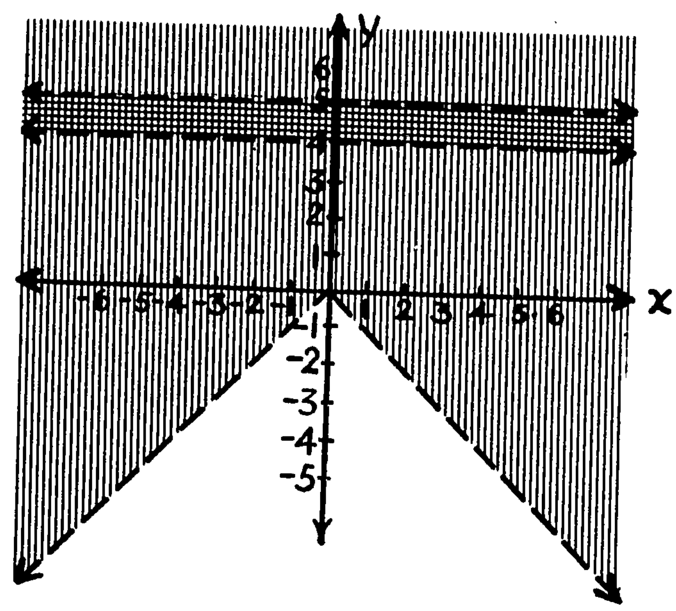
(a)



(b)



(c)



10.6 VERBAL PROBLEMS INVOLVING LINEAR EQUATIONS IN TWO VARIABLES

For each of the following problems, represent the unknown quantities in terms of two variables, write two equations in the two variables that can be used to solve the problem, and indicate the correct answer to the problem.

(1) *A businessman invested one sum of money at 5% interest and another sum at 6% interest. He received an annual income of $650 from the investments. If the rates of the two investments had been interchanged, his annual income would have been increased by $20. What was the total number of dollars he invested?*

Answer: x = amount invested at 5%
y = amount invested at 6%
x + y = total amount invested
.05x + .06y = 650.00
.06x + .05y = 670.00
x = 7,000   y = 5,000   x + y = 12,000

He invested $12,000.

Check: .05($7,000) + .06($5,000) = $650.00 true
.06($5,000) + .05($7,000) = $670.00 true
$7,000 + $5,000 = $12,000 true

(2) *A Little League baseball team purchased 8 pairs of baseball shoes and 8 baseball caps at the beginning of the season. Later in the season it was necessary to purchase 12 more pairs of shoes and 8 more caps. The first order amounted to $110 and the second order amounted to $156. If there was no change in prices during the season, find the price of a pair of baseball shoes and the price of a baseball cap.*

Answer: x = price of one pair of baseball shoes
y = price of one baseball cap
8x + 8y = 110
12x + 8y = 156
x = 11.50   y = 2.25

A pair of shoes cost $11.50 and a cap costs $2.25.

Check: 8($11.50) + 8($2.25) = $110 true
12($11.50) + 8($2.25) = $156 true

(3) *A boat goes upstream at 6 miles per hour and downstream at 9 miles per hour. What is the rate of the current and the rate of the boat in still water?*

Answer: x = rate of the boat in still water
y = rate of the current
x + y = 9
x - y = 6
x = 7.5   y = 1.5

The rate of the boat in still water is 7.5 mph.
The rate of the current is 1.5 mph.

Check: 7.5 mph. + 1.5 mph. = 9 mph. true
7.5 mph. - 1.5 mph. = 6 mph. true

(4) *A plane makes a 720-mile trip against a head wind in four and a half hours, and returns with a tail wind of the same speed in three hours. What is the speed of the wind and the speed of the plane in still air?*

Answer: x = speed of the plane in still air
y = speed of the wind
$4.5(x - y) = 720$
$3(x + y) = 720$
$x = 200 \quad y = 40$

The wind speed is 40 mph. and the speed of the plane in still air is 200 mph.

Check: $4.5(160) = 720$ true
$3(240) = 720$ true

(5) *The sum of the digits of a two-digit number is equal to one-half the number. Find the number.*

Answer: x = the units digit
y = the tens digit
$$x + y = \frac{x + 10y}{2}$$
$x = 8y$

The units digit must be 8 times the tens digit. The tens digit cannot be zero. If the tens digit is 1, the units digit is 8, and the number is 18. If x is an integer greater than 1, 8x is a two digit numeral, which cannot be a digit. Therefore, the only possible solution is that the number is 18.

Check: $1 + 8 = \frac{18}{2}$ true

Note: If the number is to be expressed as a numeral in a base other than 10, a different answer will result. Thus in base 5 the answer is 13_5 which is the numeral for the number eight. The teacher may wish to develop the pattern.

(6) *Twenty-two skilled workmen and 12 apprentices earn a total of $672 a day. At the same rates, 14 skilled workmen and 8 apprentices earn a total of $432 a day. How much does one skiled workman and how much does one apprentice earn per day?*

Answer: x = amount a skilled workman earns in one day
y = amount one apprentice earns in one day
$22x + 12y = 672$
$14x + 8y = 432$
$x = 24 \quad y = 12$

A skilled workman earns $24 a day, an apprentice earns $12 a day.

Check: 22($24) + 12($12) = $672 true
14($24) + 8($12) = $432 true

(7) *If Harry gives George 6 stamps, Harry will have $\frac{2}{3}$ as many stamps as George. If George gives Harry 9 stamps, George will have $\frac{2}{3}$ as many stamps as Harry. How many stamps does each have?*

Answer: x = the number of stamps Harry has
y = number of stamps George has

$$(x - 6) = \frac{2}{3}(y + 6)$$

$$(y - 9) = \frac{2}{3}(x + 9)$$

x = 36 y = 39

Harry has 36 stamps, George has 39 stamps.

Check: $36 - 6 = \frac{2}{3}(39 + 6)$ true

$39 - 9 = \frac{2}{3}(36 + 9)$ true

(8) *When a number is increased by one-half a second number the sum is 21. When the same first number is decreased by half the second number the result is 3. Find the two numbers.*

Answer: x = one number
y = second number

$$x + \frac{y}{2} = 21$$

$$x - \frac{y}{2} = 3$$

x = 12 y = 18

The first number is 12 and the second is 18.

Check: $12 + \frac{18}{2} = 21$ true

$12 - \frac{18}{2} = 3$ true

(9) *A chemist took 8 ounces of solution from bottle A and 8 ounces of solution from bottle B, and mixed them together. The resulting mixture was a 27.5% solution. At another time he took 12 ounces from bottle A and 8 ounces from bottle B and mixed them together. The result was a 27% solution. Find the concentration of the solution in bottle A and the concentration of the solution in bottle B.*

Answer: x = concentration of the solution in bottle A
y = concentration of the solution in bottle B

$$8x + 8y = (0.275)(16)$$
$$12x + 8y = (0.27)(20)$$
$$x = 0.25 \quad y = 0.30$$

Bottle A contains a 25% solution, bottle B, a 30% solution.

Check: $8(.25) + 8(.30) = 16(.275)$ true
$12(.25) + 8(.30) = 20(.27)$ true

Teacher Notes

UNIT TEST

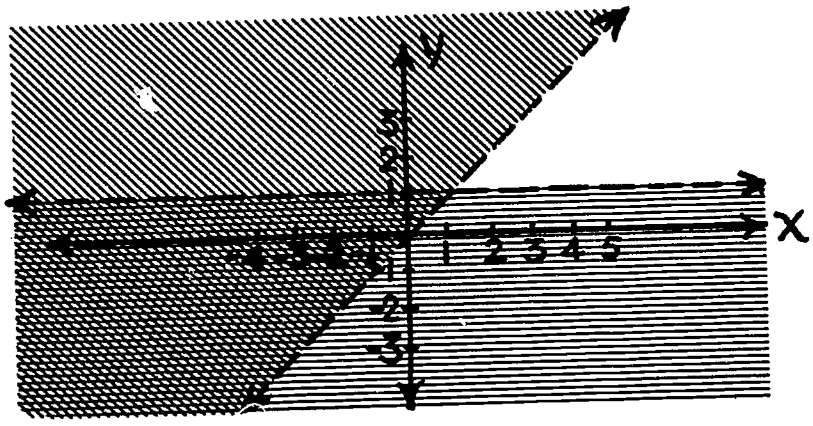
1. *Solve the following system of equations for x and y and check the answers in both of the equations.*

$\frac{5x}{3} + y = \frac{11}{3}$ $\qquad$ $\frac{x-1}{3} - \frac{y-2}{5} = 2$

Answer: x = 4, y = -3

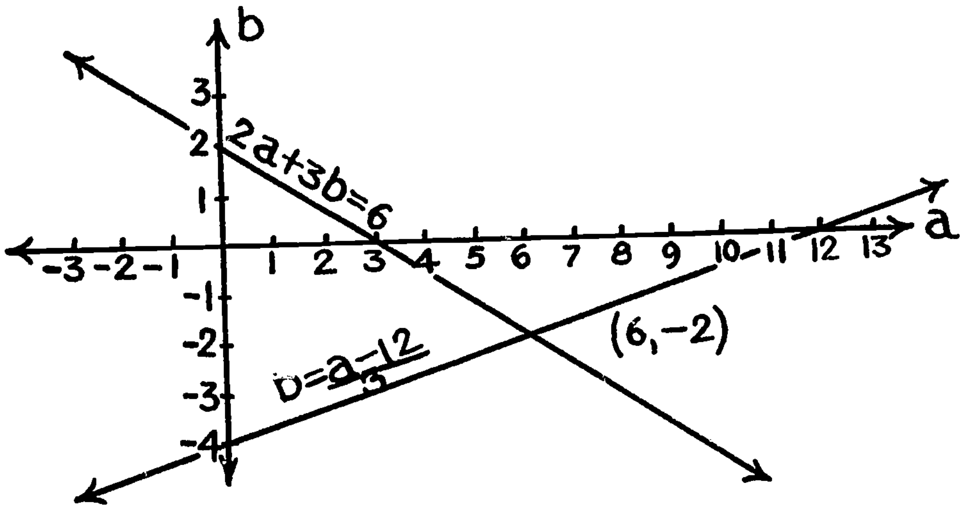
2. *Indicate on a graph the solution set of the system of inequalities $y > x$ and $y - 1 < 0$.*

Answer:



3. *Solve graph...ally and check $2a + 3b = 6$ and $b = \frac{a - 12}{3}$.*

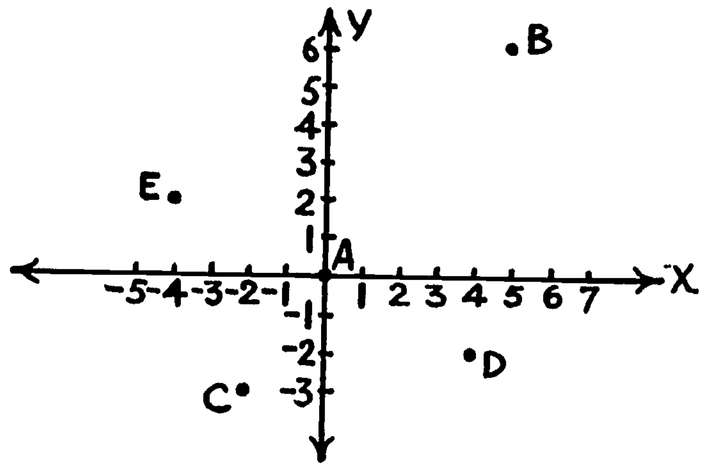
Answer: (6, -2)



4. *If the point $\left(k, \frac{3}{2}\right)$ lies on the graph of $x - 2y = 7$, find the value of k.*

Answer: k = 10

5. *On the graph at the right, identify the point whose coordinates are given.*

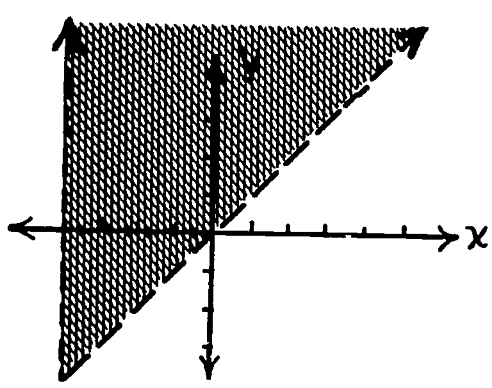


(a) (0, 0) (b) (5, 6) (c) (-2, -3) (d) (4, -2)
(e) (-4, 2)

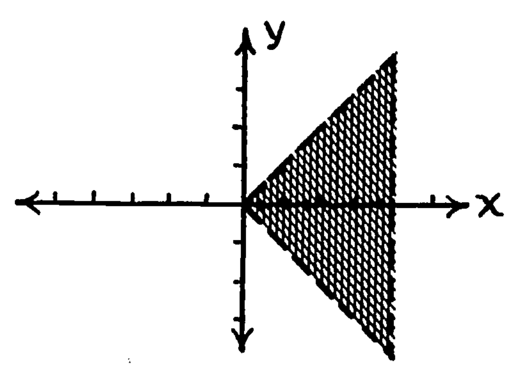
Answers: (a) Point A (b) Point B (c) Point C
(d) Point D (e) Point E

6. *Which of the following graphs is the graph of the solution set of the system of inequalities* $x > y$ *and* $x - 4 < 0$?

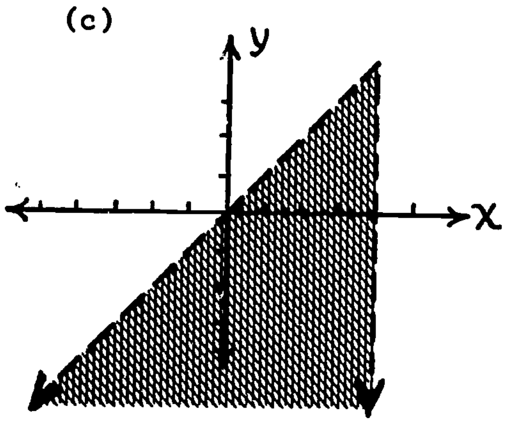
(a)



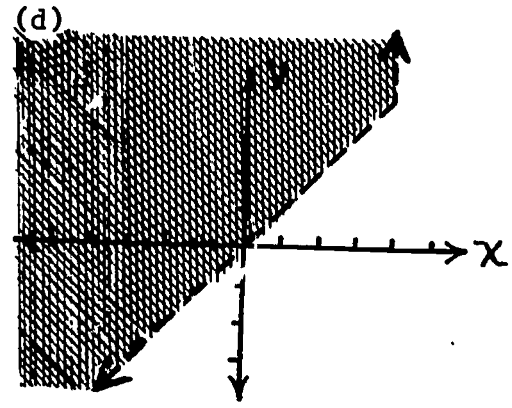
(b)



(c)

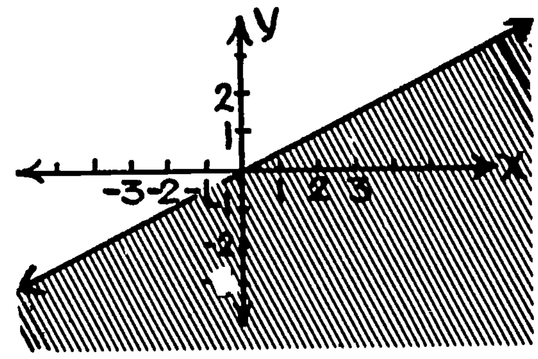


(d)



Answer: (c)

7. *Identify the inequality shown at the right.*



Answer: $x \geq 2y$

8. *Determine, without solving, whether the solution set of each of the following is a set containing one point, a set containing an infinite number of points, or the null set.*

(a) $3x - 11 = -4y$, $6x = -8y + 22$

(b) $4x - 6y = 23$, $8x - 12y = 38$

(c) $x + y = 3$, $-x - y = -3$

(d) $x + y = 3$, $-x - y = 3$

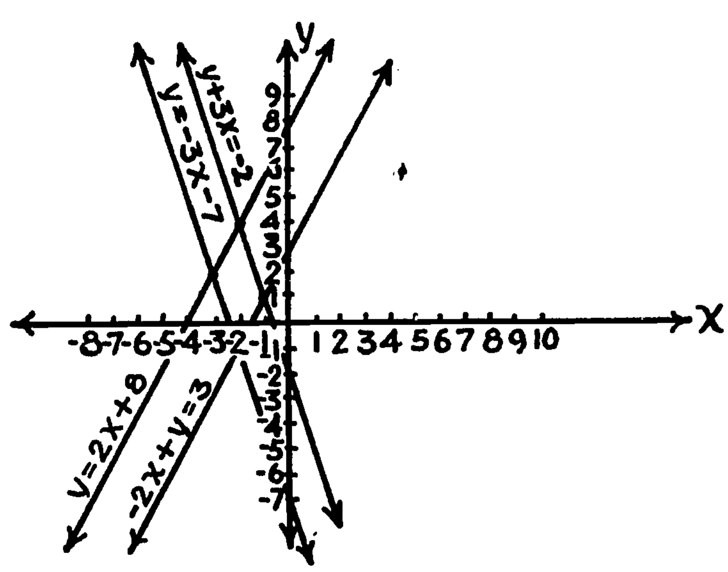
(e) $2x - 3y = 13$, $x + y = 5$

Answers:
(a) A set containing an infinite number of points
(b) The null set
(c) A set containing an infinite number of points
(d) The null set
(e) A point

9. *Graph the system and write the coordinates of the four points of intersection of the lines represented by the four equations shown at the right.*

$y = 2x + 8$
$-2x + y = 3$
$y = -3x - 7$
$y + 3x = -2$

Answers:
(-3, 2)
(-2, 4)
(-2, -1)
(-1, 1)



10. *Solve the following system of equations using the substitution method.*

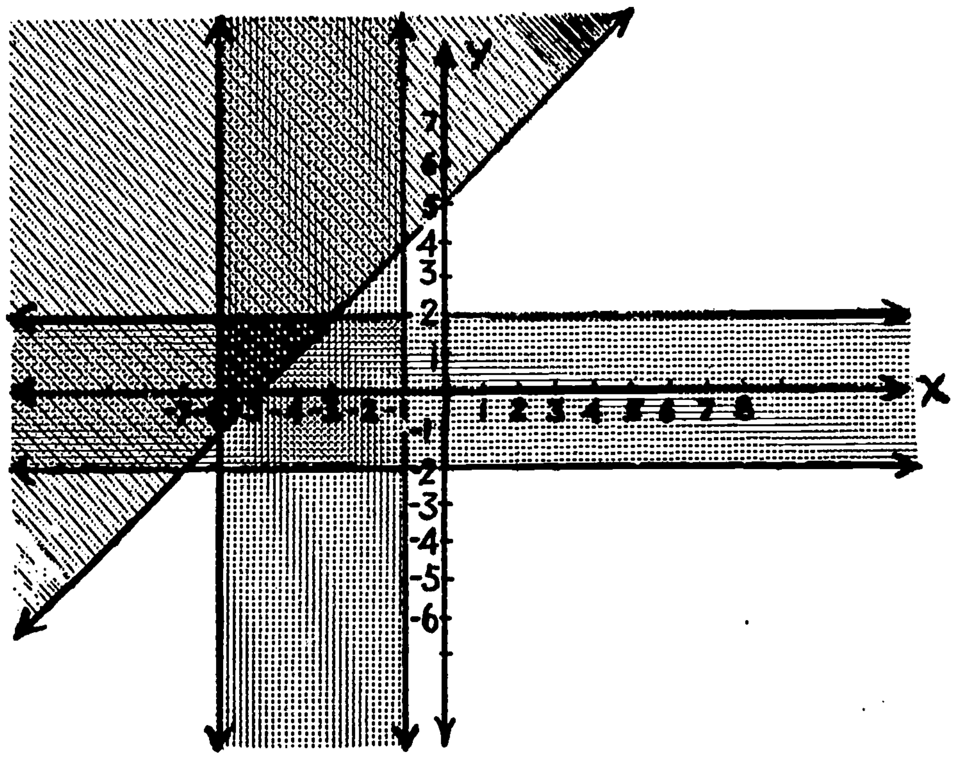
$$2x - y + z = 2$$
$$3x + y = 12$$
$$x - y = -4$$

Answer: $x = 2$, $y = 6$, $z = 4$

11. *Determine the solution set of the following set of inequalities by the graphing method.*

$$x + 5 \leq y$$
$$-6 \leq x \leq -1$$
$$-2 \leq y \leq 2$$

Answer:



UNIT 11: RELATIONS AND FUNCTIONS

Part 1. Background Material for Teachers

11.1 INTRODUCTION

One of the advantages of this unit is that many topics in the course may be applied to the study of functions. The topic, relations and functions, affords an opportunity to apply the concepts mastered earlier in the course, including sets, exponents and radicals, polynomial expressions, linear and quadratic open sentences, and coordinate geometry. In fact, the study of functions requires the applications of all these concepts. The teacher may utilize this opportunity to reinforce concepts previously mastered. Pupils may also be encouraged to discover on their own how many of the previously mastered concepts have application in the study of the topic of functions.

The concepts in this unit are essential to the study in the next unit of trigonometric functions in the unit circle, which is an optional topic. In Mathematics 8X, the sine, the cosine, and the tangent were studied as ratios and, in effect, as relations defined by tables of values. In this unit it is desirable to review the earlier work in this area in order to illustrate the important concept of defining a relation by a table. The sine, cosine, and tangent tables, at this point, are used as ordered number pairs which relate certain angles to certain ratios. These ratios should be reviewed again at this time and applied in a sufficient number of verbal problems to complete the recall. In unit 12 of this course, the sine, cosine, and tangent are studied as functions defined by other means. The pupil must have a thorough understanding of the concept of function before attempting the next unit.

11.2 RELATIONS

Any set of ordered number pairs is a relation. A set may be defined by a solution set of an equation, by a graph, by a table of values, or by a listing of the set of ordered number pairs.

If $A = \{(1, 2), (2, 4), (7, 11)\}$ then A is a relation.

Thus a table of the trigonometric ratios is a condensed chart or listing which defines the relations sine a = b, cosine a = c, tangent a = d by pairing an angle as the first member with a ratio as the second member. From a commonly used chart in which the angle is tabulated in degrees from 1° to 90°, a sine relation is the set $S = \{(1, 0.0175), (2, 0.0349), \ldots, (90, 1.0000)\}$, a cosine relation is the set $C = \{(1, 0.9998), (2, 0.9994), \ldots, (90, 0.0000)\}$, and a tangent relation is the set $T = \{(1, 0.0175), (2, 0.0349), \ldots, (89, 57.2900)\}$.

A set of ordered number pairs may be defined by listing the elements in the set of x coordinates and listing the elements in the set of y coordinates. For example, if set B is the set of x coordinates and C is the set of y coordinates and

$$B = \{1, 2, 3\}$$
$$C = \{8, 9, 10\}$$

then the largest set of ordered number pairs would be: {(1, 8), (1, 9), (1, 10), (2, 8), (2, 9), (2, 10), (3, 8), (3, 9), (3, 10)}. The set of ordered number pairs formed from the set of x coordinates B and from the set of y coordinates C may be expressed as B X C and is read "B cross C." The set of x coordinates is always given first. In the example above, the set of ordered pairs C X B would not be the same as the set B X C. The set C X B would be: {(8, 1), (8, 2), (8, 3), (9, 1), (9, 2), (9, 3), (10, 1), (10, 2), (10, 3)}. In the set C X B, set C would be the set of x coordinates and set B would be the set of y coordinates.

If the universal set U is the set of x coordinates and also the set of y coordinates, then U X U would be the set of ordered number pairs. If U = {1, 2, 3} then U X U is the set: {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}. If U is the set of all real numbers, then the set of ordered pairs U X U is called the Cartesian set. There is a one-to-one correspondence between the elements in the set U X U and the points on a plane.

Any subset of U X U is a relation. A relation may consist of the coordinates of a point, a finite set of points, an infinite set of points, or the entire number plane.

Relations are often defined by one or more equations or inequalities. Each of the following is a relation:

$$\{(x, y) \mid y = x\}$$
$$\{(x, y) \mid y \leq x\}$$
$$\{(x, y) \mid x^2 + y^2 < 16\}$$

Such equations and inequalities usually have two variables either expressed or implied. If the equation y = 2 is to be graphed on a number plane, it is implied that x may represent any real number. That is, the equation y = 2 is really just a simplified expression for y = 0x + 2. This relation may be written as: $\{(x, y) \mid y = 0x + 2\}$ or $\{(x, y) \mid y = 2\}$.

The set of x coordinates and the set of y coordinates may also be given in chart form.

x	1	1	2	2	3	3	4
y	1	2	3	4	3	2	1

Sometimes it is possible to define a relation in both chart form and also equation form. It is apparent from the chart

x	2	4	6	8
y	4	8	12	16

that y = 2x, where x and y are restricted to the values indicated in the chart. It is important to indicate clearly the restrictions on either or both variables when a relation is being defined by an equation. The equation y = 2x defines the relation $\{(x, y) \mid y = 2x\}$, and unless some restriction is indicated, it is assumed that the set of abscissas and the set of ordinates consist of all real numbers meeting the condition y = 2x. This relation is an infinite set of ordered pairs, whereas the relation defined by the chart above consists of only four ordered number pairs. When a relation is defined by a set of values given in a chart, care must be taken in defining this relation in the form of an equation to indicate clearly the restrictions on the variables. The set of abscissas, or first coordinates, of a relation is called the domain of the relation. The set of ordinates, or second coordinates, of a relation is called the range of the relation.

Given the relation A = {(2, 1), (4, 3), (6, 5)}, the domain of this relation is {2, 4, 6} and the range is {1, 3, 5}. The symbols sometimes used to indicate the domain and range of relation A are d(A) and r(A) respectively. Given the relation B = {(1, 3), (2, 2), (3, 1)}, then d(B) = {1, 2, 3} and r(B) = {1, 2, 3}. If only the domain and range of a relation are given, it is not possible to determine what the relation is. For example, both relation G and relation H, where

$$G = \{(1, 1), (2, 2), (3, 3)\}$$
$$H = \{(1, 2), (3, 3), (2, 1)\}$$

have a domain which is {1, 2, 3} and a range which is {1, 2, 3}, but these are different relations. Relations which are by no means identical may have the same range and the same domain.

11.3 FUNCTIONS

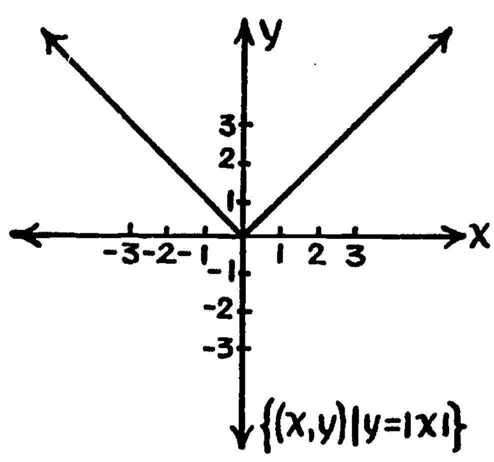
A function is a relation in which no element in the domain has more than one element in the range associated with it. An examination of a table of trigonometric ratios shows that each value of the angle in the table is associated with no more than one value of the sine, the cosine, or the tangent. Therefore, the trigonometric relations defined by the table are also functions according to the definition, and the tables are properly entitled, values of the trigonometric functions.

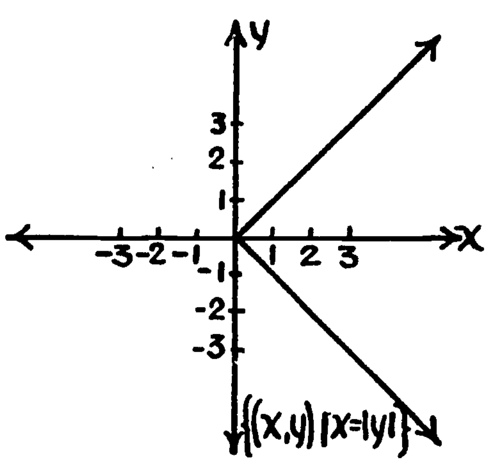
Another way of stating the definition of a function is that for every x there is one and only one value for y. Graphically, this means that a relation is a function if and only if no vertical line meets the graph of the relation at more than one point. This is sometimes called the vertical line test.

If the members of the ordered pairs of a function are interchanged and the resulting relation is also a function, then the resulting relation is called the inverse of the function. The symbol for the inverse of the function F is F^{-1}. The inverse of the function F = {(4, 1), (8, 3), (1, 2)} is

$$F^{-1} = \{(1, 4), (3, 8), (2, 1)\}$$

It is possible for a function not to have an inverse. If G = {(1, 3), (2, 3), (3, 4)} then H = {(3, 1), (3, 2), (4, 3)}. The relation H obtained from G by reversing the order of the members of each pair, does not meet the requirements of the definition of a function because the element 3 in the domain of H has two elements, 1 and 2, in the range associated with it. Another example of a function without an inverse is the function defined by the equation y = | x |. The vertical line test indicates that the relation {(x, y) | y = | x |} is a function but that the relation obtained by interchanging x and y, {(x, y) | x = | y |} is not a function. We say that F = {(x, y) | y = | x |} has no inverse. This can be seen in the following graphs.

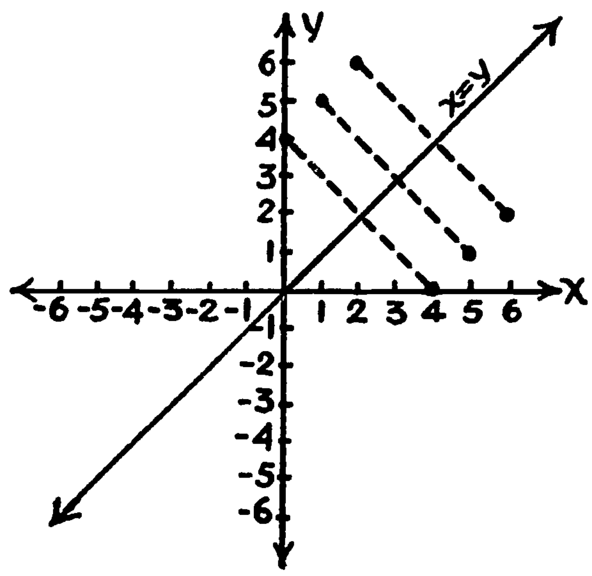


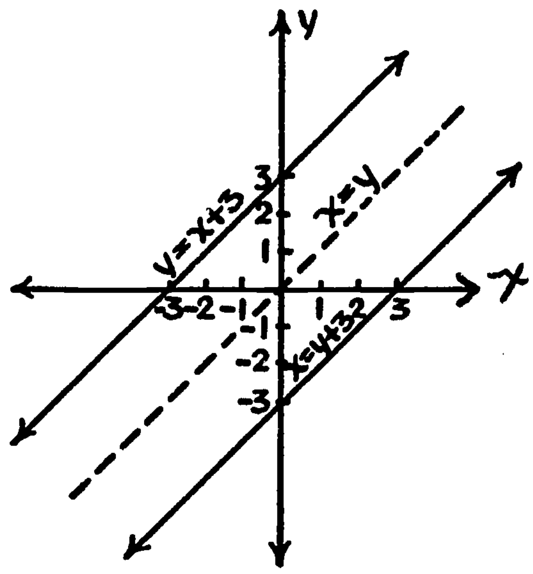


When a function and its inverse are both graphed on the same set of coordinates, the graph of the function is a reflection of its inverse with respect to the line x = y. Line segments which connect the cooresponding points of the graphs of a function and its inverse are perpendicular to the line x = y and are bisected by it.

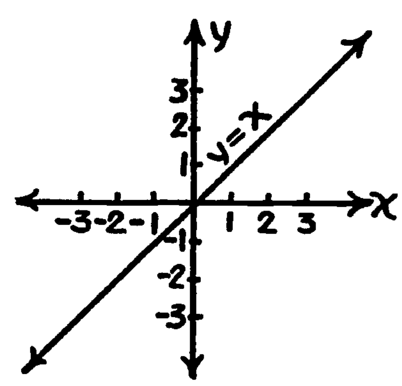
For example, if
F = {(0, 4), (1, 5), (2, 6)}
then F^{-1} = {(4, 0), (5, 1), (6, 2)}.

This can also be seen in the example G = {(x, y) | y = x + 3}
G^{-1} = {(x, y) | x = y + 3}.





A function has an inverse if and only if each y of the original function has not more than one x. This may also be stated as: a function has an inverse if and only if a line parallel to the x-axis meets the graph of the given function in not more than one point. Consider the function F = {(x, y) | y = x}. Set F is a function because any line parallel to the y-axis intersects the graph of F at no more than one point. The function has an inverse because any line parallel to the x-axis also intersects the graph of the given function at not more than one point.



The equation y = 2x + 1 defines the function {(x, y) | y = 2x + 1}. There are other ways of representing such a function. The y may be replaced by F(x) or f(x) or a symbol involving some other letter such as G(x). The equation y = 2x + 1 may be written F(x) = 2x + 1 and the symbol F(x) read "F at x." The function {(x, y) | y = 2x + 1} may be written {(x, F(x)) | F(x) = 2x + 1}.

F(x) is also called "the value of the function at x." When x is 3, F(3) = 2(3) + 1 or F(3) = 7. The value of the function at x may be determined for any given x by substituting the given value of x for x in the equation that defines the function.

A linear function is a set of ordered pairs (x, y) or (x, F(x)) for which y = ax + b or F(x) = ax + b, where a and b are real numbers. The standard form of an equation that defines a linear function is often given as y = mx + b.

The graph of a linear function is always a straight line. If the equation is expressed in the form y = mx + b, then the slope of the line is equal to m and the line intersects the y-axis at the point for which the second coordinate, y = b. This concept is developed in the questions and activities by having the pupils graph several linear functions and then discover the relationship between the slope and m and between the y-intercept and b.

The slope of a straight line graph is an indication of its steepness. If the measure of the slope is a positive number, the graph slopes upward to the right. If the measure of the slope of the graph is a negative number, the graph slopes downward to the right.

The slope of a straight line is defined as the rate of change of the ordinate with respect to the abscissa. If (x_1, y_1) and (x_2, y_2) are any two points on the straight line, then the slope is defined as $\frac{y_2 - y_1}{x_2 - x_1}$. If the points (2, 3) and (5, 9) are on the same straight line, then the slope of the line is $\frac{9 - 3}{5 - 2}$ or 2. If the points (2, 3) and (4, 0) are on some straight line, then the slope of the line is $\frac{0 - 3}{4 - 2}$ or $-\frac{3}{2}$.

Another topic in this unit is that of graphing a quadratic function. A quadratic function is a set of ordered pairs (x, y) for which $y = ax^2 + bx + c$, a, b, and c being real numbers and $a \neq 0$. The graphs of such functions are parabolas. The pupil is required to plot only enough points so that the general shape of the graph can be determined by connecting these points with a smooth curve.

The last topic of this unit is that of using the trigonometric tables in graphing the sine function, the cosine function and the tangent function where the domain of the ordered pairs is a limited subset of {1, 2,..., 89, 90}. Graphic representation of tabular data is often an aid to understanding the relation between the quantities involved. Pupils should realize that the plotted points obtained from the trigonometric tables are tc be connected by a smooth curve and that using more points results in a better indication of this curve.

UNIT 11: RELATIONS AND FUNCTIONS

Part 2. Questions and Activities for Classroom Use

11.1 INTRODUCTION

In presenting the concepts of this unit through the use of the questions and activities, the problem often arises as to which set of symbols to use for functional notation. A function may be defined by a table. A function may also be defined by an equation in the form $y = 3x + 1$, or this same function may be defined by $F(x) = 3x + 1$ or $f(x) = 3x + 1$. If function A has an inverse, then the inverse of function A may be indicated by A^{-1}.

The teacher must decide which set of symbols to use and then it is best to use this set of symbols consistently throughout the unit. After the pupils have mastered this set of symbols, they may be shown alternate ways of representing functions so that they will recognize these different symbols when they are confronted with them.

The discussion of relations is introduced by means of the trigonometric tables. The teacher should provide enough review to recall and fix again the definitions of the sine, cosine, and tangent of an angle.

11.2 RELATIONS

Concept: Definition of relation.

(1) *Problems which require that the length of a side of a right triangle be found when the length of another side and the measure of an acute angle of the triangle are given, are commonly solved by using one of the three trigonometric ratios. What is the name of each of these ratios?*

Answer: The sine of the angle, the cosine of the angle, and the tangent of the angle

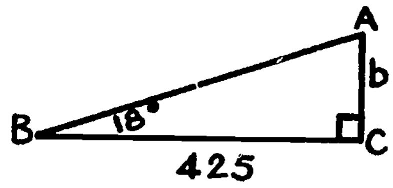
(2) *What is the definition of each of these three trigonometric ratios?*

Answer: $\text{sine of the angle} = \frac{\text{opposite arm}}{\text{hypotenuse}}$

$\text{cosine of the angle} = \frac{\text{adjacent arm}}{\text{hypotenuse}}$

tangent of the angle = $\frac{\text{opposite arm}}{\text{adjacent arm}}$

(3) *Using the data shown at the right, write an equation which could be used to find b.*



Answer: $\tan 18^\circ = \frac{b}{425}$

(4) *Recall that for a fixed acute angle in a right triangle, these three trigonometric ratios are fixed quantities which do not vary with the size of the right triangle, and which therefore can be tabulated. One such table gives the measure of the angle in degrees from .° to 90° and the values of the trigonometric ratios as decimals rounded off to the nearest ten thousandth. Use a table of trigonometric ratios to complete each of the following:*

(a) $\sin 20^\circ$ = ____ (d) $\tan 83^\circ$ = ____
(b) $\tan 35^\circ$ = ____ (e) $\sin 89^\circ$ = ____
(c) $\cos 78^\circ$ = ____

Answers:
(a) $\sin 20^\circ = 0.3420$ (d) $\tan 83^\circ = 8.1443$
(b) $\tan 35^\circ = 0.7002$ (e) $\sin 89^\circ = 0.9998$
(c) $\cos 78^\circ = 0.2079$

(5) *Given the equation tan 18° = $\frac{b}{425}$, write the steps used to find b and evaluate b to the nearest integer.*

Answer: $0.3249 = \frac{b}{425}$

$b = 425(0.3249)$

$b = 138.0825$

$b \approx 138$

(6) *In discussing graphs, the points in the number plane are named by writing within a pair of parentheses, two numbers separated by a comma. What name did we give this symbol?*

Answer: An ordered number pair

(7) *The following expressions which contain ordered number pairs are called relations:* $A = \{(1, 2), (3, 5), (4, 11)\}$, $B = \{(\frac{1}{2}, 1), (\frac{1}{4}, 2), (1, 1)\}$, $C = \{(1, 0.44), (2, 0.56), (2, 0.60), (4, 0.62)\}$. *Describe a relation.*

Answer: A relation is a set of ordered number pairs.

(8) *In a relation the members of one set of numbers are paired with the members of another set of numbers. Write the relation S in which each of the numbers, 2, 3, $5\frac{1}{2}$, 7, and 10 is paired with its double in that order.*

Answer: $S = \{(2, 4), (3, 6), (5\frac{1}{2}, 11), (7, 14), (10, 20)\}$

(9) *Various tables of numbers such as appear in mathematical handbooks, may pair the member of one set of numbers with the members of another set of numbers. A table of the trigonometric ratios makes such an association or pairing. If the first member of the pair is the measure of an angle in degrees, we may write a variety of ordered number pairs, for example, (10, 0.9848) in which we pair 10° with a definite cosine ratio. Check this pairing in a table of trigonometric ratios and then write the ordered number pairs which associate 20° with a definite tangent ratio and 50° with a definite sine ratio.*

Answer: (20, 0.3640), (50, 0.7660)

(10) *If* $A = \{5, 10, 15, 20\}$ *is a set of angle measures in degrees, use a table of trigonometric ratios to write the relation S in which each member of A is paired with a definite sine ratio as a second member.*

Answer: $S = \{(5, 0.0872), (10, 0.1736), (15, 0.2588), (20, 0.3420)\}$

(11) *If* $B = \{81, 82, 83\}$ *is a set of angle measures in degrees, use a table of trigonometric ratios to write the relation T in which each member of B is paired with a definite tangent ratio as a second member.*

Answer: $T = \{(81, 6.3138), (82, 7.1154), (83, 8.1443)\}$

(12) *If a relation contains many elements, we may sometimes list the first few elements and then the last few elements, separating the two groups by 3 dots. Use a table of trigonometric ratios to define a cosine relation C by listing its elements.*

Answer: C = {(1, 0.9998), (2, 0.9994),...,(89, 0.0175), (90, 0.0000)}

(13) *What is another name for the replacement set of the variables in open sentences such as* $5x + 4 = 23$

$$|y| \leq 11$$

$$2a + b > 14?$$

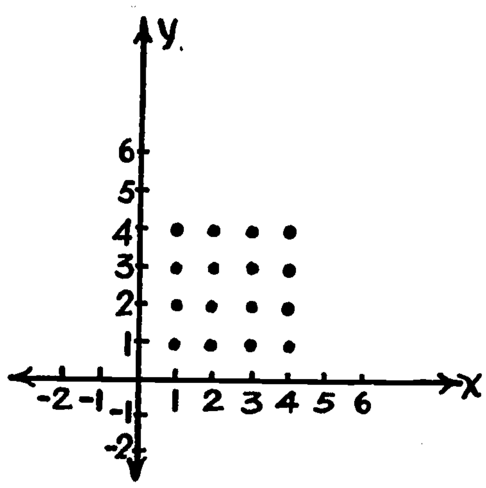
Answer: The universal set or the universe

(14) *What is the symbol for the universal set?*

Answer: U

(15) *The set of all ordered number pairs such that each of the members of each of the pairs is an element of U may be indicated by U X U and this is read "U cross U." Graph U X U when U = {1, 2, 3, 4}.*

Answer:

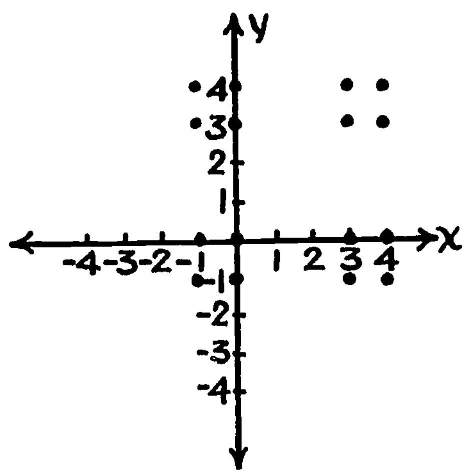


(16) (a) *List all the elements in the set U X U when U = {-1, 0, 3, 4}.*

Answer: U X U = {(-1, -1), (-1, 0), (-1, 3), (-1, 4),
(0, -1), (0, 0), (0, 3), (0, 4),
(3, -1), (3, 0), (3, 3), (3, 4),
(4, -1), (4, 0), (4, 3), (4, 4)}

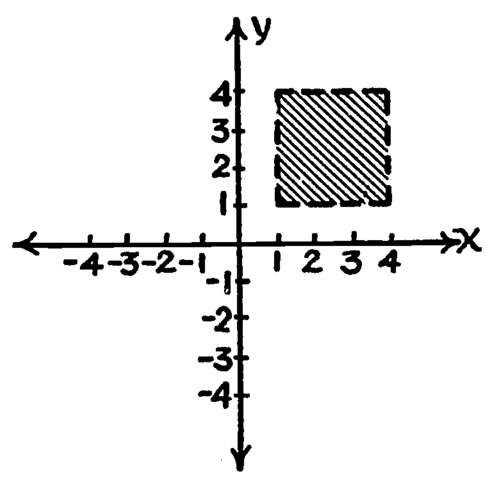
(b) *Graph U X U as given in part (a).*

Answer:



(c) *Graph U X U when U is the set of all real numbers greater than 1 and less than 4.*

Answer:



(17) *If U is the set of all real numbers, the set U X U is called the Cartesian set, named after the famous mathematician Descartes. What is the relationship between the elements of the Cartesian set and the points of a plane?*

Answer: There is a one-to-one correspondence between the elements in the Cartesian set and the points of a plane.

(18) *Given any set U, describe the elements that make up the set U X U.*

Answer: U X U will always consist of a set of ordered pairs.

(19) (a) *A relation may also be defined as any subset of $U \times U$. Is any set of ordered numbers pairs a relation?*

Answer: Any set of ordered number pairs is a relation because every set of ordered number pairs must be a subset of U X U where U is the set of all real numbers.

(b) *If U is the set of all real numbers, is the solution set of $y = x - 1$ a set of ordered pairs? Is the solution set of $y = x - 1$ a relation?*

Answer: The solution set of $y = x - 1$ is a set of ordered pairs and is therefore a relation.

(c) *Is the set of $y > x - 1$ a relation?*

Answer: The solution set of $y > x - 1$ is a set of ordered pairs and it is therefore a relation.

(d) *Is the solution set of $x + \frac{1}{x} = 2$ a relation?*

Answer: The solution set of the equation is {1}. Since the solution set is not a set of ordered number pairs, it is not a relation.

(e) *Is the solution set of the system of equations $x + \frac{1}{x} = 2$ and y = any real number, a relation?*

Answer: Yes. The solution set of this system of equations is a set of ordered number pairs, the first member or abscissa being 1 and the second member or ordinate being any real number. The solution set of this system of equations is therefore a relation.

(f) *Is the solution set of the inequality $2x + 3 > x + 5$ a relation?*

Answer: No. The solution set of this inequality is $\{x \mid x > 2\}$. Since the solution set is not a set of ordered number pairs, it is not a relation.

(g) *The solution sets of what types of open sentences are always relations?*

Answer: The solution sets of open sentences in two variables are relations.

Concept: Meaning of locus.

(20) *The graph of a relation is called a locus. Describe the locus determined by the first two relations and plot the locus determined by the third relation.*

(a) $\{(x, y) \mid y = 0x + 2\}$

(b) $\{(x, y) \mid x = 0y + 1\}$

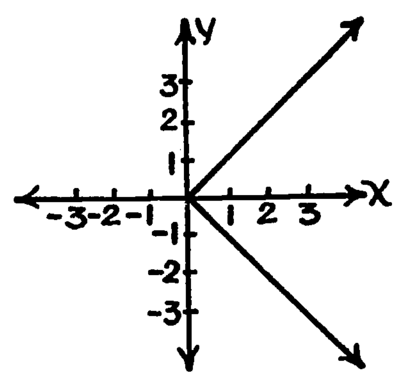
(c) $\{(x, y) \mid x = |y|\}$

Answers:

(a) A straight line parallel to and two units above the x-axis

(b) A straight line parallel to and one unit to the right of the y-axis

(c)



Concept: Meaning of domain and range.

(21) *The set of abscissas or first coordinates of the elements of a relation is called the domain of the relation. The set of ordinates or second coordinates of the elements of a relation is called the range of the relation. Indicate the domain and range of each of the following relations.*

(a) {(2, 1), (4, 3), (6, 5), (8, 7), (10, 9), (12, 11)}

(b) {(-1, -3), (0, -2), (1, -1), (2, 0)}

(c) {(1, 4), (2, 3), (3, 2), (4, 1)}

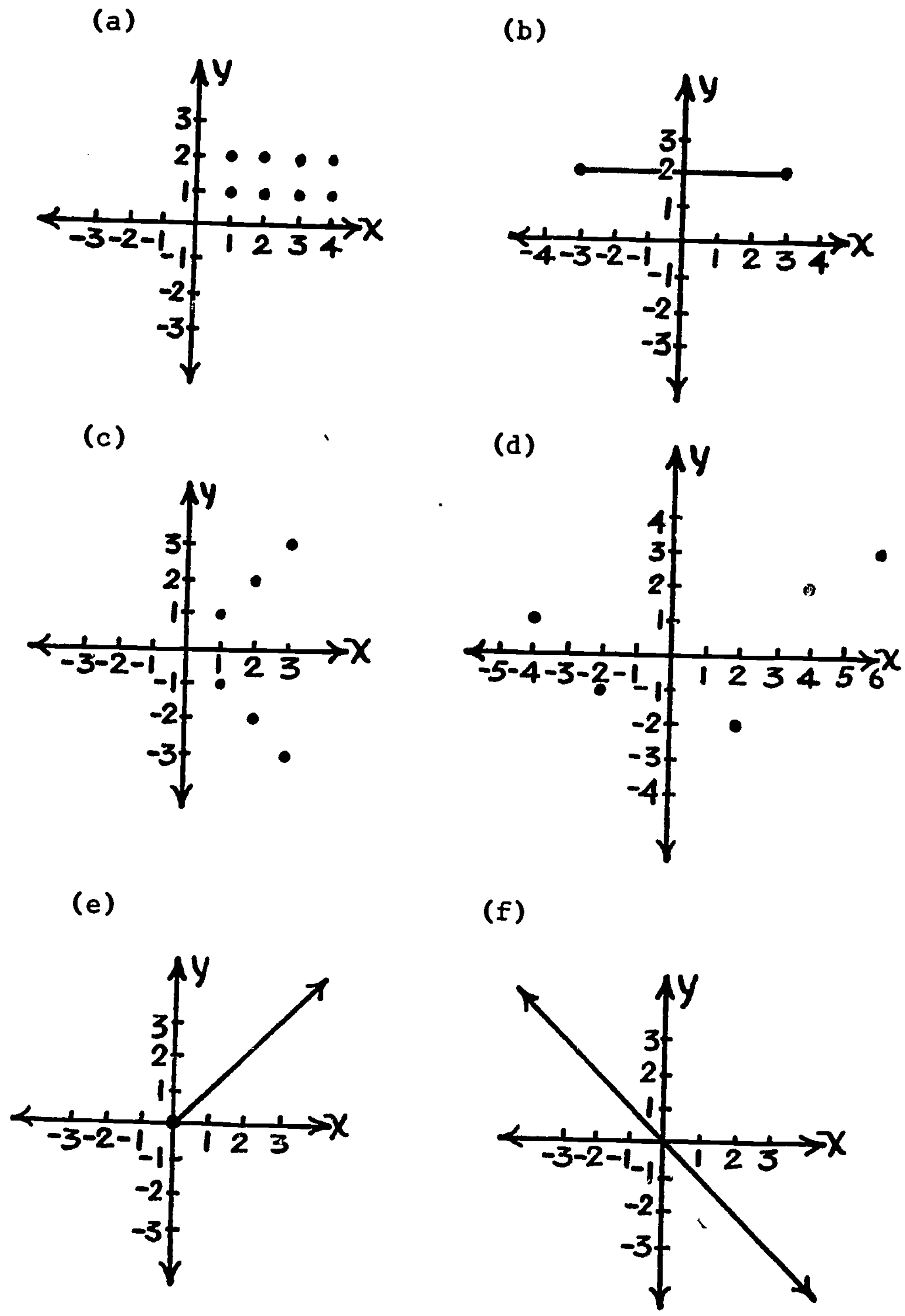
Answers:

(a) domain: {2, 4, 6, 8, 10, 12}
    range: {1, 3, 5, 7, 9, 11}

(b) domain: {-1, 0, 1, 2}
    range: {-3, -2, -1, 0}

(c) domain: {1, 2, 3, 4}
    range: {1, 2, 3, 4}

(22) *Indicate the domain and range of each of the following relations.*



Answers:

(a) domain: {1, 2, 3, 4}
    range: {1, 2}

(b) domain: $\{x \mid -3 \leq x \leq 3\}$
    range: {2}

(c) domain: {1, 2, 3}
    range: {-3, -2, -1, 1, 2, 3}

(d) domain: {-4, -2, 2, 4, 6}
range: {-2, -1, 1, 2, 3}

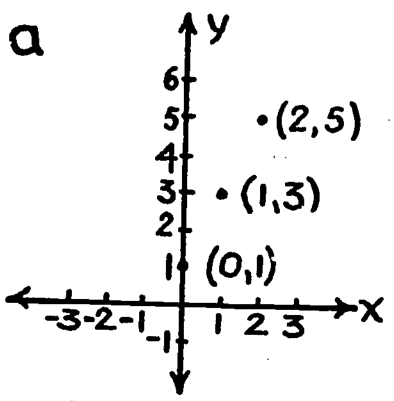
(e) domain: {x | x ≥ 0} or {non-negative real numbers}
range: {y | y ≥ 0} or {non-negative real numbers}

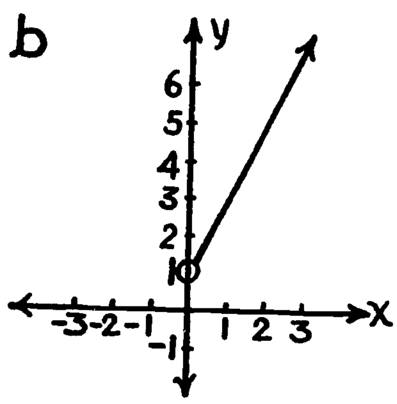
(f) domain: {all real numbers}
range: {all real numbers}

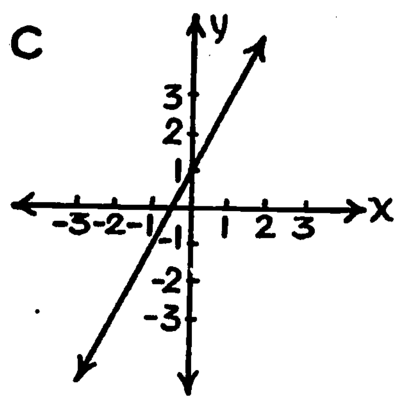
(23) *Graph the locus of the relation defined in U X U by the equation y = 2x + 1 when U is as indicated below:*

(a) U = {0, 1, 2, 3, 4, 5, 6}
(b) U = all real numbers greater than zero
(c) U = all real numbers

Answers:







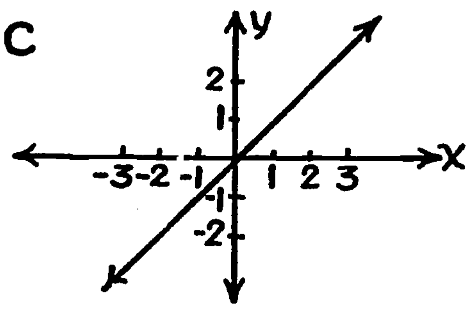
11.3 FUNCTIONS

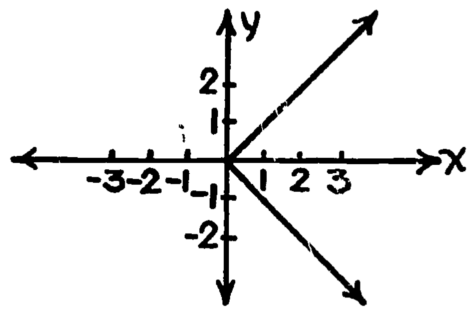
Concept: Definition of a function.

(1) *A function is a certain type of relation. Below in parallel columns are six relations that are functions and six relations that are not functions. By observing the differences in these two types of relations, what does the definition of a function appear to be?*

Relations that are functions	Relations that are not functions
(a) {(1, 2), (2, 2), (3, 4)}	{(1, 2), (1, 3), (3, 4)}
(b) {(1, 2), (3, 3)}	{(1, 2), (2, 2), (2, 3)}

c





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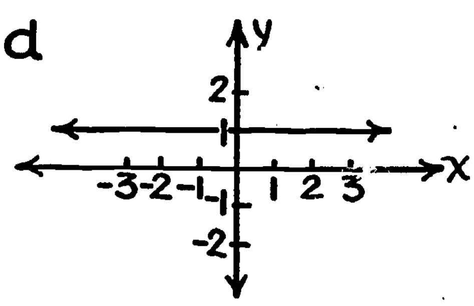

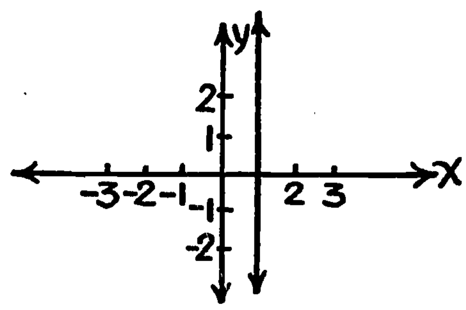

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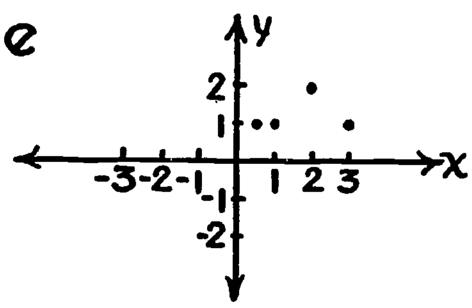

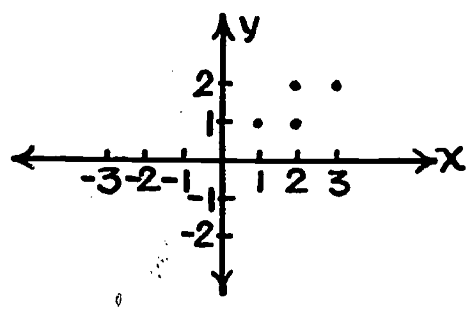

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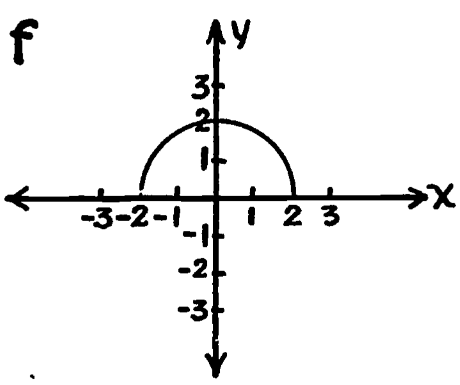
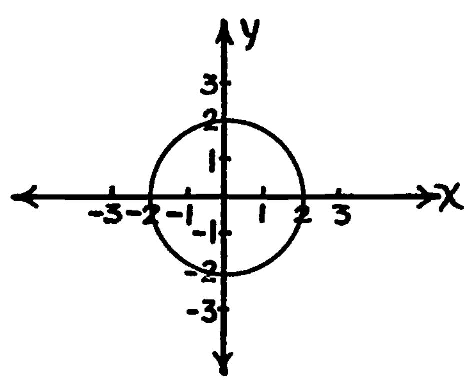




Answer: A function is a relation in which no element in the domain has more than one element in the range associated with it. For every x there is at most one y associated with it.

(2) *Examine a table of values of the trigonometric ratios. Is there more than one value of the sine, cosine, and tangent of an angle associated with a given angle?*

Answer: No

(3) *May we say that these tables define certain trigonometric functions?*

Answer: Yes, since no more than one value of the sine, cosine, and tangent of an angle is associated with a given angle, these tables may be named values of the trigonometric functions.

(4) *Any vertical line intersects the graph of a function at no more than how many points? (This is called the vertical line test for a function.)*

Answer: At no more than one point

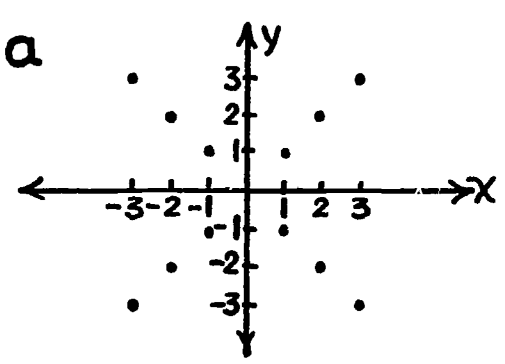
(5) *Define function in terms of this vertical line test.*

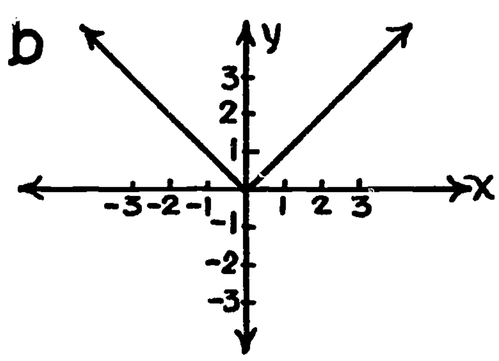
Answer: A relation is a function if and only if any vertical line intersects the graph of the relation at no more than one point.

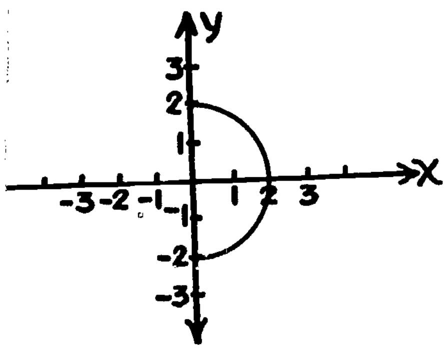
(6) *If U is the set of all real numbers, of what will the graph of U X U consist?*

Answer: The entire number plane

(7) *Below are the graphs of nine relations. Indicate whether each relation is a function.*







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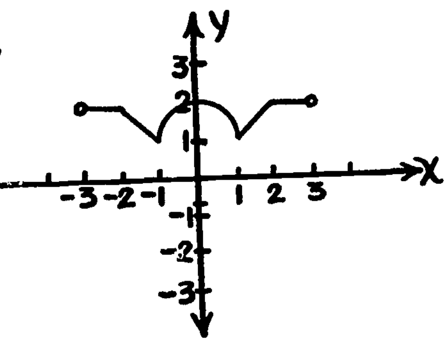

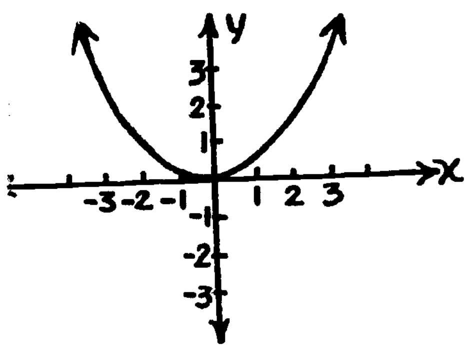

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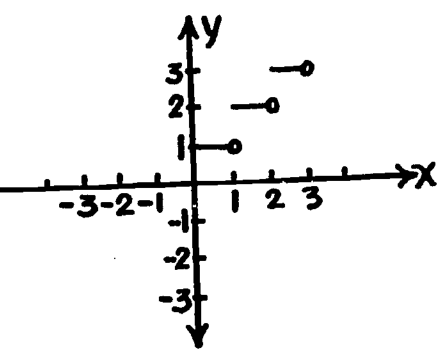

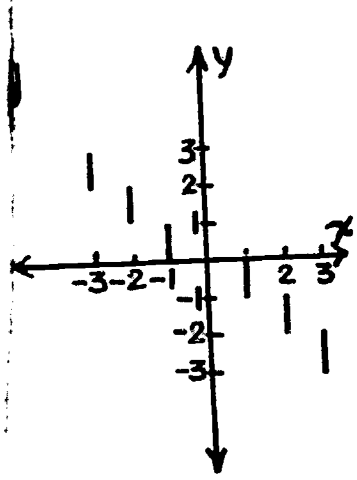

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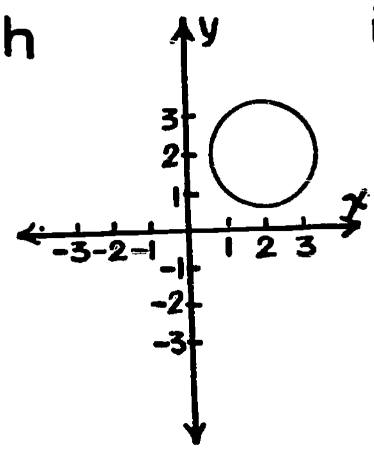

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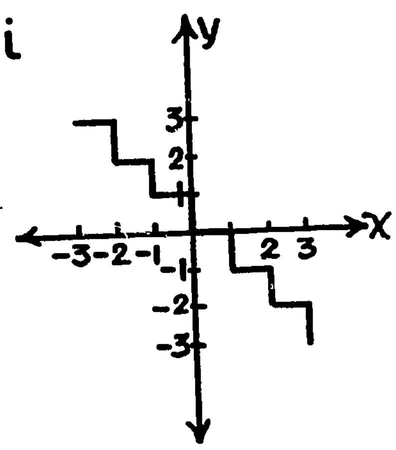


Answers:

(a) Not a function	(f) A function
(b) A function	(g) Not a function
(c) Not a function	(h) Not a function
(d) A function	(i) Not a function
(e) A function	

(8) *Indicate whether or not each of the following is function.*

(a) {(1, 0), (2, 1), (3, 1), (4, 2)}
(b) {(3, 3)}
(c) {(3, 4), (4, 3), (2, 3), (3, 2), (5, 4)}

Answers:
(a) A function (b) A function (c) Not a function

(9) *For each of the following, indicate the set A X B and state whether or not the resulting relation is a function.*

(a) A = {1, 2, 3} B = {4, 5}
(b) A = {0} B = {1, 2, 3, 4, 5, 6}
(c) A = {0, 1, 2} B = {0}

Answers:
(a) {(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)}
Not a function
(b) {(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6)}
Not a function
(c) {(0, 0), (1, 0), (2, 0)}
A function

Concept: Inverse of a function.

(10) *If the members of the ordered pairs of A = {(2, 1), (3, 2), (4, 3), (5, 4)} are interchanged we get B = {(1, 2), (2, 3), (3, 4), (4, 5)}. Is A equal to B?*

Answer: No. The elements of B are not the same as the elements of A. In general, if the members of an ordered pair are interchanged, the new ordered pair is different from the original ordered pair.

(11) *For each of the following functions, indicate the domain and range of the function, indicate the relation obtained by interchanging the members of the ordered pairs of the function, and indicate the domain and range of the resultant relation.*

(a) A = {(4, 1), (5, 3), (6, 3)}
(b) B = {(1, 3), (2, 2), (3, 1)}
(c) E = {(x, y) | y = x + 1}

Answers:
(a) A = {(4, 1), (5, 3), (6, 3)}
domain: {4, 5, 6}
range: {1, 3}

X = {(1, 4), (3, 5), (3, 6)}
domain: {1, 3}
range: {4, 5, 6}

(b) B = {(1, 3), (2, 2), (3, 1)}
domain: {1, 2, 3}
range: {1, 2, 3}

Y = {(3, 1), (2, 2), (1, 3)}
domain: {1, 2, 3}
range: {1, 2, 3}

(c) E = {(x, y) | y = x + 1}
domain: {all real numbers}
range: {all real numbers}

Z = {(x, y) | x = y + 1}
domain: {all real numbers}
range: {all real numbers}

(12) (a) *Graph each of the functions given in the previous question and also the relations obtained by interchanging the members of the ordered pairs of the given function.*

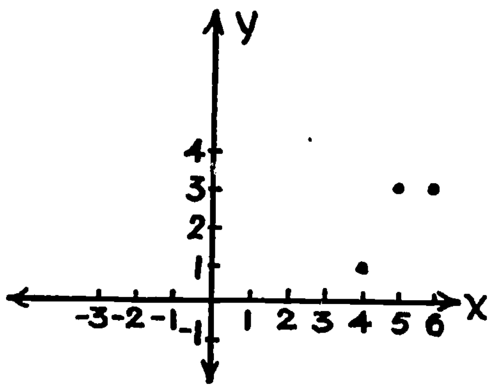
(b) *Is the resultant relation always a function?*

(c) *If the resultant relation is also a function, we say that the original function has an inverse. Which function has no inverse?*

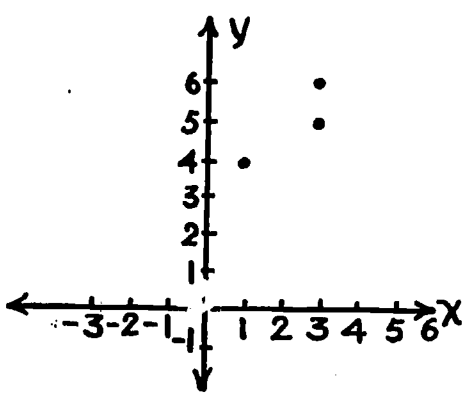
(d) *If a function F has an inverse, the inverse function may be designated as F^{-1}. Indicate the inverse functions in part (a) using this symbol.*

Answers:

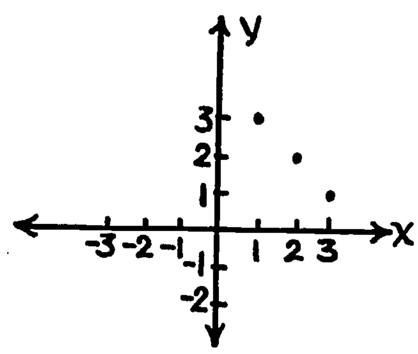
(a) A = {(4, 1), (5, 3), (6, 3)}

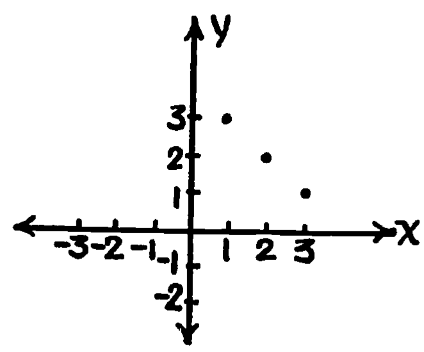


X = {(1, 4), (3, 5), (3, 6)}

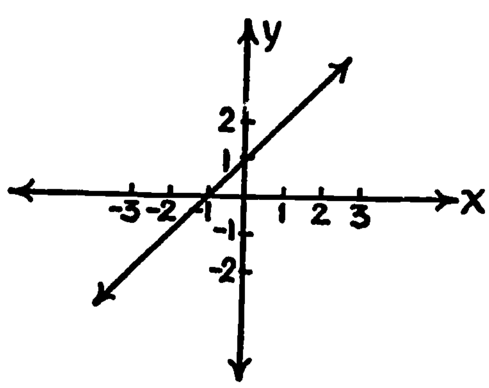


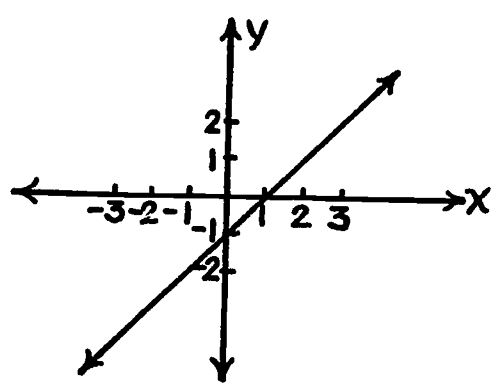
B = {(1, 3), (2, 2), (3, 1)}   Y = {(3, 1), (2, 2), (1, 3)}





E = {(x, y) | y = x + 1}, Z = {(x, y) | x = y + 1}





(b) No. The relation obtained by interchanging the members of the ordered pairs of a function is not necessarily a function.

(c) The first given function has no inverse.

(d) B^{-1} = {(3, 1), (2, 2), (1, 3)},
E^{-1} = {(x, y) | x = y + 1}

(13) *For each of the following functions, U is the set of all real numbers. Graph each function and its inverse on the same axes.*

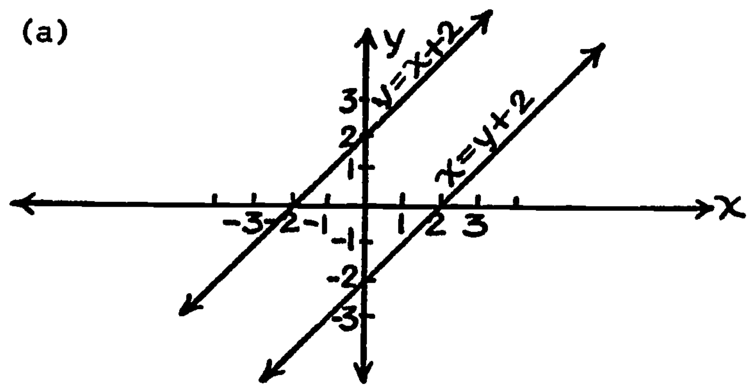
(a) A = {(x, y) | y = x + 2}, A^{-1} = {(x, y) | x = y + 2}

(b) B = {(x, y) | y = 3x}, B^{-1} = {(x, y) | x = 3y}

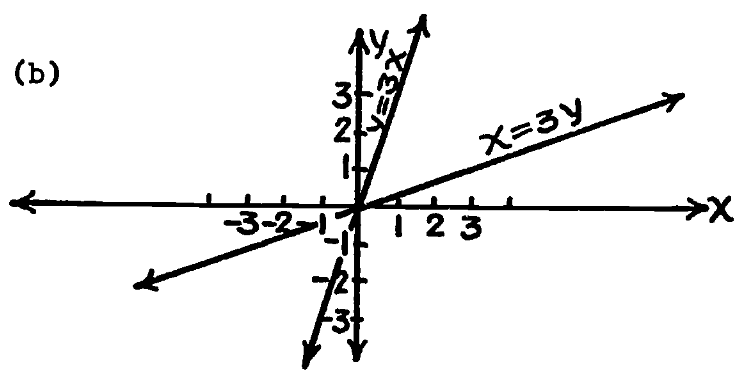
(c) C = {(x, y) | 2x = y - 2}, C^{-1} = {(x, y) | 2y = x - 2}

Answers:

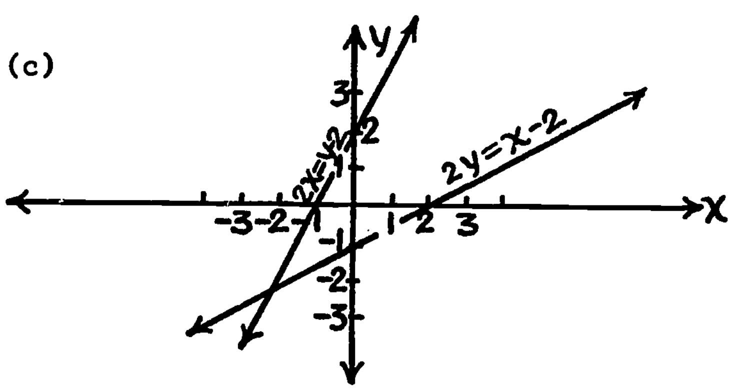
(a)



(b)



(c)



(14) *Graph any function having an inverse and on the same coordinate system graph the inverse of that function. Draw the line $x = y$. What relationship is observed between the graph of the function, the graph of its inverse, and the line $x = y$?*

<u>Answer</u>: The graph of the function is a reflection of its inverse with respect to the line x = y.

Concept: Functional notation.

(15) (a) *The function* $\{(x, y) \mid y = 2x + 1\}$ *may also be written* $\{(x, F(x)) \mid F(x) = 2x + 1\}$ *where* $F(x)$ *and* y *are used as two different symbols for the same thing.* $F(x)$ *is read "F at x" and denotes the second element in the ordered pair whose first element is x. Write in terms of x and y the equation that defines the function* $A = \{(x, F(x) \mid F(x) = 3x^2 - 6\}$.

Answer: $y = 3x^2 - 6$

(b) *If* $y = 3x^2 - 6$, *what is the value of y at x when x is 2?*

Answer: $y = 3(4) - 6$
$y = 6$

(c) $F(x)$ *is also called "the value of the function at x," the value being determined by substituting for x in the equation, the value given for x. If* $F(x) = 3x^2 - 6$, *find* $F(3)$.

Answer: $F(3) = 3(9) - 6$
$F(3) = 21$

(16) *Compute* $F(0)$, $F(-1)$, $F(1)$, $F(2)$, *and* $F(\sqrt{2})$ *for each of the following:*

(a) $F(x) = 4x - 2$

(b) $F(x) = x^2 + |x|$

(c) $F(x) = x^2 - |x| + 2x$

Answers:

(a) $F(0) = -2$
$F(-1) = -6$
$F(1) = 2$
$F(2) = 6$
$F(\sqrt{2}) = 4\sqrt{2} - 2$

(b) $F(0) = 0$
$F(-1) = 2$
$F(1) = 2$
$F(2) = 6$
$F(\sqrt{2}) = 2 + \sqrt{2}$

(c) $F(0) = 0$
$F(-1) = -2$
$F(1) = 2$
$F(2) = 6$
$F(\sqrt{2}) = 2 + \sqrt{2}$

(17) *Given* $U = \{0, 1, 2, 3, 4, 5, 6\}$ *determine the solution set in* $U \times U$, *draw the graph and determine the range and domain of each of the following functions.*

(a) $F(x) = \frac{x}{2}$ (b) $G(x) = x - 2$

(c) $H = \{(x, y) \mid x^2 + y^2 = 25\}$

(d) $J = \{(x, y) \mid x + y = 5 \text{ and } xy = 6\}$

Answers:

(a) $F(x) = \frac{x}{2}$

$F(0) = 0$

$F(1) = \frac{1}{2}$*

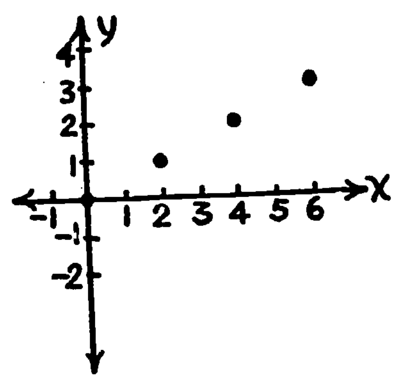
$F(2) = 1$

$F(3) = \frac{3}{2}$*

$F(4) = 2$

$F(5) = \frac{5}{2}$*

$F(6) = 3$



*Not in the universal set, therefore omit.

Solution set: {(0, 0), (2, 1), (4, 2), (6, 3)}

domain: {0, 2, 4, 6}

range: {0, 1, 2, 3}

(b) $G(x) = x - 2$

$G(0) = -2$*

$G(1) = -1$*

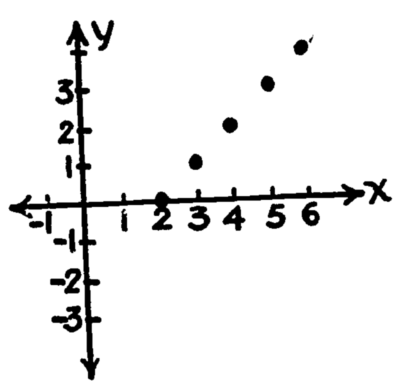
$G(2) = 0$

$G(3) = 1$

$G(4) = 2$

$G(5) = 3$

$G(6) = 4$



*Not in the universal set, therefore omit.

Solution set: {(2, 0), (3, 1), (4, 2), (5, 3), (6, 4)}

domain: {2, 3, 4, 5, 6}

range: {0, 1, 2, 3, 4}

(c) $H = \{(x, y) \mid x^2 + y^2 = 25\}$

$y = H(x) = \pm \sqrt{-x^2 + 25}$

$H(0) = 5$ and -5*

$H(1) = \pm 2\sqrt{6}$ *

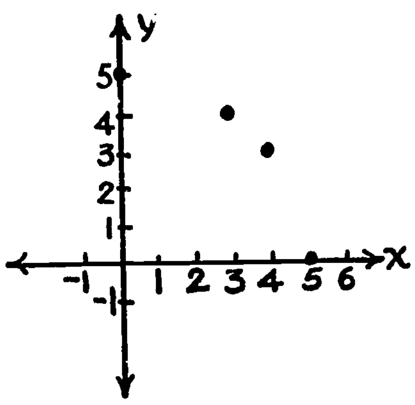
$H(2) = \pm \sqrt{21}$*

$H(3) = 4$ and -4*

$H(4) = 3$ and -3*

$H(5) = 0$

$H(6) = \sqrt{-9}$*



*Not in the universal set, therefore omit.

Solution set: {(0, 5), (3, 4), (4, 3), (5, 0)}

domain: {0, 3, 4, 5}

range: {0, 3, 4, 5}

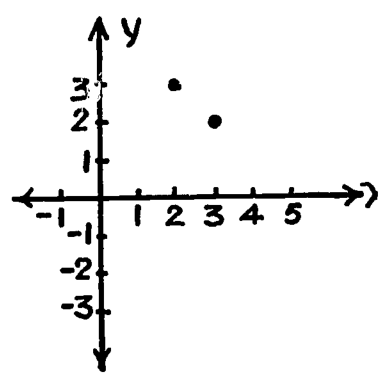
(d) $J = \{(x, y) \mid x + y = 5 \text{ and } xy = 6\}$

$x + y = 5$	$xy = 6$
$y = 5 - x$	$x(5 - x) = 6$
	$x = 3 \quad x = 2$
	$y = 2 \quad y = 3$

Solution set: {(2, 3), (3, 2)}

domain: {2, 3}

range: {2, 3}



Concept: Graph of a linear function.

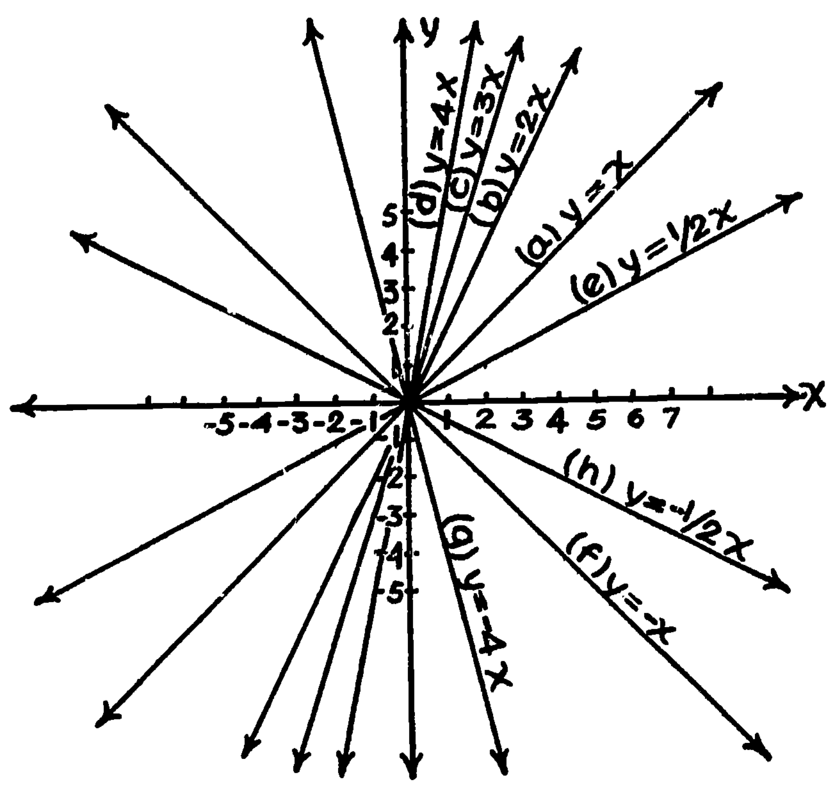
(18) *The set of ordered pairs (x, y) for which $y = mx + b$, where m and x are real numbers, is called a linear function. The graph of a linear function is always a straight line. Each of the equations below is in the form $y = mx + b$, where $b = 0$. On the same set of coordinates axes graph the functions defined by the equations below.*

(a) $y = x$ (b) $y = 2x$ (c) $y = 3x$ (d) $y = 4x$

(e) $y = \frac{1}{2}x$ (f) $y = -x$ (g) $y = -4x$ (h) $y = -\frac{1}{2}x$

What is the relationship between the steepness of the graph and the coefficient of x in each of the equations in (18)? What is the relationship between the direction of the slope of each graph and the fact that the coefficient of x is positive or negative?

Answer:



If the coefficient is a positive number, the graph slopes upward to the right, and the greater the coefficient, the greater is the steepness of the graph.

If the coefficient is a negative number, the graph slopes downward to the right, and the more negative the coefficient, the greater is the steepness of the graph.

(19) *The steepness of a graph is indicated by its slope. The slope is defined as the rate of change of abscissas with respect to change in ordinates. That is, if (x_1, y_1) and (x_2, y_2) are two points on the graph of a linear function, the slope is defined as the ratio*

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

For each of the following equations, graph the function defined by the equation, select two points on the graph of the function and determine the slope of the graph by applying the definition of slope just described.

(a) y = 3x + 4 (d) y = -3x + 6
(b) y = 5x - 2 (e) y = -5x - 6
(c) y = x + 9 (f) y = -x + 3

Answers:
(a) Slope is 3 (b) Slope is 5 (c) Slope is 1
(d) Slope is -3 (e) Slope is -5 (f) Slope is -1

(20) *Each of the equations in the previous question is in the form y = mx + b. What is the relationship between the slope of the graph and the number represented by m?*

Answer: The slope of the graph is equal to the number represented by m.

(21) *The ordinate of the point where the graph intercepts the y-axis is called the y-intercept. Determine the y-intercept of each of the graphs of the functions defined by the equations in question 19. What is the relationship between the y-intercept and the number represented by b when the the equation defining the function is written in the form y = mx + b?*

Answers:
(a) y-intercept is 4 (b) y-intercept is -2
(c) y-intercept is 9 (d) y-intercept is 6
(e) y-intercept is -6 (f) y-intercept is 3
The y-intercept is equal to the number represented by b in the equation y = mx + b

(22) *Indicate the slope and y-intercept of the graphs of each of the functions defined by the following equations.*
(a) y = -x + 11 (b) y = 7x - 21 (c) 4x + 2y = 2

Answers:
(a) Slope is -1 and y-intercept is 11
(b) Slope is 7 and y-intercept is -21
(c) Slope is -2 and y-intercept is 1

(23) *Write the equation that defines the function whose graph has the slope and y-intercept as indicated below.*
(a) Slope 3 and y-intercept -6
(b) Slope -2 and y-intercept 1
(c) Slope 1 and y-intercept 5

Answers:
(a) y = 3x - 6 (b) y = -2x + 1 (c) y = x + 5

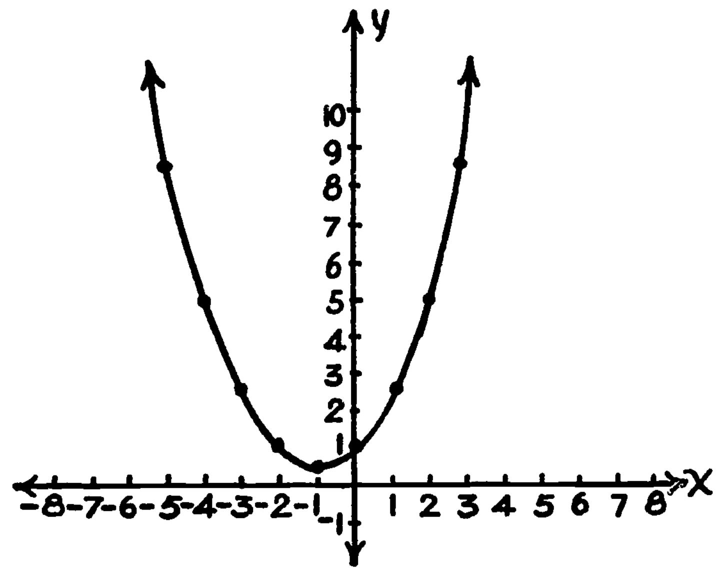
Concept: Graph of a quadratic function.

(24) *A quadratic function is a set of ordered number pairs defined by an equation $F(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. If $F(x) = \frac{1}{2}x^2 + x + 1$, find F(0), F(1), F(2), F(3), F(-1), F(-2), F(-3), F(-4), and F(-5). Graph the resulting ordered pairs and connect these points with a smooth curve which goes somewhat beyond the last point on each side and terminates in an arrowhead. A curve of this type is called a parabola.*

Answer:

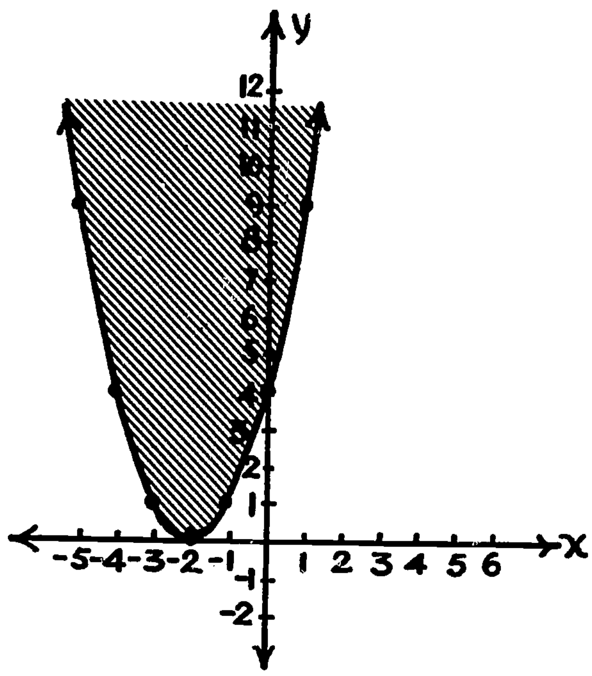
$F(0) = 1$, $F(1) = 2\frac{1}{2}$, $F(2) = 5$, $F(3) = 8\frac{1}{2}$, $F(-1) = \frac{1}{2}$,

$F(-2) = 1$, $F(-3) = 2\frac{1}{2}$, $F(-4) = 5$, $F(-5) = 8\frac{1}{2}$



(25) *Graph the function defined by the equation $y = x^2 + 4x + 4$ by plotting enough points to determine the general shape of the graph. Indicate the portion of the number plane that would be the solution set of the relation defined by the inequality $y \geq x^2 + 4x + 4$.*

Answer: $y \geq x^2 + 4x + 4$



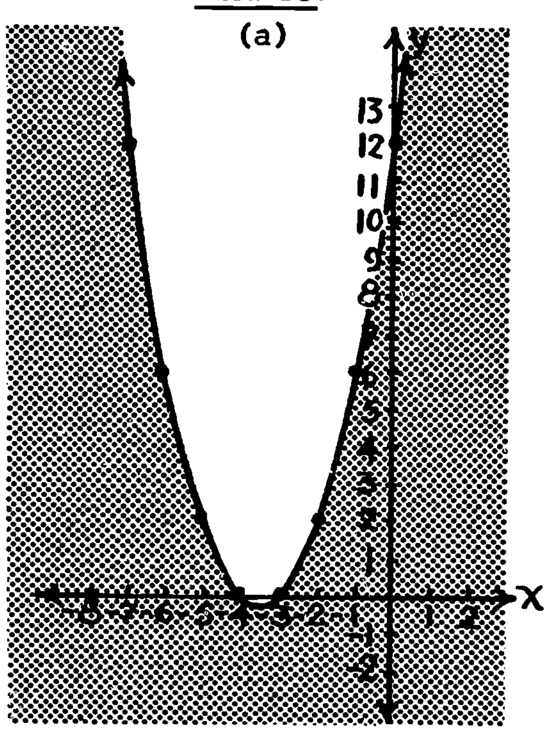
(26) *Graph the relations defined by each of the following inequalities:*

(a) $y \leq x^2 + 7x + 12$ (c) $y < x^2 + 2x + 1$

(b) $y > x^2 - 7x + 12$

Answers:

(a)



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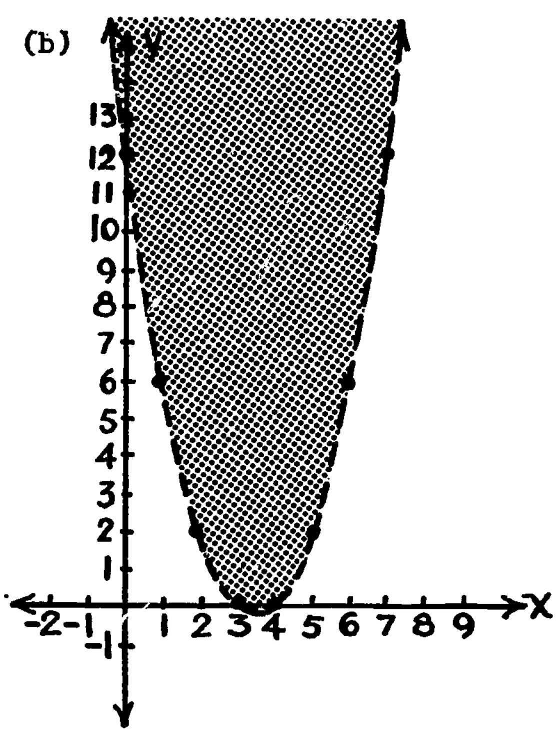

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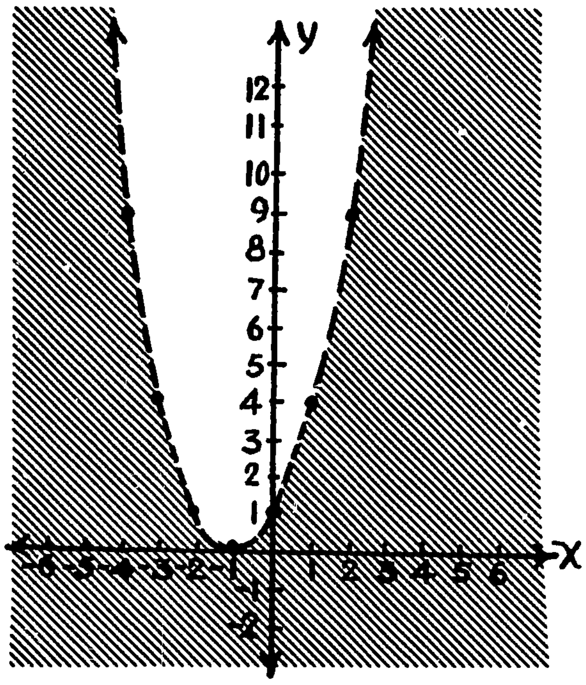
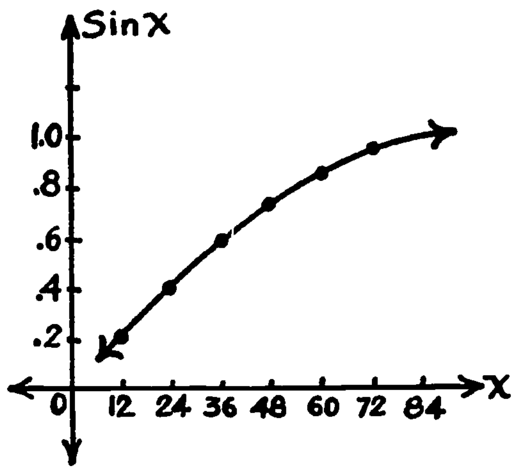


Concept: Graphs of portions of the sine function, the cosine function, and the tangent function.

(27) *The entries in the various kinds of numerical tables, including the tables of the trigonometric functions, can be used to form sets of ordered number pairs, and these pairs can be graphed. A problem which arises in making graphs of relations or functions described by numerical tables, particularly tables of some of the empirical formulas, is the scale to be used on each axis. In general, the scales on the two axes will be different and will be chosen so as to make the graph neither too compressed nor too expanded. In these exercises, we shall start each scale at 0. If A = {12, 24, 36, 48, 60, 72, 84} is a set of angle measures in degrees, use a table of trigonometric functions to write the function S in which member of A is paired with its sine ratio, rounded to the nearest hundredth, as a second member. Graph the function S. The set S is only part of a more complete set whose members are represented by a curve rather than a set of isolated points. Connect the points which were plotted with a smooth curve which goes a little beyond the last point on each side and is terminated in an arrowhead.*

Answer:

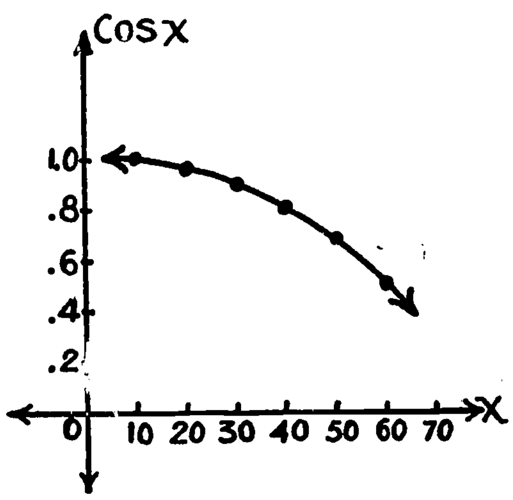
S = {(12, 0.21),
(24, 0.41),
(36, 0.59),
(48, 0.74),
(60, 0.87),
(72, 0.95),
(84, 0.99)}



(28) *If $x \in U = \{10, 20, 30, 40, 50, 60\}$, graph the function $C = \{(x, F(x)) \mid F(x) = \cos x\}$, in which each member of U is paired with its cosine ratio, rounded to the nearest hundredth, as a second member. Connect the points in the graph with a smooth curve which goes a little beyond the last point on each side and is terminated by an arrowhead. (The angle is measured in degrees.)*

Answer:

C = {(10, 0.98),
(20, 0.94),
(30, 0.87),
(40, 0.77),
(50, 0.64),
(60, 0.50)}



(29) (a) *If $B = \{60, 65, 70, 75, 80, 85\}$, write the function T in which each member of B is paired with its tangent ratio, rounded to the nearest tenth, as a second member. (The angle is measured in degrees.)*

(b) *Draw the graph of the function T.*

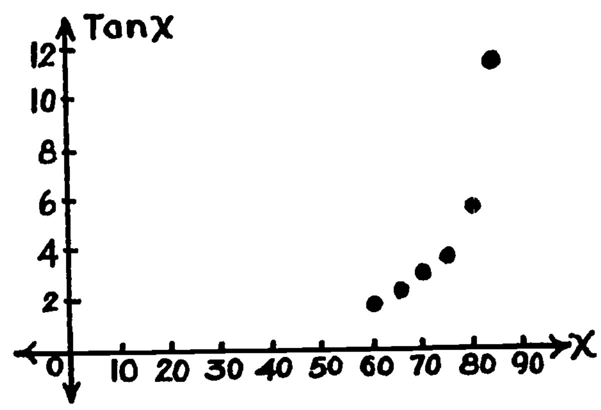
(c) *Is ($\tan 65° - \tan 60°$) less than, equal to, or greater than ($\tan 85° - \tan 80°$)?*

Answers:

(a)

T = {(60, 1.7),
(65, 2.1),
(70, 2.7),
(75, 3.7),
(80, 5.7),
(85, 11.4)}

(b)



(c) $\tan 65^\circ - \tan 60^\circ = 2.1 - 1.7 = 0.4$
$\tan 85^\circ - \tan 80^\circ = 11.4 - 5.7 = 5.7$
$(\tan 65^\circ - \tan 60^\circ) < (\tan 85^\circ - \tan 80^\circ)$

Note: Although $T = \{(x, F(x)) \mid F(x) = \tan x\}$ was graphed here as a set of discrete points because that was how the question was phrased, in this case too, T is part of a more complete set whose members are represented by a curve rather than a set of isolated points.

Teachers Notes

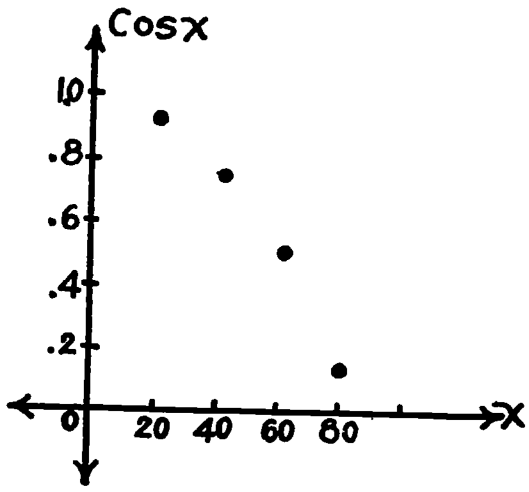
UNIT TEST

1. a. *If U = {20, 40, 60 80} is a set of angle measures in degrees, use a table of values of the trigonometric functions to write the function C in which each member of U is paired with its cosine ratio as a second member.*
   b. *Make a graph of C.*

Answers:

(a) C = {(20, 0.9397), (40, 0.7660), (60, 0.5000), (80, 0.1736)}

(b)



2. *If A = {-3, 0, 5, 7}*
   B = {-3, -4}
   C = {0, 6}, find:

(a) A X B (b) C X A (c) B X B (d) A X C
(e) *Write the equation for which set C is the solution set.*

Answers:

(a) A X B = {(-3, -3), (-3, -4), (0, -3), (0, -4), (5, -3), (5, -4), (7, -3), (7, -4)}
(b) C X A = {(0, -3), (0, 0), (0, 5), (0, 7), (6, -3), (6, 0), (6, 5), (6, 7)}
(c) B X B = {(-3, -3), (-3, -4), (-4, -3), (-4, -4)}
(d) A X C = {(-3, 0), (-3, 6), (0, 0), (0, 6), (5, 0), (5, 6), (7, 0), (7, 6)}

(e) Set C is the solution set of the equation $x^2 - 6x = 0$.

3. *What is the slope and y-intercept of the graph of the function defined by the equation $2y - 4x = 2x + y - 4$?*

Answer: $y = 6x - 4$
The slope is 6 and the y-intercept is -4.

4. *Determine F(0), F($\sqrt{2}$), F($-\frac{1}{2}$), and F(0.1) if* $F(x) = \dfrac{2x + 3}{x - x^2}$.

Answer: F(0) is undefined

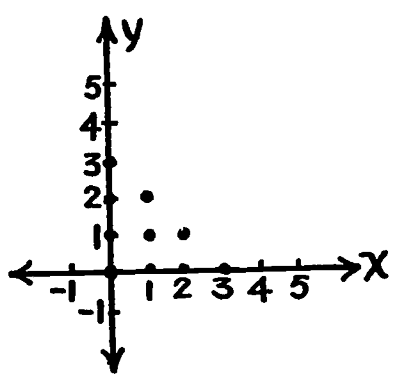
$F(\sqrt{2}) = -\dfrac{7\sqrt{2} + 10}{2}$

$F\left(-\dfrac{1}{2}\right) = -\dfrac{8}{3}$

$F(0.1) = \dfrac{320}{9}$

5. *If* $U = \{0, 1, 2, 3\}$, *graph* $\{(x, y) \mid x + y < 4\}$ *in* $U \times U$.

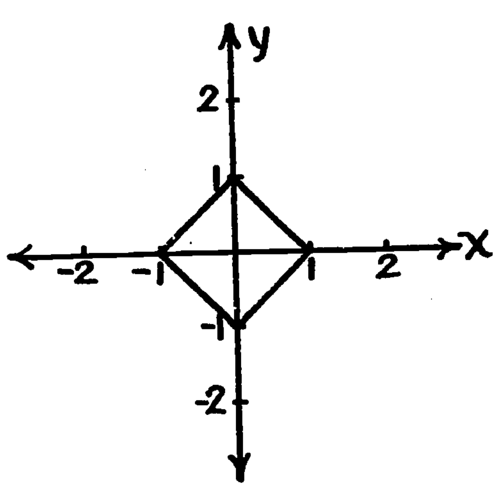
Answer:



6. *Graph the relation* $\{(x, y) \mid |x| + |y| = 1\}$ *and determine whether it is a function.*

Answer:

The relation is not a function.



7. *If A = {(1, 2), (2, 2), (3, 4), (5, 3)} find B such that the members of the ordered pairs of A are interchanged. Is A a function? Is B a function? Does A have an inverse?*

<u>Answer</u>: B = {(2, 1), (2, 2), (4, 3), (3, 5)}
A is a function and B is not a function. No, A has no inverse.

8. *Graph the relation* $\{(x, y) \mid y = 2^x\}$. *Is this relation a function? Using the horizontal line test, determine whether the new relation obtained by interchanging the members of the ordered pairs of the original relation is a function.*

<u>Answer</u>: The original relation is a function.
The new relation is also a function and therefore is the inverse of the original function.