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AN EXPERIMENTAL COURSE IN MATHEMATICS FOR THE NINTH YEAR. UNITS 8 AND 9, POLYNOMIAL EXPRESSIONS AND POLYNOMIAL EQUATIONS.

NEW YORK STATE EDUCATION DEPT., ALBANY

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THIS TEACHING GUIDE IS THE THIRD OF FIVE EXPERIMENTAL EDITIONS CONCERNING MATERIALS AND METHODS FOR TEACHING A REVISED MATHEMATICS PROGRAM IN GRADE 8. BACKGROUND MATERIAL FOR TEACHERS AS WELL AS QUESTIONS AND ACTIVITIES FOR CLASSROOM PRESENTATIONS ARE PROVIDED IN THE CONTENT AREAS OF POLYNOMIAL EXPRESSIONS (UNIT 8) AND POLYNOMIAL EQUATIONS (UNIT 9). UNIT 8 CONTAINS SECTIONS ON ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION OF POLYNOMIAL EXPRESSIONS, AND FACTORING POLYNOMIAL EXPRESSIONS. UNIT 9 INCLUDES SECTIONS ON SOLUTION BY FACTORING, SOLUTION BY COMPLETING THE SQUARE, SOLUTION BY QUADRATIC FORMULA, GRAPHING QUADRATIC EQUATIONS, AND SIMPLE PROOFS. (RP)

Units 8 and 9

an experimental course in MATHEMATICS FOR THE NINTH YEAR

THE UNIVERSITY OF THE STATE OF NEW YORK / THE STATE EDUCATION DEPARTMENT
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AN EXPERIMENTAL COURSE
IN
M A T H E M A T I C S
FOR THE
NINTH YEAR

Unit 8. Polynomial Expressions

Unit 9. Polynomial Equations

The University of the State of New York
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AN EXPERIMENTAL COURSE IN MATHEMATICS FOR THE NINTH YEAR

Mathematics 9X

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Mathematics 9X

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FOREWORD

In April 1961, an advisory committee on secondary school mathematics convened at the Department to discuss the direction that secondary mathematics curriculum revision should take. This committee consisted of college and secondary school teachers, supervisors, administrators, and a consultant from one of the national curriculum programs. As a result of this meeting, the recommendation was made that a revision of the mathematics 7-8-9 program be undertaken immediately.

This publication represents the third of a series of experimental units for a course in mathematics for the ninth grade. The first publication consisted of units 1-4; the second, units 5, 6, and 7; and this one contains materials for units 8 and 9. The remaining units, 10-12, will be distributed in one final publication during the 1964-65 school year when it is completed.

The materials in the 9X experimental syllabus are based upon the foundations laid in the 7X and 8X experimental syllabuses. Therefore, it is to be understood that the 7X and 8X experimental courses are a prerequisite to the 9X experimental course. As in the 7X and 8X syllabuses, the chief emphasis is placed upon the understanding of basic mathematical concepts as contrasted with the all-too-frequently used program in which the mechanics of mathematics receives the greatest stress. The general approach and content used is that agreed upon by leading mathematical authorities as the most desirable. In the actual teaching of the program major emphasis is placed upon the "discovery process." The principal function of the teacher is to carefully set the stage for learning in an organized fashion such that the pupils will "discover" for themselves the fundamental concepts involved.

The materials in the mathematics 7X, 8X, and 9X experimental syllabuses include much of what today are called the basic ideas and concepts of mathematics. These concepts are those which the pupils will use throughout their study in mathematics. With this material the teacher should be able to aid the pupils to see the beauty of mathematics in terms of the fundamental structure found in mathematical systems. The important unifying concepts included in the new course of study for the ninth grade are:

- Algebraic Expressions and Open Sentences
- Analysis of Algebraic Problems
- The Set of Real Numbers
- Properties of Exponents and Radicals
- Operations with Polynomial Expressions
- Quadratic Equations
- Open Sentences in Two Variables
- Relations and Functions
- Trigonometric Functions

A new mathematical curriculum is not the sole answer to the improvement of mathematics instruction. Most important perhaps is the method of presenting the material. If the teacher develops lesson plans that will allow the pupils to discover concepts for themselves, the teaching and learning of mathematics will become excitingly different and no longer remain the dissemination of rules and tricks.

A special committee was formed to review the 9X syllabus and to make recommendations for the writing of materials. This committee consisted of the following: David Adams, Liverpool High School; Benjamin Bold, Coordinator of Mathematics, High School Division, New York City Board of Education; Mary Challis, Plattsburgh High School, Francis Foran, Garden City Junior High School; Eleanor Maderer, Coordinator of Mathematics, Board of Education, Utica; William Mooar, Benjamin Franklin Junior High School, Kenmore; Verna Rhodes, Corning Free Academy; Leonard Simon, Curriculum Center, New York City; Joan Vodek, Chestnut Hill Junior High School, Liverpool; Frank Wohlfort, Coordinator of Mathematics, Junior High School Division, New York City Board of Education.

The materials for the 9X syllabus were written by Charles Burdick, coordinator and teacher of mathematics, Oneida Junior High School, Schenectady. The project has been developed under the joint supervision of this Bureau and the office of Frank Hawthorne, Chief, Bureau of Mathematics Education, who guided the planning. Aaron Buchman, associate in mathematics education, reviewed and revised the original manuscript. Herbert Bothamley, acting as temporary curriculum associate, edited and prepared the final manuscript for publication.

Gordon E. Van Hooft
Chief, Bureau of Secondary
Curriculum Development

William E. Young
Director, Curriculum
Development Center

UNIT 8: POLYNOMIAL EXPRESSIONS

PART 1. BACKGROUND MATERIAL FOR TEACHERS

8.1 INTRODUCTION

One of the first difficulties to be encountered in teaching this unit is that of explaining the meaning of the term "polynomial." The technical definition is that a polynomial is a rational integral algebraic expression of the form: (1) a_n or (2) $a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ where n is any positive integer and $a_0 \neq 0$. The form a_n is included for completeness. In the standard form x itself shall be simple.

This definition requires that the expression be rational. This means that no variable is in an irreducible radical or is under a fractional exponent. The expression $2x^2 + 1$ and $2x + \frac{1}{x}$ are rational expressions but $\sqrt{x + 1}$ and $x^{\frac{1}{2}} + 1$ are not. The variable cannot be under a radical nor have a fractional exponent.

This definition requires that the expression be integral. This means the variable when written with a positive exponent cannot appear in any denominator. This of course also means that any variable written with a negative exponent cannot appear in any numerator. However, to be a polynomial, the variable must be expressed with only positive exponents.

The expression $2x + \frac{1}{x}$ is a rational expression, but it is not an integral expression because the variable appears in a denominator.

The three major restrictions contained in the definition are:

(1) The exponents of the variable must be positive integers.
(2) The variable cannot be under an irreducible radical.
(3) The variable cannot be in a denominator.

The following are examples of expressions which are not polynomials.

(a) $\frac{x^2 + 1}{x - 1} + 3$ The variable is in a denominator.

(b) $2\sqrt{x} + 9$ The variable is under a radical.

(c) $2x^{-2} + 3x - 6$ The variable has an exponent which is not a positive integer.

(d) $3x^{\frac{2}{3}} + 4$ The variable has a fractional exponent. This is the same as being under a radical.

Notice that the restrictions apply only to the variable. They do not apply to the coefficients or the constant term. In the general definition of polynomials, no restrictions are placed on the coefficients, but such restrictions may be imposed to limit the polynomial to any desired set of numbers. If the coefficients are to be restricted to the set of integers, rationals, or real numbers, then the polynomial being defined is called "a polynomial over the integers," "a polynomial over the rationals," or "a polynomial over the reals" respectively.

In this unit, the topics relating to polynomials have been restricted almost entirely to rational coefficients which for the most part are also integral. However, just a few of the exercises involve irrational coefficients to clearly indicate to the pupils that there is no inherent prohibition against the coefficients being irrational numbers. Such exercises give the teacher an opportunity to explain the difference between $\sqrt{2}\,x$ and $\sqrt{2x}$ as far as the definition of polynomial is concerned. $\sqrt{2}\,x$ is a polynomial but $\sqrt{2x}$ is not a polynomial.

The question sometimes arises as to whether or not expressions such as 3x or 19 are polynomials. Monomials are polynomials; therefore, both 3x and 19 are polynomials.

Much of the work in this unit on polynomials, particularly the work on factoring, forms the foundations for the study of the topic of solving polynomial equations in the next unit.

8.2 ADDITION AND SUBTRACTION OF POLYNOMIALS

This unit begins with a development of the concept of what is meant by the term "polynomial." The definition developed is not the technical definition just described, but it is mathematically correct and sufficient for this course. The simpler definition is restricted to a polynomial over the reals. Polynomial is defined as a term or sum of terms, each of which is a real number, or an integral rational algebraic expression consisting of a product of a real number and a positive integral power of the variable. This definition requires in its development explanation of the phrase "integral rational algebraic expression." The meaning of this phrase was discussed in the introduction to this unit.

A polynomial, as defined above, must be a term or sum of terms. Therefore, an expression such as $y^2 - 5y - 3$ must be considered as being $y^2 + (-5y) + (-3)$ before it can be accepted as a polynomial. This is an important consideration in determining the coefficient of a variable. For example, a pupil might consider the coefficient of the x term in the expression $x^2 - 5x - 3$ to be 5, when actually the coefficient is -5. If the expression is first considered as being $x^2 + (-5x) + (-3)$, there is no such confusion as to the coefficients of any variable. The same applies to the constant term. The constant term in $x^2 - 5x - 3$ is -3, not 3.

The above definition of a polynomial also means that an expression such as $(x + 2)(x + 3)$ is a polynomial but it is not in standard polynomial form when written in this form. However, when it is written in the form $x^2 + 5x + 6$, it is in standard polynomial form. This is a technical point, but it does have an important application later in the unit when the expression "degree of a polynomial" is defined.

The restrictions on the variable contained in the definition of a polynomial and the lack of such restrictions on the coefficients is a very common point of confusion to many pupils. Some time should be devoted to preventing or clearing up any such confusion. The difficulty is usually in identifying expressions which are polynomials but which resemble expressions which are easily recognized as not being polynomials. Below are a few examples of such pairs of expressions.

Not polynomials	Polynomials
$10x^{-4}$	$10^{-4}x$
$3\sqrt{x}$	$\sqrt{3}\,x$
$6x^{\frac{2}{3}}$	$6^{\frac{2}{3}}x$
$\sqrt{3x} + 2\sqrt{3}$	$\sqrt{3} + 2\sqrt{3}$

The terms monomial, binomial, and trinomial each describe a particular type of polynomial. As indicated by their prefixes, each is a name for a polynomial of one term, two terms, and three terms respectively. Defining a monomial as a polynomial is consistant with the definition of a polynomial because a polynomial may be a term as simple as merely any real number.

The degree of a polynomial is the greatest exponent of the variable contained in the polynomial when written in standard polynomial form. For example, the degree of $y^2 - 6y^5 + 7y - 9$ is 5. There is sometimes confusion at first as to the degree of expressions such as $(x^2 + 5)(x^2 - 5)$ or $(x^3 - 6)^3$. A pupil may indicate that the degree of $(x^2 + 5)(x^2 - 5)$ is 2 because 2 is the greatest exponent contained in the expression, and that the degree of $(x^3 - 6)^3$ is 3 because 3 is the greatest exponent. Such confusion can be quickly cleared up by reminding the pupil that these expressions are not written in standard polynomial form as the sum of terms. When written in standard polynomial form, the degree of the polynomial is easily determined.

The topic of addition and subtraction of polynomials is not new as it was introduced in previous units. Performing addition and subtraction of polynomials simply requires the application of the commutative and distributive principles and application of the concept of subtraction being performed by carrying out the equivalent addition. For example,

$$(3x^2-6x+9) - (2x^2+4x-3) = 3x^2+(-6x)+9+(-1)[2x^2+4x+(-3)]$$
$$= 3x^2+(-6x)+9+(-2x^2)+(-4x)+3$$
$$= 3x^2+(-2x^2)+(-6x)+(-4x)+9+3$$
$$= x^2+(-10x)+12$$
$$= x^2-10x+12$$

It is not necessary that the pupils perform all of the steps indicated above when performing such addition or subtraction of polynomials. The pupils should be encouraged to use efficient short cuts as long as they understand the mathematics involved and could furnish all such steps if required to do so.

8.3 MULTIPLICATION OF POLYNOMIALS

Multiplication of a polynomial by a monomial is simply an application of the distributive principle. For example, $(\triangle)(2x^2 + 3x + 6) = (\triangle)(2x^2) + (\triangle)(3x) + (\triangle)(6)$, where $\triangle$ represents any monomial.

The principle applies to the multiplication of a polynomial by any binomial. If, in the above equation, $\triangle$ represents the binomial $x + 3$, then

$$(x + 3)(2x^2 + 3x + 6) = (x + 3)(2x^2) + (x + 3)(3x) + (x + 3)(6)$$
$$= 2x^3 + 6x^2 + 3x^2 + 9x + 6x + 18$$
$$= 2x^3 + 9x^2 + 15x + 18$$

The multiplication may also be performed by reversing the order of the two factors.

$$(2x^2 + 3x + 6)(x + 3) = (2x^2 + 3x + 6)(x) + (2x^2 + 3x + 6)(3)$$
$$= 2x^3 + 3x^2 + 6x + 6x^2 + 9x + 18$$
$$= 2x^3 + 9x^2 + 15x + 18$$

In each of the above examples, the final addition is performed horizontally. The addition may also be performed vertically as shown below.

$$\begin{array}{l} x + 3 \\ \underline{2x^2 + 3x + 6} \\ 2x^3 + 6x^2 \\ \quad\quad 3x^2 + 9x \\ \underline{\quad\quad\quad\quad 6x + 18} \\ 2x^3 + 9x^2 + 15x + 18 \end{array} \qquad \begin{array}{l} 2x^2 + 3x + 6 \\ \underline{x + 3} \\ 2x^3 + 3x^2 + 6x \\ \underline{\quad\quad 6x^2 + 9x + 18} \\ 2x^3 + 9x^2 + 15x + 18 \end{array}$$

Any of the four methods shown previously is correct, but it is the last method indicated which is the most common. The factor of highest degree is written first and the addition is performed vertically. This method may also be used for multiplying a polynomial by a trinomial.

A few exercises requiring the multiplication of polynomials with irrational coefficients have been included in the questions and activities to give the pupils an opportunity to maintain their skills in operating with irrational numbers.

The last part of this section is a study of the patterns in the multiplication of two binomials. This study has a twofold purpose. First, it teaches skill in multiplying two binomials on sight. Secondly, it teaches recognition of the patterns in polynomials necessary in factoring polynomials. Factorization of polynomials is the next topic in the unit. Factorization of polynomials is, in turn, used in solving some quadratic equations efficiently. Therefore, the study of patterns in the multiplication of binomials bears a very important relation to the work that follows.

The first pattern studied is:
$(x + \square)(x + \triangle) = x^2 + (\square + \triangle)(x) + \square \cdot \triangle$, where $\square$ and $\triangle$ represent the constant terms in the binomials $x + \square$ and $x + \triangle$. Special cases of this pattern are:

$$(x + \square)^2 = x^2 + 2\square x + \square^2$$

and $(x - \triangle)(x + \triangle) = x^2 - \triangle^2$

The next pattern studied is:
$(ax + \square)(bx + \triangle) = abx^2 + (a\triangle + \square b)x + \square \cdot \triangle$
Special cases of this pattern are:

$$(ax + \square)^2 = a^2x^2 + 2a\square x + \square^2$$

and $(ax + \triangle)(ax - \triangle) = a^2x^2 - \triangle^2$

Mastery of these patterns enables a pupil to multiply two binomials on sight. Even though a pupil is not able to master these latter patterns to such a degree that he can multiply two such binomials on sight, at least he should be able to perform the multiplication with much greater ease and efficiency than he would have been able to do without the study of such patterns. A reasonable effort should be made to have the pupils master these patterns so that they will be able to perform the factoring of polynomials in the next section.

By applying the concept that $x^2 = (-x)^2$, pupils can learn to identify as identities on sight such equations as:

$$(x - 3)^2 = (3 - x)^2$$

and $(x + 3)^2 = (-x - 3)^2$

The proof of the theorem $x^2 = (-x)^2$ is quite simple.

$$\begin{aligned}(-x)^2 &= (-x)(-x)\\ &= (-1)(x)(-1)(x)\\ &= (-1)(-1)(x)(x)\\ &= (1)(x)(x)\\ &= x^2\end{aligned}$$

8.4 FACTORING POLYNOMIALS

A factor of a polynomial is defined as one of two or more polynomials whose product is the given polynomial. To factor means to resolve into factors such that the coefficients lie in a certain set of numbers. In this course the set is usually the set of integers, sometimes the rationals.

The main purpose of the material in this section is to teach factoring of polynomials by applying the distributive principle and applying knowledge of the patterns of multiplication of binomials studied in the previous section.

Factoring is used in the next unit in the solution of polynomial equations, and such factoring does not require use of irrational coefficients. For this reason, the coefficients in the polynomials and the coefficients in the factors in these exercises have been limited to rational numbers, particularly integers. For example, although $x^2 - 2$ may be factored into $(x + \sqrt{2})(x - \sqrt{2})$, knowledge and experience in such factorization is not necessary in solving polynomial equations in this course, thus such factorization is not developed in this unit. It may, however, be mentioned by the teacher.

Further factorization of factors that are constants is of no value in this work and such factorization is not performed. For example, $12x^2 + 24x - 60$ may be factored into $(12)(x^2 + 2x - 5)$, but it is of no value to factor 12 into its integral factors. This would only complicate the expression. Such factorization is called trivial factorization. Again it is emphasized that factorization is performed so that the factors are in a certain set of numbers, usually the integers. In the set of rationals even 1 can be factored; for instance, $1 = \frac{4}{3}\cdot\frac{3}{4}$. In the set of reals, more complicated factors of 1 can be written; thus, $1 = (7 + 4\sqrt{3})(7 - 4\sqrt{3})$.

When directions are given to factor a polynomial as completely as possible, it is understood that the polynomial is to be expressed as a product of polynomials with integral, or sometimes rational, coefficients, such that no factor can be further factored except in a trivial way. If one of the factors is a constant, it is not intended that this be factored further.

The factorization considered in this unit is that requiring the application of the distributive law. This is often referred to as taking out all common factors. An example is: $3x^3 + 6x^2 + 9x = (3x)(x^2 + 2x + 3)$

The remaining factorizations are applications of the patterns of multiplication of binomials studied in the previous section. These are:

$(x + \square)(x + \triangle) = x^2 + (\square + \triangle)x + \square \cdot \triangle$

$(x + \triangle)^2 = x^2 + 2\triangle x + \triangle^2$

$(x - \square)(x + \square) = x^2 - \square^2$

$(ab + \square)(bx + \triangle) = abx^2 + (a\triangle + \square b)x + \square \cdot \triangle$

$(ax + \square)^2 = a^2x^2 + 2a\square + \square^2$

$(ax + \triangle)(ax - \triangle) = a^2x^2 - \triangle^2$

In applying these patterns to the topic of factoring, the polynomial indicated as the right member of one of the above equations is given and the left member must be determined. Sufficient time should be devoted to this topic to enable the pupils to acquire reasonable skill in such factoring.

8.5 DIVISION OF POLYNOMIALS

The questions and activities develop the common method used for performing the division of one polynomial by another polynomial. The sequence of exercises used to develop this concept begins with the application of the distributive principle.

If $2x + 5$ is a factor of $6x^3 + 23x^2 + 30x + 25$, then their quotient is a polynomial which may be represented by $\square + \frown + \triangle$; that is, $\frac{6x^3 + 23x^2 + 30x + 25}{2x + 5} = \square + \frown + \triangle$.

This may be written in the equivalent relation,

$(2x + 5)(\square + \frown + \triangle) = 6x^3 + 23x^2 + 30x + 25$.

Applying the distributive principle,

$(2x + 5)(\square) + (2x + 5)(\frown) + (2x + 5)(\triangle) = 6x^3 + 23x^2 + 30x + 25$.

If $2x + 5$ is to divide into $6x^3 + 23x^2 + 30x + 25$, then the first term of the quotient must be $\frac{6x^3}{2x}$ or $3x^2$. Thus $\square$ must equal $3x^2$.

$$(2x + 5)(3x^2) + (2x + 5)(\frown) + (2x + 5)(\triangle) = 6x^3 + 23x^2 + 30x + 25$$
$$(6x^3 + 15x^2) + (2x + 5)(\frown) + (2x + 5)(\triangle) = 6x^3 + 23x^2 + 30x + 25$$
$$(2x + 5)(\frown) + (2x + 5)(\triangle) = 8x^2 + 30x + 25$$

It is important to explain this last step clearly. The divisor and the first term of the quotient are multiplied and the product subtracted from the dividend. It is this step that is the basis for the formal method of dividing polynomials which the pupils will use eventually in performing such division.

Following the same analysis, $\frown$ must equal $\frac{8x^2}{2x}$ or $4x$.

$$(2x + 5)(4x) + (2x + 5)(\triangle) = 8x^2 + 30x + 25$$
$$(8x^2 + 20x) + (2x + 5)(\triangle) = 8x^2 + 30x + 25$$
$$(2x + 5)(\triangle) = 10x + 25$$

Finally $\triangle$ must equal $\frac{10x}{2x}$ or 5.

Therefore, $\frac{6x^3 + 23x^2 + 30\,x + 25}{2x + 5} = 3x^2 + 4x + 5.$

Each time a new term of the quotient is determined, the product of this term and the divisor is subtracted from the new dividend. This process can be simplified by performing it in a systematic vertical arrangement as follows.

$$\begin{array}{r|l} & 3x^2 + 4x + 5 \\ \hline 2x + 5 & 6x^3 + 23x^2 + 30x + 25 \\ & \underline{6x^3 + 15x^2} \\ & \quad 8x^2 + 30x + 25 \\ & \quad \underline{8x^2 + 20x} \\ & \qquad 10x + 25 \\ & \qquad \underline{10x + 25} \end{array}$$

In order to keep the arrangement as simple and as orderly as possible, terms in the divisor and dividend should be arranged in descending order of exponents. If either polynomial has fewer terms than one more than the greatest exponent, the missing terms may be inserted into the polynomial and given a coefficient of zero. For example, $\frac{x^5 + 6x^4 + 6x - 3}{x^3 - 4}$ may be written as $\frac{x^5 + 6x^4 + 0x^3 + 0x^2 + 6x - 3}{x^3 + 0x^2 + 0x - 4}$ before the devision is performed.

Teacher Notes

UNIT 8: POLYNOMIAL EXPRESSIONS

PART 2. QUESTIONS AND ACTIVITIES FOR CLASSROOM USE

8.1 INTRODUCTION

In the questions and activities in this unit, the coefficients and constant terms in the polynomials have been limited almost entirely to integers and in some cases to rational numbers. The reason for this is that the unit is primarily a preparation for the following unit on solving quadratic equations, particularly by the factoring method. It is not necessary to work with irrational numbers for such factoring; therefore, the teacher may feel free to delete all exercises involving irrational numbers contained in this unit. However, the teacher may prefer to have the pupils maintain their skills in operations with irrationals and should feel free to implement the exercises in this unit with additional exercises involving irrational numbers. In all cases, it would be wise to bring out that factoring depends on the set of numbers used. For example, while studying the topic of factoring, the polynomial $x^2 - 3$ may be considered as being the product of two binomials in the set of real numbers. The pupil might well be able to see that one pair of factors could be $x + \sqrt{3}$ and $x - \sqrt{3}$ but may not at once realize that in this enlarged set there are now an unlimited number of pairs of factors, another representative pair being, $2x + \sqrt{3}$ and $\frac{x}{2} - \frac{\sqrt{3}}{2}$. Again, this area is optional.

8.2 ADDITION AND SUBTRACTION OF POLYNOMIALS

Concept: Definition of polynomial.

(1) Indicate the base, the exponent, and the coefficient in the expression $3x^7$.

Answer: The base is x, the exponent is 7, and the coefficient is 3.

(2) Indicate the coefficients and the constant term in the expression $-4x^2 + 5x + 7$.

Answer: The coefficient of x^2 is -4, the coefficient of x is 5, and the constant term is 7.

(3) Each of the following expressions is called a polynomial.

$x^3 + 11x^2 + \frac{5}{9}x + 6$, $\sqrt{3}\ A^3 + 12\ A + 1$, $-3b + 9$,

$-141\ K$, $3\sqrt{2}$, 62

Answer the following questions.

(a) If the variables represent real numbers, is each term in every expression above a real number?

(b) Is every expression a term or sum of terms?

(c) If an integral algebraic expression is one which does not contain a variable in a denominator nor a negative exponent in the numerator, is every algebraic term in each of the above expressions an integral algebraic expression?

(d) Does every term in each of the above expressions which contains a variable have a positive integer as an exponent?

(e) To be classified as a polynomial in the set of real numbers, an expression must meet all of the following requirements:
  (1) It must be a term or sum of terms.
  (2) Coefficients and constant terms must be real numbers.
  (3) Variables, if present, must have positive integral exponents. Such exponents cannot be negative nor fractional, nor can the variable be written in any way which is equivalent to having a negative or fractional exponent.

Summarize these three points to form a definition of a polynomial in the set of real numbers in a single sentence.

Answers:

(a) Yes (b) Yes (c) Yes (d) Yes

(e) A polynomial, in the set of real numbers, is a term or sum of terms, each of which is a real number or an integral algebraic expression consisting of a product of a real number and a positive integral power of the variable.

(4) Indicate for each of the following whether or not the expression is a polynomial. If the expression is not a polynomial, state the reason why it fails to meet the definition of a polynomial. (In this exercise and the following ones, it is understood that we are referring to polynomials in the set of real numbers.)

(a) $2x^2 - 5x + \frac{6}{7}$

Answer: This is a polynomial. It is equivalent to the sum of terms $2x^2 + (-5x) + \frac{6}{7}$.

(b) $(2x + 2)(3x - 6)$

Answer: This is a polynomial but it is not in standard polynomial form. This product expressed as $6x^2 - 6x - 12$ would be in standard polynomial form.

(c) $\sqrt{3}\,x^2 - \sqrt[3]{7}x - 6$

Answer: This is a polynomial. The coefficients are real numbers.

(d) $\frac{x + 1}{x - 1}$

Answer: This is not a polynomial. The term is not an integral algebraic expression. The variable is in the denominator.

(e) $6x^{\frac{3}{2}}$

Answer: This is not a polynomial. The exponent of the variable is not an integer.

(f) 10^{-4}

Answer: This is a polynomial. It is a real number.

(5) Opposite each of the following polynomials is an indication of its "degree." By observing the pattern of the polynomial and its degree, determine what is meant by the degree of a polynomial (in one variable).

$16x^5 - 14x^4 + 13x^2 - 9$ has degree 5

$-9a^6$ has degree 6

$7x^2 + 6x - 4$ has degree 2

$6d - 9$ has degree 1

Answer: The degree of a polynomial is the highest power of any variable appearing in the polynomial.

(6) Answer the following.

(a) What do the prefixes mon, bi, and tri mean?

(b) What kind of polynomials would each of the terms monomial, binomial, and trinomial seem to indicate?

Answers:

(a) Mon means consisting of one.
Bi means consisting of two.
Tri means consisting of three.

(b) A monomial (note only one n) is a polynomial consisting of one term, a binomial of two terms, and a trinomial of three terms.

(7) Classify each of the following as either a monomial, binomial, or trinomial.

(a) $-\sqrt{17}x^2yz$ (b) $\frac{2x + 3y}{5}$ (c) $3x^2 + 5x - 6$

Answers:

(a) Monomial

(b) Written in standard polynomial form $\frac{2x}{5} + \frac{3}{5}y$ it is a binomial.

(c) Trinomial

Concept: Addition and subtraction of polynomials.

(8) Add the following pairs of polynomials, applying where necessary the commutative and distributive principles, and the principle that subtraction may be performed by carrying out the equivalent addition.

(a) $(6a^2 - 4a + 10) + (10a^2 + 4)$

(b) $(-5y^2 + 3y - 6) + (5y^2 - 3y + 6)$

(c) $\begin{array}{r} 3x^2 + 2x + 5 \\ \underline{5x^2 - 6x - 6} \end{array}$ (d) $\begin{array}{r} -7v^3 - 10v + 12 \\ \underline{8v - 14} \end{array}$

Answers:

(a) $16a^2 - 4a + 14$ (c) $8x^2 - 4x - 1$

(b) 0 (d) $-7v^3 - 2v - 2$

(9) Answer the following.

(a) Is it possible for the sum of two polynomials to be of higher degree than the highest degree of the two polynomials being added?

(b) Is it possible for the sum of two polynomials to be of lesser degree than the degree of either polynomial being added? Give an example.

Answers:

(a) No

(b) Yes. For example, $(2x^2 - 6x) + (-2x^2 + 10x) = 4x$.

(10) Determine the additive inverse of each of the following.

(a) $a^5 - 3a^2 + 9$ (b) $k^5 + 10^6$ (c) $-3x^3 - m^7$

Answers:

(a) $-a^5 + 3a^2 -9$ (b) $-k^5 - 10^6$ (c) $3x^3 + m^7$

(11) How can the principle $-x = (-1)(x)$ be applied to performing the subtraction $x^2 - (3x^2 - 7)$?

Answer: $x^2 - (3x^2 - 7) = x^2 + {}^-(3x^2 - 7)$

$= x^2 + (-1)[3x^2 + (-7)]$

$= x^2 + (-3x^2) + 7$

$= -2x^2 + 7$

(12) In each of the following perform the indicated subtraction.

(a) $(y^3 - y^2 + 1) - (y - 6)$

(b) $(7z^2 + 13z - 12) - (z^2 + z + 1)$

(c) $(-4p^2 + 5p - 11) - (4p^2 - 5p + 11)$

Answers:

(a) $y^3 - y^2 - y + 7$

(b) $6z^2 + 12z - 13$

(c) $-8p^2 + 10p - 22$

(13) In each of the following, subtract the lower polynomial from the upper one, by performing the equivalent addition.

(a) $-6p^3 + 3p^2 - p + 9$
$\underline{3p^3 - p^2 + 4p - 6}$

(c) $k^3 - 2k^2 + k - 7$
$\underline{4k^3 - k^2 + k - 7}$

(b) $-3t^2 + 5t + 8$
$\underline{3t^2 + 5t - 8}$

Answers:

(a) $-9p^3 + 4p^2 - 5p + 15$

(b) $-6t^2 + 16$

(c) $-3k^3 - k^2$

(14) Express as a polynomial, the perimeter of each of the following.

(a) A triangle whose sides are $2x^2 + 6$, $6x - 4$, and $6x + 4$

(b) A rectangle whose width is $7y + 7$ and whose length is $y^2 + 2y - 6$

(c) A square one of whose sides is $y^2 + y$

(d) By how much does the perimeter of the square in exercise (c) exceed the perimeter of the rectangle in exercise (b)?

Answers:

(a) $2x^2 + 12x + 6$ (c) $4y^2 + 4y$

(b) $2y^2 + 18y + 2$ (d) $2y^2 - 14y - 2$

(15) The surface of a cube $3x$ units in length on an edge is painted red. The cube is then cut into smaller cubes each x units on an edge. For each of the following, express the number of small cubes

which have the property. Also write the sum of all the painted areas of these cubes (a-e) and in (f) check by computing the total painted area of the original cube.

(a) Four faces painted
(b) Three faces painted
(c) Two faces painted
(d) One face painted
(e) No face painted

Answers:

(a) 0, 0 (b) 8, $24x^2$ (c) 12, $24x^2$ (d) 6, $6x^2$

(e) 1, 0 (f) $6(3x)(3x) = 54x^2$,
$24x^2 + 24x^2 + 6x^2 = 54x^2$

8.3 MULTIPLICATION OF POLYNOMIALS

Concept: Multiplication of a polynomial by a monomial.

(1) By the application of what principle could the product $(3x^2)(2x^2 - 3x + 7)$ be determined?

Answer: By the application of the distributive principle

(2) Determine the product of the following pairs of polynomials by applying the distributive principle.

(a) $7x^3\left(\frac{1}{21}x^2 - \frac{2}{7}x + 4\right)$

(b) $-\frac{3}{5}y^3(10y^2 - 5y - 6)$

(c) $100x^2(0.17x^3 - 0.03x^2 + 0.33x - 0.9)$

Answers:

(a) $\frac{1}{3}x^5 - 2x^4 + 28x^3$

(b) $-6y^5 + 3y^4 + \frac{18}{5}y^3$

(c) $17x^5 - 3x^4 + 33x^3 - 90x^2$

(3) Perform the indicated operations and express the answer in simplest form.

(a) $3x(2x^3-4x^2+6x-2)-5x(x^3+7x^2-5x+7)$

(b) $5c^7(19c^5-4c^3+1)+3c^2(c^{10}-2c^7-5c^6-5c^5+6)$

(c) $3k^2(k^6-k^4+k^2-1)+3(k^6-k^4+k^2-1)-3(k^8-1)$

Answers:

(a) $x^4 - 47x^3 + 43x^2 - 41x$

(b) $98c^{12} - 20c^{10} - 6c^9 - 15c^8 - 10c^7 + 18c^2$
(c) 0

(4) <u>Write an expression for the area of a rectangle whose width is $6x^2$ and whose length is $x^3 - 6x^2 + 6$.</u>

<u>Answer</u>: $6x^5 - 36x^4 + 36x^2$

(5) <u>Write an expression for the area of a triangle whose base is $3x^2 - x - 6$ and whose altitude to that base is $7x$.</u>

<u>Answer</u>: $\frac{21}{2}x^3 - \frac{7}{2}x^2 - 21x$

<u>Concept</u>: Multiplication of a polynomial by a binomial.

(6) <u>Answer the following.</u>

(a) Express the following in simplest form, applying the distributive principle: $(\square)(3x^2 + 4x + 5)$

<u>Answer</u>: $\square 3x^2 + \square 4x + \square 5$

(b) Replace $\square$ by $x + 2$ and express the product in simplest form.

<u>Answer</u>: $(x + 2)(3x^2) + (x + 2)(4x) + (x + 2)(5) =$
$3x^3 + 6x^2 + 4x^2 + 8x + 5x + 10 =$
$3x^3 + 10x^2 + 13x + 10$

(c) Express the following in simplest form, applying the distributive principle: $(\triangle)(x + 2)$

<u>Answer</u>: $\triangle x + \triangle 2$

(d) Replace $\triangle$ with $3x^2 + 4x + 5$. Write the product of the second pair of factors under the product of the first pair of factors in such a way that like terms are in vertical columns. Express the answer in simplest form.

<u>Answer</u>: $(3x^2 + 4x + 5)(x) + (3x^2 + 4x + 5)(2)$

$$\begin{array}{rrrr} 3x^3 & + 4x^2 & + 5x & \\ & 6x^2 & + 8x & + 10 \\ \hline 3x^3 & + 10x^2 & + 13x & + 10 \end{array}$$

(e) Perform the multiplication $(-6x^2 + x - 3)(x + 2)$ first by horizontal addition and then perform the same multiplication by vertical addition.

<u>Answer</u>: $(-6x^2 + x - 3)(x) + (-6x^2 + x - 3)(2) =$
$-6x^3 + x - 3x + (-12x^2) + 2x - 6$
$-6x^3 - 11x^2 - x - 6$

$$\begin{array}{r} -6x^3 + x^2 - 3x \qquad \\ \underline{\qquad -12x^2 + 2x - 6} \\ -6x^3 - 11x^2 - x - 6 \end{array}$$

(f) Which is easier to perform, the horizontal addition or the vertical addition?

<u>Answer</u>: The vertical addition

(g) Perform the multiplication $(-3x^2 + 2x - 5)(x - 3)$ by the vertical addition method, first expressing $x - 3$ as the equivalent addition $x + (-3)$.

<u>Answer</u>: $-3x^3 - 7x^2 - 11x + 15$

(7) <u>Perform each of the following multiplications, using either the horizontal or vertical method. Express the final answer in descending order of exponents.</u>

(a) $(x + 1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)$

(b) $(4a^2 - a + 3)(3a + 7)$

(c) $(2y + 5y^2 - 5)(3 + 2y)$

(d) $(3x + 2)(4x - 5)(3x - 4)$

<u>Answers</u>:

(a) $x^7 + 1$

(b) $12a^3 + 25a^2 + 2a + 21$

(c) $10y^3 + 19y^2 - 4y - 15$

(d) $36x^3 - 69x^2 - 2x + 40$

<u>Concept</u>: Multiplication of two trinomials.

(8) <u>Determine the following products, using the vertical addition method.</u>

(a) $(r^2 + 7r - 3)(3r^2 + 5r - 8)$

(b) $(3y^2 + 6y - 4)(-3y^2 - 6y + 4)$

(c) $(2x^4 + 3x^2 - 2)(x^2 - x + 1)$

Answers:

(a) $3r^4 + 26r^3 + 18r^2 - 71r + 24$

(b) $-9y^4 - 36y^3 - 12y^2 + 48y - 16$

(c) $2x^6 - 2x^5 + 5x^4 - 3x^3 + x^2 + 2x - 2$

Concept: Multiplication of polynomials containing irrational numbers.

(9) Express each of the following products in simplest form.

(a) $\sqrt{3}\,(\sqrt{2}x^2 + 6x - \sqrt{3})$

(b) $(\sqrt{2}x + 7)(3x^2 - \sqrt{2}x - \sqrt{18})$

Answers:

(a) $\sqrt{6}x^2 + 6\sqrt{3}x - 3$

(b) $3\sqrt{2}x^3 + 19x^2 - 7\sqrt{2}x - 6x - 21\sqrt{2}$

Concept: Patterns in multiplication of binomials.

(10) Compute the product of each of the following.

(a) $(x + 4)(x + 2)$

(b) $[x + (-2)]\ (x + 9)$

(c) $[y + (-3)]\ [y + (-7)]$

Answers:

(a) $x^2 + 6x + 8$

(b) $x^2 + 7x - 18$

(c) $y^2 - 10\ x + 21$

(11) In each of the exercises in (10), the product of the two constant terms is equal to what term in the polynomial?

Answer: The last or constant term

(12) In each of the exercises in (10), the sum of the two constant terms is the same as what coefficient in the polynomial?

Answer: The coefficient of the middle or x term

(13) Answer the following.

(a) Express the product $(x + \square)(x + \triangle)$ as a polynomial.

(b) Combine the second and third terms in this expression, applying the distributive principle.

(c) Describe how the product $(x + \square)(x + \triangle)$ can be expressed as a polynomial on sight, where x represents the variable and $\square$ and $\triangle$ represent the constant terms.

Answers:

(a) $x^2 + \square x + \triangle x + \square \cdot \triangle$

(b) $x^2 + (\square + \triangle)x + \square \cdot \triangle$

(c) The first term of the polynomial will be equal to x^2, or the product of the first terms of the factors. The second term will be the product of the sum of the constant terms and the variable. The third term will be the product of the constant terms.

(14) Applying the principle that $(x + \square)(x + \triangle) = x^2 + (\square + \triangle)x + \square \cdot \triangle$, perform the following multiplications by inspection.

(a) $(y + 7)(y - 3)$ (c) $(k + 9)(k + 9)$

(b) $(x - 3)(x - 4)$ (d) $(x - 6)^2$

Answers:

(a) $y^2 + 4y - 21$ (c) $k^2 + 18x + 81$

(b) $x^2 - 7x + 12$ (d) $x^2 - 12x + 36$

(15) Express the product $(x + \triangle)(x - \triangle)$ in simplest form.

Answer: $x^2 - \triangle^2$

(16) Applying the principle that $(x + \triangle)(x - \triangle) = x^2 - \triangle^2$, determine by inspection the following products.

(a) $(x + 9)(x - 9)$ (c) $(x + 1)(x - 1)$

(b) $(x - 11)(x + 11)$

Answers:

(a) $x^2 - 81$ (b) $x^2 - 121$ (c) $x^2 - 1$

(17) Answer the following.

(a) Determine the simplest form of the product $(-1)(x)^2$.

Answer: $(-1)(x)^2 = (-1)(x)(-1)(x)$

$= (-1)(-1)(x)(x)$

$= x^2$

(b) $(-1)(x)$ is equivalent to what simpler expression?

Answer: $-x$

(c) What is the relationship between $(-x)^2$ and x^2?

Answer: $(-x)^2 = x^2$

(d) Express the relationship $(-x)^2 = x^2$ in the form of a word sentence, x representing any polynomial.

Answer: The square of a polynomial and the square of its additive inverse are equivalent.

(e) Can the equation $(x - 3)^2 = (3 - x)^2$ be recognized as an identity without actually performing the indicated multiplication?

Answer: Yes. It can be recognized as an identity as it is an example of the identity $x^2 = (-x)^2$. $(3 - x)$ is the additive inverse of $(x - 3)$. That is, $-(x - 3) = -x + 3 = 3 - x$

(f) Is the equation $(h + 5)^2 = (-h -5)^2$ an identity?

Answer: Yes. $(-h -5)$ is the additive inverse of $(h + 5)$; therefore, $(h + 5)^2 = (-h -5)^2$.

(18) Answer the following.

(a) Express the product $(ax + \square)(bx + \triangle)$ as a polynomial, a and b representing the coefficients of the variable x and $\square$ and $\triangle$ representing the constant terms.

(b) Combine the second and third terms into a single term by applying the distributive principle.

Answers:

(a) $abx^2 + b\square x + a\triangle x + \square \cdot \triangle$

(b) $abx^2 + (b\square + a\triangle)x + \square \cdot \triangle$

(19) Applying the principle $(ax + \square)(bx + \triangle) = abx^2 + (b\square + a\triangle)x + \square \cdot \triangle$, express by inspection the following products as polynomials.

(a) $(3x + 4)(6x + 1)$ (c) $(3k - 5)(7k - 11)$

(b) $(2y + 7)(y - 3)$ (d) $(4y - 7)^2$

Answers:

(a) $18x^2 + 27x + 4$ (c) $21k^2 - 68k + 55$

(b) $2y^2 + y - 21$ (d) $16y^2 - 56y + 49$

(20) Express the product $(ax + \square)(ax - \square)$ in simplest form.

Answer: $a^2x^2 - \square^2$

(21) Applying the principle $(ax + \square)(ax - \square) = a^2x^2 - \square^2$, determine by inspection the products of the following.

(a) $(5x - 7)(5x + 7)$ (c) $(-x + 8)(-x - 8)$

(b) $(\frac{1}{2}x + 4)(\frac{1}{2}x - 4)$ (d) $(0.1x + 0.1)(0.1x - 0.1)$

Answers:

(a) $25x^2 - 49$ (c) $x^2 - 64$

(b) $\frac{1}{4}x^2 - 16$ (d) $.01x^2 - .01$

(22) Replace the frames in the following equations with numbers to make each equation an identity.

(a) $(x - 3)^2 = x^2 + \triangle x + 9$

(b) $(x + \square)^2 = x^2 + 8x + 16$

(c) $(y - \square)^2 = y^2 - \triangle y + \frac{1}{4}$

(d) $(k + \square)^2 = k^2 + \triangle k + 81$

Answers:

(a) $(x-3)^2 = x^2-6x+9$ (c) $(y-\frac{1}{2})^2 = y^2-y+\frac{1}{4}$

(b) $(x+4)^2 = x^2+8x+16$ (d) $(k+9)^2 = k^2+18k+81$

(23) Indicate what number must replace the frame in each of the following if the polynomial is to be the square of a binomial.

(a) $s^2 + \triangle s + 1$ (c) $y^2 - .2y + \square$

(b) $t^2 + \frac{5}{3} t + \bigcirc$ (d) $x^2 - 3x + \triangle$

Answers: (a) 2 (c) 0.01

(b) $\frac{25}{36}$ (d) $\frac{9}{4}$

(24) Express each of the following products as a polynomial.

(a) $(x + \triangle)^2$ (b) $(ax + \square)^2$

Answers:

(a) $x^2 + 2\triangle x + \triangle^2$ (b) $a^2x^2 + 2a\square x + \square^2$

(25) Summarize, by listing, the patterns found in the multiplication of two binomials.

Answer:

(1) $(x + \square)(x + \triangle) = x^2 + (\square + \triangle)x + \square \cdot \triangle$

(2) $(x + \triangle)^2 = x^2 + 2\triangle x + \triangle^2$

(3) $(x + \square)(x - \square) = x^2 - \square^2$

(4) $(ax + \square)(bx + \triangle) = abx^2 + (b\square + a\triangle)x + \square \cdot \triangle$

(5) $(ax + \triangle)^2 = a^2x^2 + 2a\triangle x + \triangle^2$

(6) $(ax + \square)(ax - \square) = a^2x^2 - \square^2$

8.4 FACTORING POLYNOMIALS

Concept: Factoring by applying the distributive principle.

(1) Below are several polynomials and opposite each are two or more polynomials which are called factors of the given polynomial. What does the phrase "factor of a polynomial" seem to mean?

Given polynomial	Factors of the given polynomial
21	3 and 7
17yz	17, y, and z
ac + bc	c and (a + b)
$x^2 + 7x + 12$	$(x + 3)$ and $(x + 4)$
$x^3 - x$	x, $(x + 1)$, and $(x - 1)$
30	2, 3, and 5

Answer: A factor of a polynomial is one of two or more polynomials whose product is the given polynomial.

(2) What principle could be applied to the expression ak + bk so that it could be resolved into two factors?

Answer: The distributive principle. $ak + bk = (k)(a + b)$

Note: In the previous and the following exercises in factoring, if the given expression contains only variables and integers, then it shall be required that the factors contain only variables and integers. Denominators in factors of expressions containing fractions shall not be greater than those given.

(3) Resolve each of the following into two factors by applying the distributive principle

(a) $ax^2 + bx$ (b) $3y^3 - 6y^2$ (c) $5abc + 10cde$

(d) $4x^4 - 14x^3 + 22x^2$ (e) $c^{99} + c^{100}$

(f) $\frac{3m^3}{7} - \frac{2m^2}{7}$

<u>Answers</u>:

(a)	$(x)(ax + b)$	(d)	$(2x^2)(2x^2 - 7x + 11)$
(b)	$(3y^2)(y - 2)$	(e)	$(c^{99})(1 + c)$
(c)	$(5c)(ab + 2de)$	(f)	$\frac{m^2}{7}(3m - 2)$

<u>Concept</u>: Factoring an integral (rational) polynomial that is the product of two integral (rational) binomials.

(4) <u>Answer the following</u>.

(a) If the expression $x^2 + 13x + 42$ is the product of two binomial factors containing only variables and integers, what must be the product of the two constant terms in the factors? What must be the sum of the two constant terms?

(b) What two integers have a product of 42 and a sum of 13?

(c) If the constant terms of the factors are 6 and 7, express $x^2 + 13x + 42$ as the product of two binomial factors containing only variables and integers.

<u>Answers</u>:

(a) The product of the two constant terms must be 42 and their sum must be 13.

(b) 6 and 7

(c) $(x + 6)(x + 7)$

(5) <u>Applying the principle</u> $x^2 + (\square + \triangle)x + \square \cdot \triangle = (x + \square)(x + \triangle)$, <u>resolve each of the following into two integral factors (containing only variables and integers)</u>.

(a)	$h^2 - 7h + 6$	(e)	$x^2 + 8x + 12$
(b)	$x^2 + 7x - 18$	(f)	$x^2 - 11x + 18$
(c)	$x^2 + x - 12$	(g)	$z^2 - z - 30$
(d)	$y^2 - 3y - 18$	(h)	$f^2 - 13f + 36$

<u>Answers</u>:

(a)	$(h - 6)(h - 1)$	(e)	$(x + 6)(x + 2)$
(b)	$(x + 9)(x - 2)$	(f)	$(x - 9)(x - 2)$
(c)	$(x + 4)(x - 3)$	(g)	$(z - 6)(z + 5)$
(d)	$(y - 6)(y + 3)$	(h)	$(f - 9)(f - 4)$

(6) Answer the following.

(a) If the polynomial $w^2 + 6w + 9$ is the square of an integral binomial, what information concerning the constant term of the binomial can be determined by inspecting the polynomial?

Answer: The square of the constant term must be 9 and two times the constant term must be 6.

(b) What must the constant term of the binomial be in the above example?

Answer: 3

(c) Write the trinomial form $w^2 + 6w + 9$ in the form of a binomial squared.

Answer: $(w + 3)^2$

(7) Indicate the two equal factors of each of the following by writing the equivalent binomial squared.

(a) $x^2 + 4x + 4$ (d) $d^2 + 22d + 121$

(b) $a^2 + 12a + 36$ (e) $c^2 - 12c + 36$

(c) $y^2 - 10y + 25$

Answers:

(a) $(x + 2)^2$ (c) $(y - 5)^2$ (e) $(c - 6)^2$

(b) $(a + 6)^2$ (d) $(d + 11)^2$

(8) Answer the following.

(a) If the expression $x^2 - 9$ is the product of two integral binomials, what must be the sum of the two constant terms?

(b) What must be the product of the two constant terms?

(c) What two integers have a product of -9 and a sum of zero?

(d) What are the two factors of $x^2 - 9$?

Answers:

(a) Zero

(b) -9

(c) 3 and -3

(d) $(x + 9)$ and $(x - 9)$

(9) Express $x^2 - \triangle^2$ as the product of two factors.

Answer: $(x + \triangle)(x - \triangle)$

(10) Following the note preceding exercise (3) of this section, express each of the following as the product of two integral or rational binomials.

(a) $x^2 - 4$ (c) $k^2 - 16$ (e) $z^2 - \frac{4}{9}$

(b) $y^2 - \frac{1}{4}$ (d) $y^2 - 81$ (f) $w^2 - 0.04$

Answers:

(a) $(x + 2)(x - 2)$ (d) $(y + 9)(y - 9)$

(b) $(y + \frac{1}{2})(y - \frac{1}{2})$ (e) $(z + \frac{2}{3})(z - \frac{2}{3})$

(c) $(k + 4)(k - 4)$ (f) $(w + 0.2)(w - 0.2)$

(11) Answer the following.

(a) Express $(ax + \square)(bx + \triangle)$ as a trinomial.

Answer: $abx^2 + (\square b + a\triangle)x + \square\cdot\triangle$

(b) If the expression $10x^2 + 21x + 9$ is the product of two binomials, what must be the product of the two constant terms?

Answer: 9

(c) What must be the product of the two coefficients?

Answer: 10

(d) What pair of integral constant terms whose product is 9 and what pair of integral coefficients whose product is 10 can be chosen so that $(\square b + a\triangle)$ equals 21?

Answer: If the two constant terms are 3 and the coefficients 5 and 2 so that $(5x+3)$ and $(2x+3)$ are the two factors, the conditions will be satisfied.

(12) Applying the principle in (11), resolve each of the following into two integral binomial factors.

(a) $16x^2 - 8x + 1$ (c) $5y^2 + 11y + 6$

(b) $6y^2 + 25y + 25$ (d) $3x^2 + 2x - 21$

Answers:

(a) $(4x - 1)(4x - 1)$ (c) $(5y + 6)(y + 1)$

(b) $(2y + 5)(3y + 5)$ (d) $(3x - 7)(x + 3)$

(13) Answer the following.

(a) Express $\square^2 - \triangle^2$ as the product of two binomials.

Answer: $(\square + \triangle)(\square - \triangle)$

(b) Express $9x^2 - 100$ as the product of two integral binomials.

Answer: $(3x + 10)(3x - 10)$

(14) Following the note preceding exercise (3) of this section, resolve each of the following into two integral or rational binomial factors (product form).

(a) $4k^2 - 121$ (c) $49x^4 - 36$

(b) $\frac{1}{9}f^2 - 0.01$ (d) $9a^2 - \frac{1}{4}$

Answers:

(a) $(2k + 11)(2k - 11)$ (c) $(7x^2 + 6)(7x^2 - 6)$

(b) $(\frac{1}{3}f + 0.1)(\frac{1}{3}f - 0.1)$ (d) $(3a - \frac{1}{2})(3a + \frac{1}{2})$

(15) In the set of integers, factor each of the following completely (as the product of prime factors).

(a) $5ab + 10ac + 15a^2b$

(b) $x^3 + 3x^2 + 2x$

(c) $14xy + 7x^2y - 21x^3y$

(d) $6xy^2 - 24x$

(e) $5m^2 + 40m - 45$

(f) $x^4 - 16$

(g) $d^4 - 3d^2$

Answers:

(a) $(5a)(b + 2c + 3ab)$ (e) $(5)(m + 9)(m - 1)$

(b) $(x)(x + 2)(x + 1)$ (f) $(x^2 + 4)(x + 2)(x - 2)$

(c) $(7xy)(2 + 3x)(1 - x)$ (g) $d^2 (d^2 - 3)$

(d) $(6x)(y + 2)(y - 2)$

(16) Simplify each of the following by first factoring the numerator and denominator and reducing the expression to lowest terms by applying the cancellation law.

(a) $\frac{4c^2 - 9}{4c + 6}$ (b) $\frac{x^2 + 9x + 14}{x^2 + 5x + 6}$ (c) $\frac{(x + y)^2}{x^2 - y^2}$

Answers:

(a) $\frac{2c - 3}{2}$ (b) $\frac{x + 7}{x + 3}$ (c) $\frac{x + y}{x - y}$

8.5 DIVISION OF POLYNOMIALS

Concept: Division is the inverse of multiplication.

(1) <u>How can it be determined whether or not 19 is a factor of 589</u>?

<u>Answer</u>: If 19 is a factor if 589, it will divide into 589 without a remainder.

(2) <u>Answer the following</u>.

(a) Express the division $\frac{589}{19} = 31$ as an equivalent multiplication.

(b) How can this principle be used to check the accuracy of division?

<u>Answers</u>:

(a) $(31)(19) = 589$

(b) If the product of the quotient and divisor equals the dividend, the division was performed correctly.

(3) <u>Answer the following</u>.

(a) How can it be determined whether or not $(2x + 5)$ is a factor of $6x^3 + 23x^2 + 30x + 25$?

<u>Answer</u>: By dividing $6x^3 + 23x^2 + 30x + 25$ by $2x + 5$. If there is no remainder, then $2x + 5$ is a factor.

(b) If $2x + 5$ is a factor of $6x^3 + 23x^2 + 30x + 25$, then $\frac{6x^3 + 23x^2 + 30x + 25}{2x + 5} = (\square + \frown + \triangle)$ where $(\square \div \frown + \triangle)$ represents a polynomial. Express the above equation as an equivalent multiplication.

<u>Answer</u>: $(2x + 5)(\square + \frown + \triangle) = 6x^3 + 23x^2 + 30x + 25$

(c) Express the left member of the above equation in an equivalent manner, applying the distributive principle.

<u>Answer</u>: $(2x + 5)(\square) + (2x + 5)(\frown) + (2x + 5)(\triangle) = 6x^3 + 23x^2 + 30x + 25$

(d) If $2x + 5$ is a factor of the given polynomial, what must $(\square)$ represent?

<u>Answer</u>: $(\square)$ must represent $\frac{6x^3}{2x}$ or $3x^2$.

(e) Substitute $3x^2$ for $(\square)$ in the above equation and simplify the equation.

Answer: $(2x + 5)(3x^2) + (2x + 5)(\text{⌓}) + 2x + 5)(\triangle) = 6x^3 + 23x^2 + 30x + 25$

$$6x^3 + 15x^2 + (2x + 5)(\text{⌓}) + (2x + 5)(\triangle) = 6x^3 + 23x^2 + 30x + 25$$

$$(2x + 5)(\text{⌓}) + (2x + 5)(\triangle) = 8x^2 + 30x + 25$$

(f) What must (⌓) represent in the answer to (e)?

Answer: (⌓) must represent $\frac{8x^2}{2x}$ or $4x$.

(g) Substitute $4x$ for (⌓) in the equation in (f) and simplify the equation.

Answer: $(2x + 5)(4x) + (2x + 5)(\triangle) = 8x^2 + 30x + 25$

$$8x^2 + 20x + (2x + 5)(\triangle) = 8x^2 + 30x + 25$$

$$(2x + 5)(\triangle) = 10x + 25$$

(h) What must ($\triangle$) represent in the answer to (g)?

Answer: ($\triangle$) must represent $\frac{10x}{2x}$ or 5.

(i) What is the quotient of $\frac{6x^3 + 23x^2 + 30x + 25}{2x + 5}$?

Answer: $3x^2 + 4x + 5$

(4) After each new term of the quotient is determined, what is the next step?

Answer: The product of this new term and the divisor is subtracted from the portion of the dividend remaining from the previous step.

(5) In what operation with numbers is this same procedure performed?

Answer: In the procedure for performing long division of numbers.

(6) Show how the procedure for performing long division of numbers can be used to perform the long division of polynomials.

Answer:

$$\begin{array}{r} 3x^2 + 4x + 5 \\ 2x + 5 \overline{)\, 6x^3 + 23x^2 + 30x + 25} \\ \underline{6x^3 + 15x^2} \\ 8x^2 + 30x + 25 \\ \underline{8x^2 + 20x} \\ 10x + 25 \\ \underline{10x + 25} \end{array}$$

(7) Perform the following divisions using the form in (6).

(a) $\frac{12x^3 + 11x^2 - 33x - 30}{3x + 5}$

(b) $\frac{-4w^3 + 14w^2 - 20w + 16}{w - 2}$

(c) $\frac{88p^3 + 157p^2 + 142p + 63}{8p + 7}$

Answers:

(a) $4x^2 - 3x - 6$ (c) $11p^2 + 10p + 9$

(b) $-4w^2 + 6w - 8$

(8) Determine the required quotient in each of the following by performing the appropriate division. If the given binomial is not a factor of the dividend, indicate in the manner shown below the remainder that exists after the division has been performed.

(a) $x^3+6x^2+2x-4=(x+5)(\qquad)+R$

(b) $-30x^3-38x^2-60x-16=(6x^2+10x-4)(\qquad)+R$

(c) $6a^4+35a^3+39a^2-27a-28=(2a+7)(\qquad)+R$

Answers:

(a) $(x + 5)(x^2 + x - 3) + 11$

(b) $(6x^2 + 10x - 4)(-5x + 2) + (-100x - 8)$

(c) $(2a + 7)(3a^3 + 7a^2 - 5a + 4) + (-56)$

(9) Answer the following.

(a) What difficulty arises in performing the disivion $\frac{x^4 + 2x^3 + 14x + 15}{x + 3}$?

Answer: There is no x^2 term.

(b) How can an x^2 term be inserted into the polynomial without changing its value?

Answer: By expressing the x^2 term with a zero coefficient

(c) Perform the division $\frac{x^4 + 2x^3 + 14x + 15}{x + 3}$ by first inserting x^2 term with coefficient of zero into the numerator.

Answer:

$$\begin{array}{r} x^3 - x^2 + 3x + 5 \\ x + 3 \overline{)\, x^4 + 2x^3 + 0x^2 + 14x + 15} \\ \underline{x^4 + 3x^3 \qquad\qquad\qquad\quad} \\ -x^3 + 0x^2 + 14x + 15 \\ \underline{-x^3 - 3x^2 \qquad\qquad\quad} \\ 3x^2 + 14x + 15 \\ \underline{3x^2 + \;\; 9x \qquad} \\ 5x + 15 \\ \underline{5x + 15} \end{array}$$

(10) Perform the following divisions, first inserting terms with a zero coefficient where terms are missing.

(a) $8x^2 + 7 \overline{)\, 64x^4 + 8x^3 + 7x - 49}$

(b) $\dfrac{x^6 + 6x^3 - 91}{x^3 + 13}$

(c) $(a^4 - 4a^2 - 45) \div (a - 3)$

Answers:

(a) $8x^2 + x - 7$ (c) $a^3 + 3a^2 + 5a + 15$

(b) $x^3 - 7$

(11) In performing the division $\dfrac{6x^2 + 6x + 10 - 4x^3}{2x - 5}$, what should first be done to the numerator to make the division as neat and as simple as possible?

Answer: The terms of the numerator should first be arranged in descending order of exponents.

(12) Perform the following divisions, first arranging the terms of the polynomials in descending order of exponents.

(a) $\dfrac{22x - x^3 - 3x^2 - 12}{3 - x}$

(b) $(-30x + 8 + 15x^3 + 19x^2) \div (4 - 5x - 3x^2)$

(c) $\dfrac{6x^3 - x^4 - 10x^2 + 22x + 15}{5 - x}$

Answers:

(a) $x^2 + 6x - 4$ (c) $x^3 - x^2 + 5x + 3$

(b) $-5x + 2$

UNIT TEST

1. Subtract $x^2 - x + 4$ from $2x^2 - x + 3$.

   Answer: $x^2 - 1$

2. Divide $-3k^2 + 5k + 2$ by $-k + 2$.

   Answer: $3k + 1$

3. Write an expression for the perimeter of a square whose side is represented by $\frac{3s}{2} - 9$.

   Answer: $6s - 36$

4. Write an expression for the perimeter of a square whose area is $16w^2 + 40w + 25$.

   Answer: $16w + 20$

5. Write an expression for the perimeter and for the area of a rectangle whose width is $2x + 4$ and whose length is $3x^2 + 4x - 10$.

   Answer: Its perimeter is $6x^2 + 12x - 12$.
   Its area is $6x^3 + 20x^2 - 4x - 40$.

6. Factor completely in the set of integers: $3a^2 - 12$.

   Answer: $3(a + 2)(a - 2)$

7. Perform the indicated operations and express the result in standard polynomial form: $(3x + 5)(x - 2) - (x - 1)^2$.

   Answer: $2x^2 + x - 11$

8. Express $\frac{x^2 - 7x + 12}{x^2 - 3x}$ in lowest terms.

   Answer: $\frac{x - 4}{x}$

9. If x represents an integer, find the product of the next three greater integers.

   Answer: $x^3 + 6x^2 + 11x + 6$

10. Perform the indicated operations and express the result in simplest form.

$$\frac{6x^2y}{x^2 + xy} \cdot \frac{(x + y)^2}{2x} \div \frac{x + y}{3y^2}$$

Answer: $9y^3$

11. Express $\frac{y^2 - 4}{y^2 + 3y - 10}$ in lowest terms.

Answer: $\frac{y + 2}{y + 5}$

12. Is $5\sqrt{7}\,x^2 + 6^{\frac{1}{4}}x - 7^{-4}$ a polynomial?

Answer: Yes

13. If a given polynomial contains a variable, is the multiplicative inverse of the polynomial a polynomial? Why?

Answer: No. The inverse will contain a variable in a denominator.

14. Is the equation $-x^2 = (-x)^2$ an identity? Why or why not?

Answer: No. $-x^2$ means $-(x^2)$ and is therefore a negative number except when x is zero. $(-x)^2$ is a positive number except when x is zero. The equation is false except when x is zero.

15. Factor completely in the set of integers: $256x^8 - 1$.

Answer: $(16x^4 + 1)(4x^2 + 1)(2x + 1)(2x - 1)$

UNIT 9: POLYNOMIAL EQUATIONS

PART 1. BACKGROUND MATERIAL FOR TEACHERS

9.1 INTRODUCTION

Three common methods of solving polynomial equations are the factoring method, the method of completing the square, and the use of the quadratic formula. Completing the square is an application of the factoring method, and the derivation of the quadratic formula involves completing the square. Therefore, all three of these methods of solving polynomial equations are based on the process of factoring. It is essential that the pupils have mastered the concepts of factoring before attempting the work in this unit.

The concepts in this unit pertain almost entirely to the solving of quadratic equations. However, some of the concepts are applicable to equations of higher degree than quadratic equations. For example, the concept of solving quadratic equations by the factoring method is applicable to the solution of a polynomial equation of degree three that can be factored into three first degree factors. The equation $x^3 + 2x^2 + x = 0$ can be solved by first performing the factorization $(x)(x + 1)(x + 1) = 0$. Because the questions and activities contain a few equations of degree higher than the second, this unit has been entitled "Polynomial Equations" rather than "Quadratic Equations." However, this does not mean that the unit contains an extensive discussion of the solution of equations of higher degree than quadratic equations, nor that the unit contains a development of the concepts involved in the solution of such equations.

9.2 SOLVING POLYNOMIAL EQUATIONS BY FACTORING

The concept of solving a polynomial equation by factoring is based on the principle that the equation $(\square)(\triangle) = 0$ if $(\square) = 0$ or if $(\triangle) = 0$. The equation is true if $(\square) = 0$, regardless of what number $(\triangle)$ represents, and it is true if $(\triangle) = 0$, regardless of what number $(\square)$ represents. The product of any number and zero is zero. The roots of the equation $(\square)(\triangle) = 0$ are determined by solving each of the equations $(\square) = 0$ and $(\triangle) = 0$. The union of the solution sets of these equations forms the solution set of the given equation. If the left member of an equation is a polynomial and the right member is zero, and if the polynomial can be factored into two or more first degree factors, the equation can be solved in the manner described.

The first type of polynomial equation to be solved by factoring is the quadratic equation. A quadratic equation in standard form is one in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. It is important to emphasize that an equation of the type $6x^2 - 3x - 6 = 0$ which involves subtraction must first be written as the equivalent equation involving only addition. That is, only if the equation is expressed as $6x^2 + (-3)x + (-6) = 0$ is it in the standard form of a quadratic equation. The importance of this arises later in the section pertaining to the use of the quadratic formula which involves the coefficients of the x term and the constant term. The coefficient of x is -3 and the constant term is -6. If $6x^2 - 3x - 6 = 0$ were considered to be in the standard form of a quadratic equation, the pupils would be justified in assuming that the coefficient of x is 3 and the constant term is 6. The quadratic formula would then have to be modified before it could be used for solving such equations. For this reason, it is important to make it clear to the pupils that the standard form of a quadratic equation does not involve subtraction. An equation involving subtraction must be replaced by its equivalent form involving only addition.

Any quadratic equation that can be factored into the form $(\square)(\triangle) = 0$ where $\square$ and $\triangle$ each involves the placeholder, can be solved by the factoring method. Each factor is written equal to zero, and the union of the solution sets of both of these supplementary equations forms the solution set of the original equation. For example,

$$x^2 + 7x + 12 = 0$$
$$(x + 4)(x + 3) = 0$$
$$x + 4 = 0 \qquad x + 3 = 0$$
$$x = -4 \qquad x = -3$$
$$\{-4, -3\}$$

The equation must be written in standard quadratic form before the factoring is performed. The equation $x^2 + 4x = 12$ should first be replaced by the equivalent equation $x^2 + 4x + (-12) = 0$ before factoring the left member of the equation.

If the coefficient of the x term in the quadratic equation $ax^2 + bx + c = 0$ is zero and the constant term is negative, there is a second method that may be used for solving a quadratic equation. If $x^2 - \triangle = 0$, then $(x + \sqrt{\triangle})(x - \sqrt{\triangle}) = 0$ and $x = -\sqrt{\triangle}$, $x = \sqrt{\triangle}$. Thus in the second method if $x^2 - \triangle = 0$ then $x^2 = \triangle$, and $x = \pm\sqrt{\triangle}$. If $x^2 - 7 = 0$, then $x^2 = 7$ and $x = \pm\sqrt{7}$. This development may be carried one step further in applying the alternate principle that if $\square^2 - \triangle = 0$, then $\square^2 = \triangle$ and $\square = \pm\sqrt{\triangle}$. We may let $\square$ represent a binomial. If $\square$ represents $x + 5$ and $\triangle$ represents 36, then

$$(x + 5)^2 = 36$$
$$x + 5 = \pm\sqrt{36}$$
$$x + 5 = 6 \qquad x + 5 = -6$$
$$x = 1 \qquad x = -11$$
$$\{1, -11\}$$

A quadratic equation of the type $\square^2 - \triangle = 0$ may be solved by either the factoring method or the square root method just described, which is an application of the factoring method. It probably is best for the pupils to have experience in the use of both methods and to be given free choice in the use of either method in solving equations of this type. The factoring method of solving quadratic equations may be applied to solving polynomial equations of higher degree than quadratic if the pupils are able to factor the given polynomial into first degree factors.

The equation $(\square)(\triangle)(\bigcirc) = 0$ is true when $\square = 0$, when $\triangle = 0$, or when $\bigcirc = 0$, regardless of what numbers the other two factors represent. Such an equation may be solved by writing each factor equal to zero and solving each of the three resulting equations. The union of the three resulting solution sets forms the solution set of the given equation. The equation $x^3 + 5x^2 + 6x = 0$ may be solved in this manner. The left member of the equation may be factored into $(x)(x + 3)(x + 2)$.
If $x^3 + 5x^2 + 6x = 0$, then $(x)(x + 3)(x + 2) = 0$. This is true if $x = 0$, or $x + 3 = 0$, or $x + 2 = 0$

$$x = -3 \qquad x = -2$$

The solution set is $\{0, -2, -3\}$.

The questions and activities in unit 8 did not contain exercises requiring factoring polynomials whose factors contained irrational numbers. The equation $\square^2 - \triangle = 0$ may be solved, as just explained, by writing the equation in equivalent form $\square^2 = \triangle$, the roots of which are $\square = \sqrt{\triangle}$ and $\square = -\sqrt{\triangle}$. However, in the set of real numbers this quadratic equation may also be solved by factoring, and the factors may contain irrational numbers. Such factoring is quite simple, and this affords an excellent opportunity for the pupils to gain experience in the use of such irrational factors. Thus, if $\square^2 - \triangle = 0$, then $(\square + \sqrt{\triangle})(\square - \sqrt{\triangle}) = 0$

$$\square + \sqrt{\triangle} = 0 \qquad \square - \sqrt{\triangle} = 0$$
$$\square = -\sqrt{\triangle} \qquad \square = \sqrt{\triangle}$$

The solution set is $\{\sqrt{\triangle}, -\sqrt{\triangle}\}$.

9.3 COMPLETING THE SQUARE

Solving a quadratic equation by completing the square is an application of the principle that if $\square^2 = \triangle$, then $\square = \pm\sqrt{\triangle}$ where $\square$ represents a binomial and $\triangle$ is a positive constant. The equation $(x - 4)^2 = 2$ may be solved as follows:

$$(x - 4)^2 = 2$$
$$x - 4 = \pm\sqrt{2}$$
$$x - 4 = \sqrt{2} \qquad x - 4 = -\sqrt{2}$$
$$x = 4 + \sqrt{2} \qquad x = 4 - \sqrt{2}$$

The solution set is $\{4 + \sqrt{2}, 4 - \sqrt{2}\}$.

This method is used when the left member of the equation is the square of a binomial or when some number can be added to both members of the equation to make the left member of the resulting equivalent equation the square of a binomial. The left member of the equation equivalent to $x^2 + 8x + 10 = 0$ will be the square of the binomial $(x + 4)$ if 6 is added to both members of the given equation.

$$x^2 + 8x + 10 = 0$$
$$x^2 + 8x + 16 = 6$$
$$(x + 4)^2 = 6$$
$$x + 4 = \pm \sqrt{6}$$
$$x + 4 = \sqrt{6} \qquad x + 4 = -\sqrt{6}$$
$$x = -4 + \sqrt{6} \qquad x = -4 - \sqrt{6}$$

The solution set is $\{-4 + \sqrt{6}, -4 - \sqrt{6}\}$.

The method of solving a quadratic equation by completing the square has the advantage that it can be used to solve equations which cannot be solved by the first factoring described, and it usually is a much easier and faster method than use of the quadratic formula which is described in the next section. Of course, its application is limited to those quadratics in which the polynomial can be replaced by one which is a square of a binomial, by adding the same number to each side of the equation. There are instances in which completing the square can be used to solve a quadratic equation by subtracting the same number from each side of the equation. However, this must be done with caution so that the number on the right side of the equation does not become a negative number. If $\square^2 = \triangle$, where $\triangle$ is a negative number, then $\square$ cannot represent a number in the set of real numbers, as the square of any real number is non-negative. One important reason for studying the use of the method of completing the square is that it is used in the derivation of the quadratic formula.

9.4 THE QUADRATIC FORMULA

The quadratic formula may be used to solve any quadratic equation. The derivation of this formula is as follows:

(1) $ax^2 + bx + c = 0$ — This is the standard form of a quadratic equation.

(2) $x^2 + \frac{bx}{a} + \frac{c}{a} = 0$ — Each member of the original equation is divided by a, yielding an equivalent equation in which the coefficient of the x^2 term is unity.

(3) $x^2 + \frac{bx}{a} = -\frac{c}{a}$ — $-\frac{c}{a}$ is added to both sides of the previous equation.

If the left side of the resulting equivalent equation is to be the square of a binomial, $\frac{b}{a}$ must be two times the constant term of such a binomial. The constant term of the binomial must be $\frac{b}{2a}$ and the constant term in the square of such a binomial would be $\frac{b^2}{4a^2}$.

(4) $x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$ — $\frac{b^2}{4a^2}$ is added to both sides of the previous equation to yield an equivalent equation in which the left member is a perfect square; this is called completing the square.

(5) $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ — The left side of the equation is expressed as a square of a binomial. The terms on the right side are combined under the operation subtraction.

(6) $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$ — This is the application of the principle that if $\square^2 = \triangle$, then $\square = \pm\sqrt{\triangle}$.

(7) $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ — $-\frac{b}{2a}$ is added to both sides of the previous equation, and the right side is simplified by applying the principle $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

(8) $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ — Both terms on the right side of the equations are combined under the operations of addition and subtraction.

Step 7 above is actually the combination of several steps that have been combined for the sake of simplicity. This step implies that $\pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$ is equal to $\pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$ which in turn is equal to $\pm\frac{\sqrt{b^2 - 4ac}}{2a}$. One point needs to be made clear. We know that $\sqrt{4a^2} = \sqrt{4}\ \sqrt{a^2} = 2\sqrt{a^2} = 2|a|$. However, in this case, we may use the denominator 2a since, whether a is positive or negative, $\pm 2|a| = \pm 2a$, in some order. Thus the two fractions in step 7 are written with the same denominator and are combined in step 8.

Below is an example of how the quadratic formula may be used to solve the equation $-4x^2 - 2x + 3 = 0$.

$$-4x^2 + (-2)x + 3 = 0 \qquad a = -4,\ b = -2,\ c = 3$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - (4)(-4)(3)}}{2(-4)}$$

$$x = \frac{2 \pm \sqrt{4 + 48}}{-8}$$

$$x = \frac{2 \pm \sqrt{52}}{-8}$$

$$x = \frac{2 \pm 2\sqrt{13}}{-8}$$

This last expression may be simplified by factoring the numerator by application of the distributive principle and then reducing to lowest terms by application of the cancellation law.

$$x = \frac{2(1 \pm \sqrt{13})}{2(-4)}$$

$$x = \frac{1 \pm \sqrt{13}}{-4} = -\frac{1 \pm \sqrt{13}}{4}$$

The solution set is $\left\{-\frac{1 + \sqrt{13}}{4}, -\frac{1 - \sqrt{13}}{4}\right\}$.

$b^2 - 4ac$ in the expression $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is called the discriminant. An examination of the discriminant of a quadratic equation furnishes much information as to the nature of the roots of the equation. There is no real number x such that $x^2 = -3$. The square of any real number is non-negative. The roots of $x^2 = -3$ are $x = \sqrt{-3}$ and $x = -\sqrt{-3}$. Therefore, $\pm\sqrt{-3}$ cannot be real numbers. The square root of any negative number cannot be a real number. Therefore, if $b^2 - 4ac$ is negative, then $\sqrt{b^2 - 4ac}$ is not a real number and $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ cannot be a real number. If the discriminant of a quadratic equation is negative, the equation has no real roots.

If $b^2 - 4ac$ is zero, then $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ becomes $-\frac{b}{2a}$.
Now, b and a are real numbers so $-\frac{b}{2a}$ is a real number.
Therefore, if the discriminant of a quadratic equation is zero, the equation has only one root and that root is the real number $-\frac{b}{2a}$.

If $b^2 - 4ac$ is positive, then both roots $x = \frac{-b + \sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ are real numbers because the square root of a positive real number is a real number.

If $b^2 - 4ac$ is the square of an integer k, then $\frac{-b \pm \sqrt{k^2}}{2a}$ will equal $\frac{-b \pm k}{2a}$. If a, b, and c are rational and if $x = \frac{-b \pm k}{2a}$ then the roots of the quadratic equation will be rational numbers. All the exercises in the questions and activities involve quadratic equations whose coefficients and constant terms are rational numbers. However, the teacher may wish to bring up for class discussion the topic of equations containing irrationals.

When both sides of an equation are multiplied by a variable, or when both sides of an equation are squared, the resulting equation may not have the same solution set as the original equation, that is, it is not an equivalent equation. The solution set of $x - 3 = 10$ is $\{13\}$. If both sides of the equation are multiplied by x, the equation $x^2 - 3x = 10x$ results. The solution set of this equation is $\{0, 13\}$. If, in solving an equation, both sides of the equation are multiplied by a variable, each element in the solution set of the resulting equation must be substituted for the variable in the original equation to determine which of these elements is in the solution set of the original equation. The solution set of the resulting equation usually has more elements than the solution set of the original equation. The additional numbers which have been involved as ostensible roots of the original equation are often referred to as extraneous roots.

This same caution applies when both sides of an equation have been squared. The equation $x + 5 = 8$ has only one root. However, if both members are squared, the resulting equation, $x^2 + 10x + 25 = 64$, has two roots, only one of which is an element of the solution set of the original equation. Again, these are not equivalent equations.

Care must be taken in solving an equation of the type $x^2 = y^2$. If $x^2 = y^2$, you cannot conclude that $x = y$. For example, $(-3)^2 = 3^2$, but it is not true that $-3 = 3$.

If $x^2 = y^2$, then $x = \pm \sqrt{y^2}$. Therefore, if $x^2 = y^2$ then $x = \pm |y|$ which is equivalent to saying $x = \pm y$. If $x^2 = y^2$ it can be concluded that $x = \pm y$, yet it cannot be concluded that $x = y$ and it cannot be concluded that $x = -y$. This is a very fine point and a cause of some serious confusion. The key to clearing up this confusion is the word or. $x = \pm y$ means that x equals y or that x equals -y. In this instance the word or means that it is possible that x equals y; it is possible that x equals -y; it is possible that x is equal to both y and -y. It does not mean that x does equal y; it does not mean that x does equal -y; it does not mean that x equals both y and -y. If the square root of each side of an equation is taken when solving an equation, it is very important to understand what conclusion can and what conclusions cannot be reached concerning the results of such an operation.

9.5 VERBAL PROBLEMS

This section consists of solving verbal problems by the use of quadratic equations. It is important for the pupils to realize that the solution set of the quadratic equation used in solving a verbal problem may contain elements which do not meet the requirements of the problem. The problem may contain stated or implied restrictions. Below are some examples of verbal problems that contain such restrictions.

(1) The product of 3 greater than a given integer and one greater than twice that integer is equal to 25. Determine the given integer.

This problem contains the stated restriction that the number must be an integer. When the problem is solved by use of the equation $(x + 3)(2x + 1) = 25$, it is found that the solution set of the equation is $\{-\frac{11}{2}, 2\}$. The number $-\frac{11}{2}$ is a root of the equation but it does not meet the requirements of the problem because it is not an integer, and must therefore be discarded as a solution to the problem.

(2) The length of a rectangle is 4 more than twice the width. The area is 48 square inches. Find the length and the width.

This problem contains the implied restriction that the measure of a dimension cannot be expressed as a negative number. A rectangle cannot have a width or length that is negative. The problem may be solved by use of the equation $(x)(2x + 4) = 48$. The roots of this equation are -6 and 4. The root -6 must be discarded because the width cannot be negative.

These two problems illustrate the importance of determining whether each of the roots of the quadratic equation used to solve the problem meets the stated or implied restrictions contained in the problem.

Almost all the verbal problems in this unit have one or two answers. However, pupils may find it interesting and challenging to solve problems whose solution set is the null set or whose solution set is an infinite set. Below are two problems, the solution set of the first being the null set and the solution set of the second being an infinite set.

(3) The square of a number is equal to the square of one more than the number decreased by twice the the number. Find the number.

If the problem is solved by use of the equation $x^2 = (x + 1)^2 - 2x$ then $0 = 1$. This is an internally inconsistent equation so therefore the equation $x^2 = (x + 1)^2 - 2x$ must be an internally inconsistent equation and the solution set is the null set. There is no number that can meet the requirements of the problem.

(4) Find three consecutive integers such that the square of the middle number is one less than half the sum of the squares of the other two numbers.

This problem may be solved as follows:

$$(x + 1)^2 + 1 = \frac{x^2 + (x + 2)^2}{2}$$

$$x^2 + 2x + 1 + 1 = \frac{x^2 + x^2 + 4x + 4}{2}$$

$$x^2 + 2x + 2 = x^2 + 2x + 2$$

This equation is an identity; therefore, its solution set is the set of all real numbers. The problem contains the restriction that the answer must be an integer so the answer to the problem is the set of all integers.

The topic of solution of quadratic inequalities is not introduced until unit 10. The solution of such inequalities is based on the same factoring method used in this unit for solving quadratic equations.

Teacher Notes

UNIT 9: POLYNOMIAL EQUATIONS

PART 2. QUESTIONS AND ACTIVITIES FOR CLASSROOM USE

9.1 INTRODUCTION

The major purpose of the questions and activities in this unit is to develop the concepts underlying the various methods that may be used to solve quadratic equations, and to give the pupils experience in solving verbal problems leading to quadratic equations. The questions and activities should be considered only as a guide to illustrate to the teacher how such concepts can be developed by a systematic series of questions, and as a guide to illustrate what types of problems the pupils should be able to solve after mastering each concept. Supplemental exercises for classroom drill, homework assignments, and testing may be obtained from available texts or developed by the teacher. The topics of graphing quadratic equations and quadratic inequalities are both introduced in a later unit. This unit does not contain a unit test. The last section on verbal problems serves as a good instrument to determine how well the pupils have mastered the concepts developed in the unit.

9.2 SOLVING POLYNOMIAL EQUATIONS BY FACTORING

(1) Solve each of the following problems.

(a) What number substituted for k in the equation $(k)(m) = 0$ will make the equation true, regardless of what number m represents?

Answer: Zero

(b) What number substituted for m in the equation $(k)(m) = 0$ will make the equation true, regardless of what number k represents?

Answer: Zero

(c) What number substituted for the first factor in the equation $(x + \square)(x + \triangle) = 0$ will make the equation true, regardless of what number the second factor represents?

Answer: Zero

(d) What number substituted for the second factor in the equation $(x + \square)(x + \triangle) = 0$ will make the equation true, regardless of what number the first factor represents?

Answer: Zero

(e) Under what two conditions is the equation $(x + 4)(x - 6) = 0$ definitely true?

Answer: The equation is definitely true when $x + 4 = 0$ or when $x - 6 = 0$.

(f) If the equation $(x + 4)(x - 6) = 0$ is true when $x + 4 = 0$ or when $x - 6 = 0$, then what numbers must x represent to make the equation true?

Answer: $x = -4$ or $x = 6$

(g) When -4 is substituted for x in the equation, is the equation true?

Answer: Yes

(h) When 6 is substituted for x in the equation, is the equation true?

Answer: Yes

(2) If an equation is in the form $(k)(m) = 0$, how can the solution set of this equation be determined?

Answer: The solution set can be determined by writing each factor equal to 0 and solving the resulting equations. The union of the two solution sets forms the solution set of the given equation.

(3) In each of the following determine what values of the variable will make the equation a true statement.

(a) $(x + 9)(x - 9) = 0$ (c) $(x - 3)(x - 11) = 0$
(b) $(x)(x + 4) = 0$ (d) $(-x + 8)(-x - 3) = 0$

Answers:

(a) 9, -9 (b) 0, -4 (c) 3, 11 (d) 8, -3

(4) How can the equation $x^2 - 11x + 30 = 0$ be solved, applying the principles just developed?

Answer: The equation can be solved by factoring the left member of the equation into two factors, setting each factor equal to zero, and solving these resulting equations. The union of the two solution sets is the solution set of the original equation.

(5) Solve each of the following equations by first factoring the polynomial which is the left member of the equation.

(a) $x^2 + 10x + 16 = 0$ (c) $w^2 - 16w + 48 = 0$
(b) $a^2 - 9a - 36 = 0$ (d) $x^2 - 25 = 0$

Answers:

(a) -8, -2 (b) 12, -3 (c) 12, 4 (d) 5, -5

(6) What must be done to the equation $x^2 + 15x = -56$ before it can be solved by the factoring method?

Answer: The equation must first be replaced by the equivalent equation having the form $x^2 + 15x + 56 = 0$ before it can be solved by factoring.

(7) Solve each of the following using the factoring method, first replacing the equation by an equivalent equation in correct form.

(a) $x^2 = -10x - 21$ (d) $x^2 + 4x - 16 = 4x$

(b) $-0.9x = -0.2 - x^2$ (e) $x^2 = 121$

(c) $x^2 + 19 = 10 - 10x$

Answers: (a) -7, -3 (b) 0.5, 0.4 (c) -9, -1 (d) 4, -4 (e) 11, -11

(8) Solve the equation $x^2 = y$ for x.

Answer: $x = \pm\sqrt{y}$

(9) Is factoring necessary to solve the equation $x^2 = 121$?

Answer: No. If $x^2 = 121$, then $x = \pm\sqrt{121}$ or $x = \pm 11$.

(10) Solve each of the following by applying the principle that if $x^2 = y$, then $x = \pm\sqrt{y}$.

(a) $x^2 - 86 = 14$ (b) $x^2 + 19 = 68$ (c) $49 - x^2 = 48$

Answers: (a) 10, -10 (b) 7, -7 (c) 1, -1

(11) Solve each of the following, applying the same principle as in (10).

(a) $x^2 - 3 = 0$ (b) $x^2 = 17$ (c) $x^2 + 17 = 22$

Answers:

(a) $\sqrt{3}, -\sqrt{3}$ (b) $\sqrt{17}, -\sqrt{17}$ (c) $\sqrt{5}, -\sqrt{5}$

(12) How can an equation in the form $(x + 5)^2 = 36$ be solved by the factoring method?

Answer: The left member is first written as a polynomial. Then 36 is subtracted from each side of the equation. The resulting equivalent equation can then be solved by the factoring method.

(13) Show how the equation $(x + 5)^2 = 36$ can be solved using the method just illustrated.

Answer: $(x + 5)^2 = 36$

$$x^2 + 10x + 25 = 36$$
$$x^2 + 10x - 11 = 0$$
$$(x + 11)(x - 1) = 0$$
$$x + 11 = 0 \qquad x - 1 = 0$$
$$x = -11 \qquad x = 1$$

The solution set is $\{-11, 1\}$.

Check: $(-11 + 5)^2 = 36$, true; $(1 + 5)^2 = 36$, true.

(14) Solve each of the following.

(a) $(x - 7)^2 = 36$ (c) $\left(k + \frac{1}{3}\right)^2 = \frac{4}{9}$

(b) $(w + 0.2)^2 = 0.25$ (d) $(x + 2)^2 = 9$

Answers:

(a) 13, 1 (b) 0.3, -0.7 (c) $\frac{1}{3}$, -1 (d) 1, -5

(15) (a) Solve the equation $\triangle^2 = y$ for $\triangle$.

Answer: $\triangle = \pm \sqrt{y}$

(b) In the equation $\triangle^2 = y$, replace $\triangle$ with $x + 5$ and replace y with 16. Solve the resulting equation.

Answer: $(x + 5)^2 = 16$

$$x + 5 = \pm \sqrt{16}$$
$$x + 5 = \pm 4$$
$$x + 5 = 4 \qquad x + 5 = -4$$
$$x = -1 \qquad x = -9$$

The solution set is $\{-1, -9\}$.

Check: $(-1 + 5)^2 = 16$, true; $(-9 + 5)^2 = 16$, true.

(16) Solve each of the following, applying the principle that if $\triangle^2 = y$, then $\triangle = \pm \sqrt{y}$.

(a) $(k + 9)^2 = 144$ (c) $(w - 11)^2 = 1$

(b) $\left(y - \frac{2}{3}\right)^2 = \frac{25}{16}$ (d) $(x + 0.1)^2 = 0.04$

Answers: (a) 3, -21 (b) $\frac{23}{12}$, $-\frac{7}{12}$ (c) 12, 10

(d) 0.1, -0.3

(17) Can the equation $2x^2 + 15x + 18 = 0$ be solved by the factoring method?

Answer: Yes.

$$2x^2 + 15x + 18 = 0$$
$$(2x + 3)(x + 6) = 0$$
$$2x + 3 = 0 \qquad x + 6 = 0$$
$$x = -\frac{3}{2} \qquad x = -6$$

The solution set is $\{-\frac{3}{2}, -6\}$.

Check: $2(-\frac{3}{2})^2 + 15(-\frac{3}{2}) + 18 = 0$, true.

$2(-6)^2 + 15(-6) + 18 = 0$, true.

(18) Solve each of the following, using the factoring method.

(a) $3x^2 - 2x - 1 = 0$ (c) $4x^2 - 7x + 3 = 0$

(b) $5x^2 + 16x = -3$

Answers: (a) $1, -\frac{1}{3}$ (b) $-3, -\frac{1}{5}$ (c) $1, \frac{3}{4}$

(19) Answer the following.

(a) Is the equation (□)(△)(⌂) = 0 true when (□) = 0, regardless of what the other two factors represent?

(b) Is the equation true when (△) = 0, regardless of what the other two factors represent?

(c) Is the equation true when (⌂) = 0, regardless of what the other two factors represent?

Answers: (a) Yes (b) Yes (c) Yes

(d) Describe how a polynomial equation in the form (□)(△)(⌂) = 0 can be solved.

Answer: Each of the three factors is written equal to zero. The union of the three solution sets is the solution set of the original equation.

(e) Show how the equation $x^3 + 5x^2 + 6x = 0$ can be solved by factoring.

Answer:

$$x^3 + 5x^2 + 6x = 0$$
$$(x)(x^2 + 5x + 6) = 0$$

$$(x)(x + 3)(x + 2) = 0$$

$x = 0 \qquad x + 3 = 0 \qquad x + 2 = 0$

$\qquad\qquad x = -3 \qquad\quad x = -2$

The solution set is $\{0, -3, -2\}$.

Check: $0^3 + 5(0)^2 + 6(0) = 0$, true.

$(-3)^3 + 5(-3)^2 + 6(-3) = 0$, true.

$(-2)^3 + 5(-2)^2 + 6(-2) = 0$, true.

(20) <u>Solve each of the following by using the factor method</u>.

(a) $y^3 + 10y^2 = -21y$ (c) $2y^4 + 10y^3 + 12y^2 = 0$

(b) $x^3 - x = 0$ (d) $(3x^3-6x^2+x)-(-x^2+3x)=0$

<u>Answers</u>: (a) 0, -3, -7 (b) 0, 1, -1

(c) 0, -2, -3 (d) $0, -\frac{1}{3}, 2$

(21) <u>Solve each of the following, applying the principle that if $\square^2 = k$, then $\square = \pm\sqrt{k}$</u>.

(a) $(a - 2)^2 = 7$ (c) $(k - 6)^2 = 5$

(b) $(x + \frac{2}{5})^2 = 2$ (d) $(y + 0.2)^2 = 3$

<u>Answers</u>:

(a) $2 + \sqrt{7}, 2 - \sqrt{7}$ (b) $-\frac{2}{5} + \sqrt{2}, -\frac{2}{5} - \sqrt{2}$

(c) $6 + \sqrt{5}, 6 - \sqrt{5}$ (d) $-0.2 + \sqrt{3}, -0.2 - \sqrt{3}$

9.3 COMPLETING THE SQUARE

(1) <u>Answer the following</u>.

(a) The polynomial $x^2 - 8x + 16$ is the square of what binomial?

<u>Answer</u>: It is the square of $x - 4$.

(b) How can the quadratic equation $x^2 - 8x + 16 = 2$ be solved by applying the principle that if $\square^2 = k$, then $\square = \pm\sqrt{k}$?

Answer: The equation may be written in the form:

$$(x - 4)^2 = 2$$
$$(x - 4) = \pm \sqrt{2}$$
$$x - 4 = \sqrt{2} \qquad x - 4 = -\sqrt{2}$$
$$x = 4 + \sqrt{2} \qquad x = 4 - \sqrt{2}$$

The solution set is $\{4 + \sqrt{2}, 4 - \sqrt{2}\}$.

(2) Solve each of the following, applying the prin- in (1) above.

(a) $y^2 - 6x + 9 = 3$ (c) $a^2 + 18a + 81 = 7$

(b) $w^2 - 2w + 1 = 5$ (d) $x + \frac{6}{5}x + \frac{9}{25} = 11$

Answers:

(a) $3 + \sqrt{3}, 3 - \sqrt{3}$ (c) $-9 + \sqrt{7}, -9 - \sqrt{7}$

(b) $1 + \sqrt{5}, 1 - \sqrt{5}$ (d) $-\frac{3}{5} + \sqrt{11}, -\frac{3}{5} - \sqrt{11}$

(3) Answer the following.

(a) Is the polynomial $x^2 + 8x + 10$ the square of a binomial whose constant term is an integer?

Answer: No. The constant term must be 16 if the first two terms are $x^2 + 8x$.

(b) What number could be added to both sides of the equation $x^2 + 8x + 10 = 0$ that would make the left side of the resulting equivalent equation the square of a binomial whose constant term is an integer?

Answer: If 6 is added to both sides of the equation, the resulting equivalent equation is $x^2 + 8x + 16 = 6$. This can be written as $(x + 4)^2 = 6$.

(c) Solving an equation by first adding some number to both sides of the equation so that the left side of the resulting equivalent equation is the square of a binomial is called completing the square. Show how the equation $x^2 + 4x + 1 = 0$ may be solved by this method of completing the square.

Answer:

$$x^2 + 4x + 1 = 0$$
$$x^2 + 4x + 4 = 3$$
$$(x + 2)^2 = 3$$
$$x + 2 = \pm \sqrt{3}$$

$x + 2 = \sqrt{3}$ $\quad$ $x + 2 = -\sqrt{3}$

$x = -2 + \sqrt{3}$ $\quad$ $x = -2 - \sqrt{3}$

The solution set is $\{-2 + \sqrt{3}, -2 - \sqrt{3}\}$.

(4) Solve each of the following by use of the method of completing the square.

(a) $z^2 + 10z + 15 = 0$ $\quad$ (c) $k^2 - 8k = 1$

(b) $a^2 + 2a - 2 = 0$ $\quad$ (d) $x^2 - 12x + 18 = 0$

Answers:

(a) $-5 + \sqrt{10}, -5 - \sqrt{10}$ $\quad$ (c) $4 + \sqrt{17}, 4 - \sqrt{17}$

(b) $-1 + \sqrt{3}, -1 - \sqrt{3}$ $\quad$ (d) $6 + 3\sqrt{2}, 6 - 3\sqrt{2}$

9.4 THE QUADRATIC FORMULA

(1) An equation in the form $ax^2 + bx + c = 0$, where the left member of the equation is a polynomial of second degree, is called a quadratic equation.

(a) If $a = 0$, is $ax^2 + bx + c = 0$ a quadratic equation?

Answer: No. The left member of the equation is not a polynomial of second degree.

(b) If a, b, and c are any real numbers provided $a \neq 0$, is $ax^2 + bx + c = 0$ a quadratic equation?

Answer: Yes. The left member of the equation is a polynomial of second degree. The coefficients and constant term in a polynomial may be any real numbers.

(2) Answer the following.

(a) What operation may be performed on the equation $ax^2 + bx + c = 0$ so that the coefficient of the x^2 term in the resulting equivalent equation is unity? Write the resulting equation.

Answer: Both sides of the equation may be divided by a. The resulting equivalent equation is $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

(b) What operation may be performed on this equation so that $\frac{c}{a}$ does not appear in the left side of the resulting equivalent equation? Write the resulting equation.

Answer: $-\frac{c}{a}$ may be added to both sides of the equation. The resulting equivalent equation is $x^2 + \frac{bx}{a} = -\frac{c}{a}$.

(c) If this equation is to be solved for x by the method of completing the square, the left side of the resulting equivalent equation must be in what form?

Answer: The left side of the equation must be the square of a binomial.

(d) If $x^2 + \frac{bx}{a} + \triangle$, the left side of the equation, is to be the square of a binomial, what must be the relationship between $\frac{b}{a}$ and the constant term of the binomial?

Answer: $\frac{b}{a}$ must be equal to twice the constant term of the binomial.

(e) What must be the constant term of the binomial?

Answer: $\frac{b}{2a}$

(f) If the square of the binomial is $x^2 + \frac{bx}{a} + \triangle$ and the binomial is $x + \frac{b}{2a}$, what must $\triangle$ represent?

Answer: $\frac{b^2}{4a^2}$

(g) What operation must be performed on the equation $x^2 + \frac{bx}{a} = -\frac{c}{a}$ so that the left side of the resulting equivalent equation is the square of a binomial? Write the resulting equation.

Answer: $\frac{b^2}{4a^2}$ must be added to each side of the equation. The resulting equation is $x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$.

(h) Write the equation in an equivalent form, combining the two terms on the right side into a single term.

Answer: $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$

(i) Solve the above equation by the principle that if $y^2 = z$, then $y = \pm \sqrt{z}$.

Answer: $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(3) Answer the following.

(a) This last equation in (i) above is called the quadratic formula and it may be used to solve any quadratic equation. In solving the equation $2x^2 + 12x + 6 = 0$ by the use of this formula, the letters a, b, and c in the formula will represent what numbers?

Answer: $a = 2$, $b = 12$, $c = 6$

(b) Solve the equation $2x^2 + 12x + 6 = 0$ by use of this formula.

Answer: $x = \frac{-12 \pm \sqrt{144 - 48}}{4} = \frac{-12 \pm \sqrt{96}}{4}$

$$= \frac{-12 \pm 4\sqrt{6}}{4}$$

(c) By the application of what principles can the expression $\frac{-12 \pm 4\sqrt{6}}{4}$ be reduced to lowest terms?

Answer: The numerator can be factored by the application of the distributive law, and then the expression can be reduced to lowest terms by the application of the cancellation law.

$$\frac{-12 \pm 4\sqrt{6}}{4} = \frac{4(-3 \pm \sqrt{6})}{4}$$

$$= -3 \pm \sqrt{6}$$

(4) Solve each of the following.

(a) In solving the equation $-4x^2 - 2x + 3 = 0$, how must this equation be rewritten in order to be in the standard form of a quadratic equation?

Answer: It must be written in the form $(-4)x^2 + (-2)x + 3 = 0$. A quadratic equation contains a polynomial which is a sum of terms.

(b) In solving the equation in (a) by the use of the quadratic formula, what numbers would the letters a, b, and c in the formula represent?

Answer: $a = -4$, $b = -2$, $c = 3$

(c) Solve the equation in (a) above by use of the quadratic formula.

Answer:

$$x = \frac{2 \pm \sqrt{4 - (-48)}}{-8} = \frac{2 \pm \sqrt{52}}{-8}$$

$$= \frac{2 \pm 2\sqrt{13}}{-8}$$

$$= \frac{2(1 \pm \sqrt{13})}{-8}$$

$$= \frac{1 \pm \sqrt{13}}{-4}$$

$$= \frac{-1 \pm \sqrt{13}}{4}$$

(5) Solve each of the following by applying the quadratic formula.

(a) $3x^2 - 5x - 9 = 0$ (c) $-8x - 2 = 5x^2$

(b) $2(3 + x) = -3x(2 - x)$

Answers: (a) $\frac{5 \pm \sqrt{133}}{6}$ (b) $\frac{4 \pm \sqrt{34}}{3}$ (c) $\frac{-4 \pm \sqrt{6}}{5}$

Concept: Use of discriminant.

(6) Answer the following.

(a) The expression $b^2 - 4ac$ in the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is called the discriminant. What restriction must be placed on the value of the discriminant if $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is to repre- a real number?

Answer: $b^2 - 4ac$ must be non-negative.

(b) If the discriminant is negative, what can be concluded as to the nature of the roots of the quadratic equation?

Answer: The roots of the equation are not real numbers.

(c) If $b^2 - 4ac$ is equal to zero, what will the quadratic formula simplify to?

Answer: If $b^2 - 4ac$ is zero, then the quadratic formula becomes $x = -\frac{b}{2a}$.

(d) Does $-\frac{b}{2a}$ represent a real number?

Answer: Yes, because b and a are real numbers.

(e) Is the square root of a positive real number always a real number?

Answer: Yes

(f) If $b^2 - 4ac$ is positive, describe the nature of the roots of $ax^2 + bx + c = 0$.

Answer: The roots will be real numbers.

(g) Summarize the description of the roots of the equation $ax^2 + bx + c = 0$ when the discriminant is negative, when it is positive, and when it is zero.

Answer: If $b^2 - 4ac$ is negative, there are no real roots. If the discriminant is zero, there is only one root and this is a real number. If the discriminant is positive, there are two roots and they are both real numbers.

(7) Determine whether the discriminant in each of the following is positive, negative, or zero and then indicate the number and nature of the roots without solving the equations.

(a) $3y^2 - 6y + 3 = 0$ (c) $17x^2 + 5x + 10 = 0$

(b) $x^2 - 2x - 5 = 0$

Answers:

(a) The discriminant is zero. The equation has one root and it is a real number.

(b) The discriminant is positive. The equation has two roots and they are real numbers.

(c) The discriminant is negative. The equation has no real root.

Concept: Extraneous roots.

(8) Answer the following.

(a) State the multiplication axiom of algebra as used in solving algebraic equations.

Answer: If both sides of an equation are multiplied by the same number, except zero, the given and resulting equations are equivalent, that is, the solution set of the resulting equation is identical to the solution set of the original equation.

(b) What is the solution set of the equation $3x = 13$?

Answer: $\left\{\frac{13}{3}\right\}$

(c) When both sides of the equation $3x = 13$ are multiplied by x, what is the resulting equation?

Answer: $3x^2 = 13x$

(d) Compare the solution set of this equation with the solution set of the original equation.

Answer: The solution set of the original equation is $\left\{\frac{13}{3}\right\}$. The solution set of the second equation is $\left\{0, \frac{13}{3}\right\}$.

(e) When both sides of an equation are multiplied by a variable, are the given and resulting equations equivalent, that is, is the solution set of the resulting equation identical to the solution set of the original equation?

Answer: No

(9) Answer the following.

(a) What is the solution set of the equation $x - 3 = 10$?

Answer: $\{13\}$

(b) What is the solution set of the equation $x - 3 = -10$?

Answer: $\{-7\}$

(c) Square both sides of the equation $x - 3 = 10$ and determine the solution set of the resulting equation.

Answer: $(x - 3)^2 = 100$

$$x - 3 = \pm \sqrt{100}$$

$$x = 3 \pm 10$$

The solution set is $\{13, -7\}$.

(d) Are both elements in the solution set of $(x - 3)^2 = 100$ also in the solution set of $x - 3 = 10$?

Answer: No. -7 is not in the solution set of $x - 3 = 10$.

(e) When both sides of an equation are multiplied by a variable or when both sides are squared, the original and resulting equations are not equivalent, that is, the solution set of the resulting equation contains an element (or elements) that is not in the solution set of the original equation. This element is often called an extraneous root of the original equation, but it is in no sense a root of the original equation; it is only an ostensible root, a number that seems to be a root because the equations are assumed to be equivalent when they are not. Solve the equation $\sqrt{x + 20} = x$ by squaring both sides. Determine whether or not the solution set of the resulting equation contains an extraneous root of the original equation.

Answer: $x + 20 = x^2$

$$x^2 - x - 20 = 0$$

Apparent solution set is $\{5, -4\}$.

The root $x = -4$ is an extraneous root because it does not check in the original equation.

The solution set is $\{5\}$.

(10) Answer the following.

(a) Is the statement "If $x = y$, then $x^2 = y^2$" always a true statement?

Answer: Yes. If $x = y$, then x may be substituted for y in the equation $x^2 = y^2$ and the equation $x^2 = x^2$ results. This resulting equation is an identity so the equation $x^2 = y^2$ must always be true when $x = y$.

(b) Is the statement "If $x = -y$, then $x^2 = y^2$" always a true statement?

Answer: Yes. If $x = -y$, then $-y$ may be substituted for x in the equation $x^2 = y^2$ and the equation $(-y)^2 = y^2$ results. This last equation is an identity. Therefore, $x^2 = y^2$ is an identity when $x = -y$.

(c) Determine whether or not the statement "If $x^2 = y^2$ then $x = y$" is always true by substituting -3 for x and 3 for y.

Answer: If $x = -3$, and $y = 3$, $(-3)^2 = 3^2$ is true; but $-3 = 3$ is false. Therefore, the statement is not always true.

(d) Determine whether or not the statement "If $x^2 = y^2$, then $x = -y$" is always true by substituting 3 for x and 3 for y.

Answer: If $x = 3$ and $y = 3$, $3^2 = 3^2$ is true; but $3 = -3$ is false. Therefore, the statement is not always true.

(11) Indicate whether each of the following statements is always true or is not always true.

(a) If $x - 3 = 10$, then $(x - 3)^2 = 100$.

(b) If $(x - 3)^2 = 100$, then $x - 3 = 10$.

(c) If $x - 3 = -10$, then $(x - 3)^2 = (-10)^2$.

(d) If $(x - 3)^2 = (-10)^2$, then $x - 3 = -10$.

(e) If $x - 3 = 10$, then $x^2 - 3x = 10x$.

(f) If $x^2 - 3x = 10x$, then $x - 3 = 10$.

Answers:

(a) Always true
(b) Not always true (False when $x = -7$)
(c) Always true
(d) Not always true (False when $x = 13$)
(e) Always true
(f) Not always true (False when $x = 0$)

(12) Indicate for each of the following whether the statement is true or false.

(a) If both sides of an equation are squared, all numbers which check in the original equation check in the resulting equation.

(b) If both sides of an equation are multiplied by the same variable, all numbers which check in the original equation check in the resulting equation.

(c) If the square root of both sides of an equation is taken, all numbers which check in the original equation check in the resulting equation.

(d) If both sides of an equation are divided by a variable, all numbers which check in the original equation check in the resulting equation, in general.

Answers:
(a) True (b) True (c) False (d) False

Note: The phrase, in general, is needed in part d because there are some special cases involving multiple zero roots; thus, all the numbers which check in $x^3 = x^2$ also check in $x^2 = x$. The solution set of each equation is $\{0, 1\}$.

9.5 VERBAL PROBLEMS INVOLVING QUADRATIC EQUATIONS

For each of the following problems, represent the unknowns as algebraic expressions, write an equation that may be used to determine the unknowns, solve the equation, and determine which root or roots meet the requirements of the problems.

(1) The product of one more than a certain number and three less than this number is 12. Determine the given number.

Answer: x = the number
$(x + 1)(2x - 3) = 12$
$x = 3$, $x = -\frac{5}{2}$. The solution set is
$\{3, -\frac{5}{2}\}$ since both roots meet the requirements of the problem.
The number is either 3 or $-\frac{5}{2}$.

(2) The product of three greater than a given integer and seven less than twice the integer is equal to 15. Determine the given integer.

Answer: x = the given integer
$(x + 3)(2x - 7) = 15$
$x = \frac{9}{2}$, $x = -4$. The solution set is $\{-4\}$
since $\frac{9}{2}$ does not meet the requirements of the problem as it is not an integer.
The integer is -4.

(3) The length of a rectangle exceeds twice the width by 5 inches. The area of the rectangle is 52 square inches. Find the length and the width.

Answer: w = the width

$2w + 5$ = the length

$w(2w + 5) = 52$

$w = 4, \quad w = -\frac{13}{2}$. The solution set is $\{4\}$ since $-\frac{13}{2}$ does not meet the requirements of the problem as the measure of the width of a rectangle cannot be a negative number.

The length is 13 inches and the width is 4 inches.

(4) If each of three consecutive integers are squared and the results added, the sum is 50. Determine the three integers.

Answer: x = first integer

x + 1 = second integer

x + 2 = third integer

$x^2 + (x + 1)^2 + (x + 2)^2 = 50$

$x = 3 \qquad x = -5$. The solution set is $\{3, -5\}$ since both roots of the equation satisfy the requirements of the problem.

The three consecutive integers are either 3, 4, 5 or -5, -4, -3.

(5) Solve the following problems.

(a) The sum of the length and width of a rectangle is what fractional part of the perimeter?

Answer: one-half

(b) If the perimeter of a rectangle is 40 feet, what is the sum of the length and width?

Answer: 20 feet

(c) If w represents the width of the rectangle whose perimeter is 40, write an expression for the length in terms of w.

Answer: 20 - w

(d) The perimeter of a rectangle is 40 feet and its area is 96 square feet. Find its length and width if the length is greater than the width.

Answer: w = width

20 - w = length

$w(20 - w) = 90$

$w = 8$ $w = 12$. The solution set is $\{8\}$ since if $w = 12$, then the length of the rectangle is 8 inches which does not meet the requirements of the problem that the length be greater than the width.

The length is 12 feet and the width is 8 feet.

(6) Find a number such that the sum of the number and 6 times its reciprocal is 7.

Answer: x = the number

$x + \frac{6}{x} = 7$

$x = 1$, $x = 6$. The solution set is $\{1, 6\}$ since both roots of the equation satisfy the requirements of the problem.

The number is either 1 or 6.

(7) The difference between two numbers is 5 and their product is 36. Find the numbers.

Answer: x = the first number

$x + 5$ = the greater number

$x(x + 5) = 36$

$x = -9$ $x = 4$. The solution set is $\{-9, 4\}$ since both roots of the equation satisfy the requirements of the problem.

The two numbers are either -9 and -4 or 4 and 9.

(8) The hypotenuse of a right triangle is 4 inches longer than one leg and 8 inches longer than the other leg. How long is each side of the triangle?

Answer: x = length of the hypotenuse

$x - 4$ = length of one leg

$x - 8$ = length of the other leg

$(x - 4)^2 + (x - 8)^2 = x^2$

$x = 20$ $x = 4$. The solution set is $\{20\}$ since the root $x = 4$ of the equation does not meet the requirements of the problem.

The hypotenuse is 20 inches and the legs are 16 inches and 12 inches.

(9) The altitude of a triangle is 8 inches longer than the base to which it is drawn. The area of the triangle is 24 square inches. Determine the length of the altitude.

Answer: x = length of the base
x + 8 = length of the altitude
$\frac{1}{2}x(x + 8) = 24$
$x = 4$ $x = -12$. The solution set is $\{4\}$ since the root $x = -12$ does not meet the requirements of the problem as the measure of a base of a triangle cannot be a negative number.
The base is 4 inches and the altitude is 8 inches.

(10) <u>Find two consecutive integers such that the square of the first integer is equal to the square of the second integer</u>.

Answer: x = first integer
x + 1 = second integer
$x^2 = (x + 1)^2$
$x = -\frac{1}{2}$. The solution set is ∅ since there is only one root to this equation and this root does not meet the requirements of the problem as it is not an integer. There is no integer that can satisfy the requirements of the problem.

(11) <u>Find three consecutive even integers such that the square of the middle integer is 4 less than half the sum of the squares of the other two integers</u>.

Answer: x = first integer
x + 2 = second integer
x + 4 = third integer
$(x + 2)^2 + 4 = \frac{x^2 + (x + 4)^2}{2}$
$x^2 + 4x + 8 = x^2 + 4x + 8$
The equation is an identity so therefore the solution set of the equation consists of the set of all real numbers. Any three consecutive even integers will meet the requirements of the problem.