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Maryland Elementary Mathematics

Inservice Program

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Final Report of Study-Demonstration Phase

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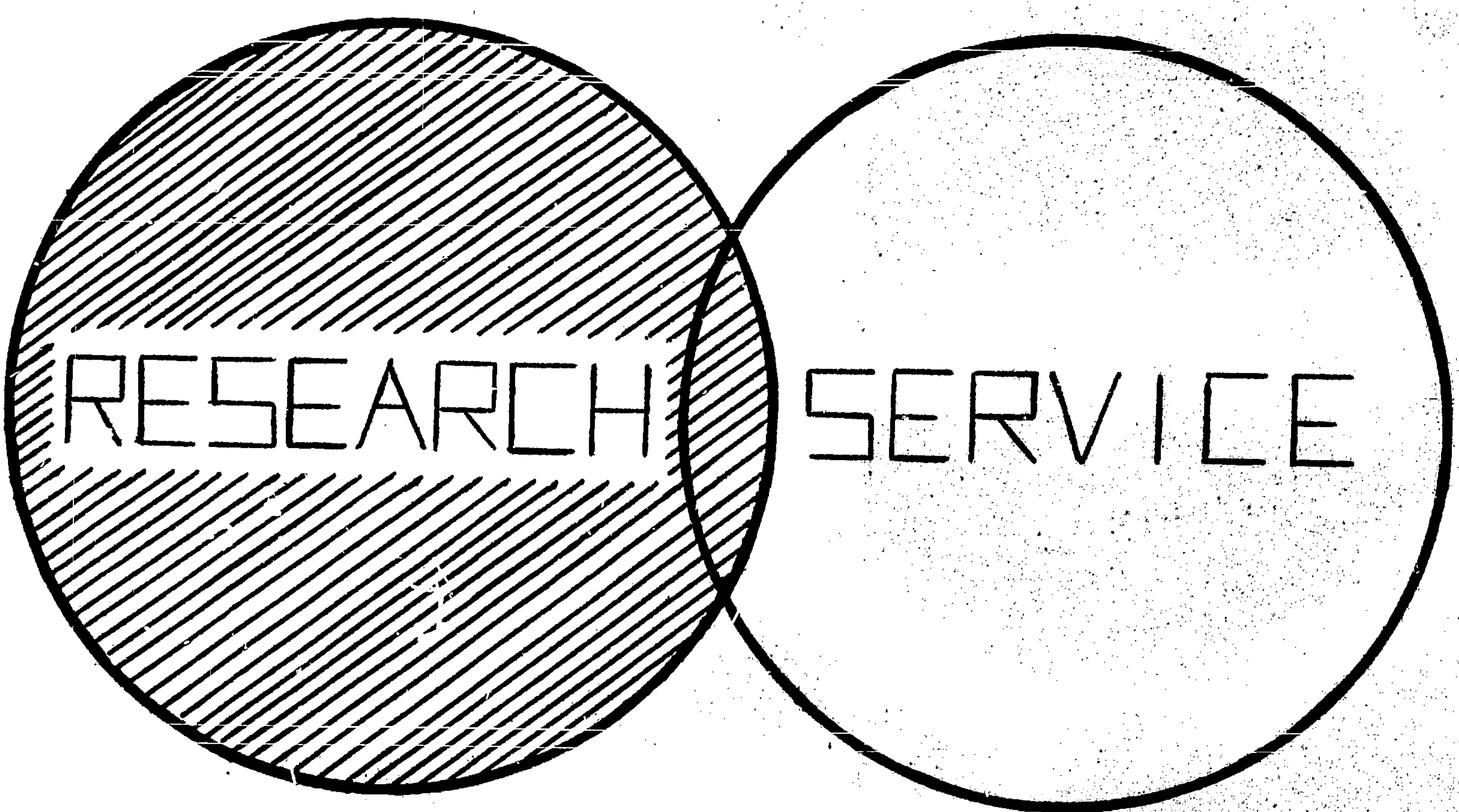


BUREAU OF EDUCATIONAL RESEARCH
AND FIELD SERVICES
College of Education
University of Maryland

MARYLAND ELEMENTARY MATHEMATICS INSERVICE PROGRAM REPORT
Study-Demonstration Phase

Summer, 1966. The Maryland Elementary Mathematics Inservice Program (MEMIP) initiated the developmental phase of its work during the summer of 1966. The work was conducted at the University of Maryland by a summer staff consisting of the Director, two Associate Directors, one Research Associate, two Graduate Assistants, and two Research Assistants. The project team included two college professors in mathematics education, the state supervisor of mathematics in Maryland, two graduate students in mathematics education, and two elementary school teachers from the Frederick, Maryland, public school system. A list of the staff members is contained in Appendix A.

The unique commitment of MEMIP is to a conceptual structure whose axioms of organization are psychological -- the organizing unit for a set of instructional materials can be reliably observable human behavior. Efforts directed at the development and implementation of instructional materials in mathematics up to the time of the MEMIP effort have not specified the desired behavioral outcomes of instruction. Make no mistake, objectives are often stated, but these descriptions are not behavioral descriptions. Furthermore, content has been the building block for curriculum. One need only ask how the decisions are made to include or exclude topics to make the case for the dominant role of content. Content selected for "historical" reasons, or "logical" reasons, or "practical" reasons, or "updating" reasons have dominated the organization of instructional materials in mathematics.

The behavioral orientation of MEMIP focuses the instructional programs unequivocally upon learner outcomes. This is not to argue that a content organization will not also accomplish this focusing upon the learner, it is merely an acknowledgment that the psychological structuring makes such an orientation inevitable. Once the behavioral expectations are specified, success or failure rests entirely upon whether the learners are able to exhibit the desired behavior after instruction. Another benefit which is realized from the behavioral description of instructional outcomes is the potential for organizing the desired behaviors in sequences designed for the high probability acquisition by the learners.

It is one thing to postulate that behavioral description for inservice mathematics instruction can be constructed and quite another to actually demonstrate such a product. Of the summer staff which began working in 1966, only two members had ever actually constructed a behavioral objective. The staff was for the most part naive as to the mechanics of behavioral description, its benefits, and its proposed relationship to the development of the instructional materials -- at the same time their mathematical sophistication was more than adequate to the proposed level of mathematical development.

The first task of the project was to help each staff member acquire the behaviors related to being able to make a behavioral description and being able to construct a behavioral hierarchy (sequence of dependent behaviors intended to optimize acquisition). The program for acquisition of these behaviors was initiated by viewing and responding to a self-instructional program on behavioral objectives. The program was presented in the form of a synchronized filmstrip and tape recording. A concurrent reading assignment included Mager's[1] programmed text on preparing behavioral objectives and Gagne's[2] volume on learning. As a result of this audio-visual program and the supplementary readings, the staff members were now able to distinguish between the description of a behavioral objective and a non-behavioral objective. This first and simplest of the behaviors prerequisite to beginning the development phase was acquired by each staff member during one working session. So much for the simplest task.

The potential use of behavioral objectives was now raised with the staff. The expected, and obtained responses, were related to minimizing the likelihood of misinterpretation in the intent of an objective and providing an observer with a means of being able to identify the learners who have successfully acquired whatever the instruction intended the learners to acquire.

The behavior of being able to construct behavioral objectives -- the basic performance which each staff member needed to perform if the project was to succeed -- was the next behavior to be acquired by each of the MEMIP staff. The first recognition elicited in the acquisition of the construction behavior was to recall that performance in English is described by means of a particular part of speech -- verbs, of course. Hence, if behavioral objectives are to be descriptions of reliably observable human performance, then every objective must contain a verb. In fact, since it is performance or action which is being described, the verb must be an active verb. The identification of active (or action) verbs as the principal source of performance description followed quickly on the heels of the verb acknowledgment. At this point, the staff had begun to actively play the "behavioral game."

But how many action verbs are there in English? Certainly there are a finite number, since the set of action verbs is a proper subset of the set of English words which is itself a finite set. Although the number of action verbs is finite, it is still a rather large number. If the purpose of behaviorally stated objectives is to reduce the amount of misinterpretation by describing specific behavior, then it would be useful to reduce the number of action verbs to a minimum collection and provide an operational definition of each verb. An added constraint on this reduction is that the consolidation must occur without reducing the variety of learner behavior it is possible to include as instructional purpose. This task was accomplished by confronting the staff with a series of structured instructional settings. Each of the instructional segments included: (1) providing a set of materials for each staff member, (2) requesting each individual to exhibit some specific performance, (3) requesting one or more additional performances using the same materials but different verbs, (4) asking the staff to identify and name the verbs which initiated each of the performances they made, (5) obtaining the acknowledgment that the performances exhibited were similar, (6) listing the action verbs which could serve as behavioral synonyms for one another, (7) operationally defining a behavioral class by selecting

[1]Robert F. Mager, Preparing Instructional Objectives, Palo Alto: Fearon Publishers, 1962.

[2]Robert M. Gagné, The Conditions of Learning, New York: Holt, Rinehart and Winston, Inc., 1965.

one verb as the class name for all of the behavioral synonyms listed, and (8) repeating the sequence of (1) through (7) with a new collection of action verb synonyms.

The action verb classes of behavior developed with the staff were adapted from the collection described by Walbesser.[3] An approximation of the procedures used to develop the behavioral action verbs is described as Sessions I and II of the Instructional Program for Teachers found in Appendix B. The operational definitions currently being used by MEMIP are described on pages six through eight of Session II. All objectives of this project now contain one of the action verbs from this list.

The construction of behavioral hierarchies remained the only behavior yet to be acquired by the working staff before the development stage could begin. One small hierarchy was constructed by each staff member who was first given six stated behavioral objectives. The six objectives were related to the description and identification of two dimensional projections of three dimensional objects. The intent of this activity was to illustrate the distinction between a psychological ordering and a logical ordering of objectives.

Now the MEMIP staff was ready to begin the identification and description of terminal tasks for the inservice instructional program. The question which confronted the MEMIP staff at this point was what terminal tasks would have the highest yield for inservice elementary teacher instruction. The first few discussions attempted to describe terminal tasks which would encompass much, if not all, of the mathematical competencies an elementary teacher should possess.

The staff activity at this time was devoted to small group (two or three individuals) or individual effort directed at the construction of a behavioral hierarchy, presentation to the entire group at one of the meetings, critical analysis of each hypothesized dependence, and termination in the rejection of the proposed hierarchy.

This strategy of exploration led to a number of specific excursions. These excursions are best characterized by saying they were attempts to construct behavioral hierarchies related to the performance of specific arithmetic operations. Although these efforts did not yield usable hierarchies, they did provide the staff with valuable experience in constructing behavioral descriptions and sequences of behavioral dependence. As is so often true in an experiment, these first attempts might be termed failures, since they did not yield behavioral hierarchies used in the program. These "failures", however, did pay handsome dividends. As a consequence of these initial probings, the first terminal tasks and the allied collection of subordinate behaviors were identified, constructed, and ordered.

The first terminal task accepted by the staff related to the presentation and explanation of algorithms. This is a reasonable choice when one considers the instructional time devoted to algorithms in the elementary grades.

[3] "Science Curriculum Evaluation: Observations on a Position," The Science Teacher, 33:34-39, February, 1966.

A first approximation of the algorithms hierarchy was presented to the staff late in the summer of 1966. The analysis, discussion, and challenges that followed its first presentation put the hierarchy through a thorough examination which led to numerous revisions. The current working edition of the algorithms process hierarchy is presented as Appendix C.

The terminal task of the algorithms hierarchy is actually a triple of behaviors that the teacher will be able to exhibit after being exposed to the algorithms instructional sequence. The three behaviors which constitute this terminal task represent the desired instructional output of the subordinate behavioral sequence. This triple includes (1) demonstrating the procedures of an algorithm as they would be carried out by a machine, (2) constructing a convincing explanation for each procedure of an algorithm which appeals to observations based upon physical situations, and (3) constructing an explanation for each procedure of an algorithm that appeals to agreed upon rules of the "convincing game."

The first of this triple of terminal task behaviors describes a familiar activity of elementary teachers -- the literal demonstration of the procedures of an algorithm with no explanation of how or why it works. Unfortunately some instruction in algorithms at the elementary school level never proceeds beyond this mechanical level. The second behavior describes the activity of explaining how an algorithm works by relating the explanation of each procedure to observations of physical situations. This is another familiar activity of the elementary teacher when teaching an algorithm. The third behavior, explaining the procedures of an algorithm by means of rules of the convincing game, represents those behaviors more characteristic of a contemporary mathematics curriculum with its appeal to the field properties and mathematical structure. This third behavior is the one which the elementary teacher has most likely not acquired and yet, in many ways, it is the most critical to successful instruction in elementary mathematics today.

The subordinate behaviors in the algorithms hierarchy reflect this same triple of constructing and demonstrating behaviors, but are associated with a particular operation within a specified number system. The final task differs from the subordinate ones in that any algorithm could be presented to the teacher and he would be expected to be able to exhibit these specified behaviors without instruction.

Subordinate to the algorithms hierarchy behaviors are the convincing game rule behaviors. The behaviors associated with the identification and naming of the field properties are developed in the context of game rules for two reasons. First, games provide a vehicle for identifying the properties in a setting which promotes individual investigation and immediate application of the identified rules. Second, the departure from a formal mathematical presentation to a game presentation reduces the "mathematical anxiety" which often accompanies mathematical instruction for the elementary teacher. The development of the game rules are found as Sessions IV and V in Appendix B.

The initial development plan which was proposed and adopted by MEMIP was first to identify the terminal tasks for the instructional materials. The identification would then be followed by the construction of a behavioral hierarchy for each terminal task. Once a hierarchy was constructed the instructional sequence would be determined by beginning with the least complex behaviors in the hierarchy, designing instructional materials to help the learner acquire the specified behaviors, and repeating the process up through the sequence until the terminal task is reached. The instructional materials were to be selected by adopting existing materials from commercially published volumes and available experimental volumes. A collection of possible sources was gathered as a library of resource volumes. A partial list of these volumes are listed as Appendix D.

The search of the resource volumes has not proven to be as useful as was anticipated. It soon became apparent in the search for appropriate algorithms material that little or no instructional material is provided in the available volumes about elementary school mathematics which would aid the elementary teacher in acquiring either of the constructing behaviors. What is more, little variety was found in the material devoted to the demonstrating algorithm behavior for any of the operations specified as the setting for the subordinate behaviors in the algorithm hierarchy. What could be done to add the needed richness in the algorithms to be used for instruction? This problem was resolved by expanding the search of the literature to include historical sources (principally mathematical texts which are not contemporary or algorithms which have historical curiosity) and the periodical literature.

Fall Semester, 1966-67. The field tryout phase of the pilot study was initiated during the fall semester of the 1966-67 school year. The first major decision to be made with respect to the field tryout phase concerned the scope of the investigation. Two strategies were considered. One called for the investigation of the validity of the entire algorithms hierarchy. The second strategy called for the investigation of select segments of the algorithms hierarchy. The purpose of the more limited investigation would be to evaluate the format of presentation and the design of the immediate assessment measures. The staff decided to adopt the second tactic since it possessed a variety of obvious advantages. Perhaps the most important one was that such an investigation would permit an adequate trial of the instructional and immediate assessment formats. Once this step was completed it would be possible to test the validity of the behavioral hierarchy without the confounding error of difficulties with the instructional format.

Segments of the algorithms process hierarchy were selected as the trial behaviors to be studied. The sections selected were the sessions on the game rules, the acquisition of the triple of behaviors related to the addition of whole numbers algorithms, and the acquisition of the triple of behaviors related to the addition of integers algorithms.

The instructional decision was made that for each algorithm triple (the two constructing behaviors and the demonstrating behavior) two different algorithms for the operation would be presented to the teachers and a third algorithm for the same operation would be used as an assessment measure.

The next decision to be made was concerned with the format of the instructional material. The instructional materials could be presented in a variety of forms, each with some advantages. After discussion and examination of various alternatives by the staff, it was decided that the instructional materials would be written in a narrative and conversational form rather than the more formal textbook dialog. A second characteristic agreed upon was that the reader would be required to respond at various intervals. The responses would not be as frequent as a self-instructional program, but would rather focus upon the key decisions to be made during instruction by the learner. The advantages of this form are two-fold: (1) the active rather than passive participation of the learner is promoted, and (2) the learner (the elementary teacher) will be able to review the area of instruction weeks or months later with the key decisions contained instructionally in the body of the text. These instructional materials could be used after a session had been taught to the elementary teacher, or the instructional materials could be used independently. Each of the sessions used in the pilot study are included in Appendix B as second experimental editions. Four additional sessions are included in Appendix B as first experimental editions.

The critical involvement of behavioral description led to the decision that the first three sessions for teachers be devoted to the description and construction of behavioral statements, the operational definitions of the action verbs used by the project, and the construction of behavioral hierarchies. The rules of the convincing game are developed in the next two sessions with the use of two games. The remainder of the sessions for the tryout consist of the algorithms for adding whole numbers and adding integers. For each of these operations on sets of numbers two different algorithms are demonstrated along with the constructing of convincing explanations based upon physical situations and the rules of the convincing game. A third algorithm is then used for assessment of each operation.

An assessment of behavioral acquisition is provided after each instructional segment. These measures of behavioral acquisition involve new materials so as to provide a change of stimulus. Direct recall of the materials in the instructional segment is not tested. The behavioral acquisition of the learner is assessed, since the concern is with behavioral acquisition.

The fall staff of MEMIP consisted of the summer staff and two new graduate assistants. The two elementary teachers from Frederick met once a week with other members of the staff before the tryout of materials. Two classes of elementary teachers from the Frederick schools were taught most of the pilot study materials by the two Frederick teachers who served as project staff members. Their performance was observed by another member of the project and a recording of various participation dimensions was tallied. A copy of the instructional observation data sheet is included as Appendix E. The data provided by these observations served as one source of objective feedback which has helped to guide the revision of the instructional materials.

Characteristics of the 28 Frederick teachers participating in the pilot study are summarized in the following data descriptions. The median number of semester hours of mathematics courses was 5 hours with a range from 0 hours to 27 hours. The mathematics methods hours revealed a median of 2 hours and a range from 0 hours to 6 hours. The median year in which the last mathematics course was taken was 1964. Finally, the number of years of teaching experience varied from 0 years to 30 years with a median of 8 years. The teacher data suggests that the pilot study sample of elementary teachers was reasonably representative with the variety of mathematics preparation and teaching experience one would encounter throughout the State of Maryland.

The instructional materials were used in two different ways. Session I was distributed to both groups of elementary teachers without instruction. Session II was distributed to one group without instruction while a member of the staff taught the material to the other group. The elementary teachers reacted much more favorably to the staff instruction.

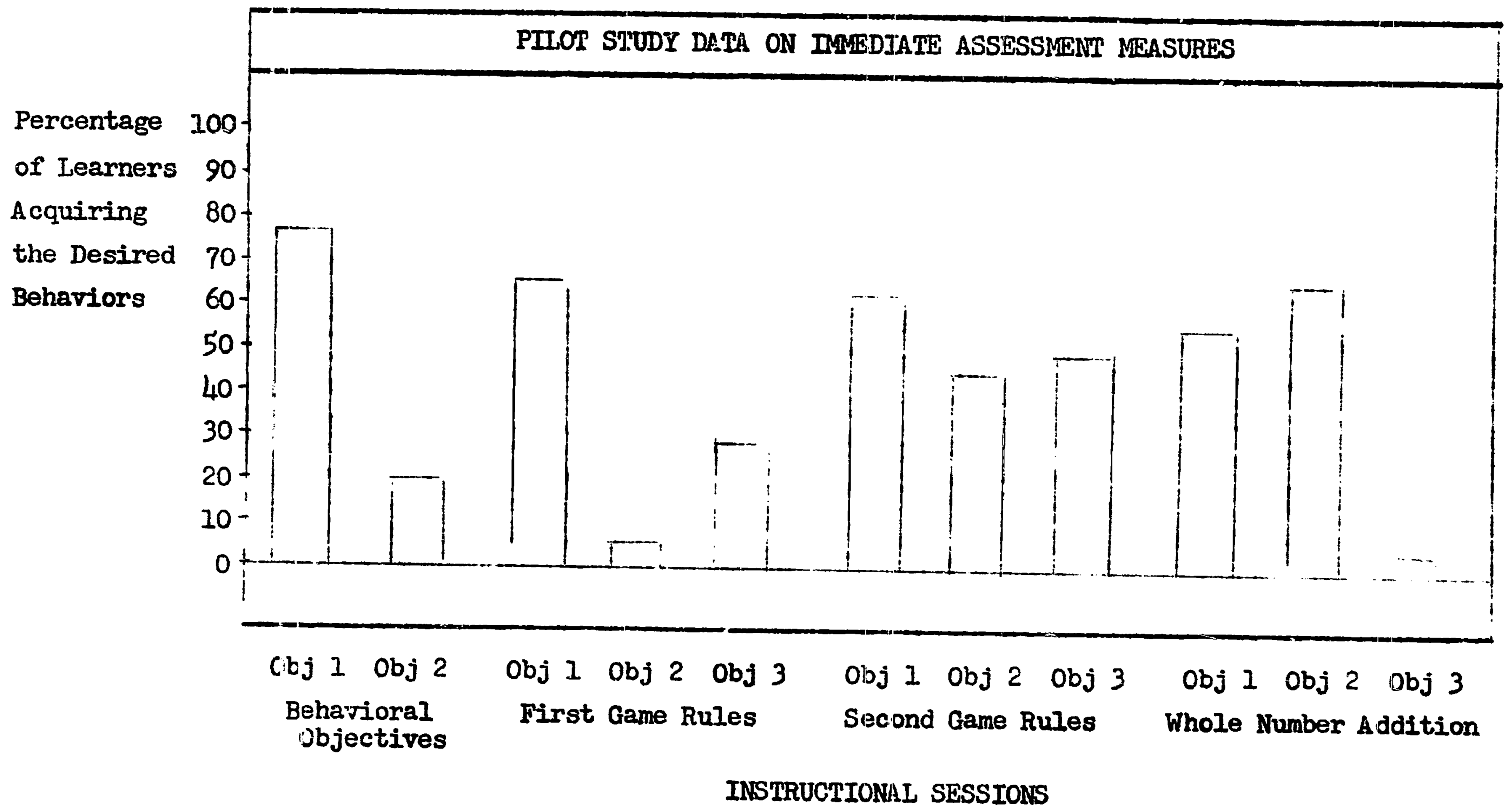
The format of the instructional materials was also very useful for the two elementary teachers who taught the remaining sessions in the pilot study. However, the two elementary teachers also felt that it was important that the sessions be taught to them by other members of staff before they taught the two groups of elementary teachers.

The sessions which were assessed included those which were developed for instruction in the following areas: behavioral objectives, the game rules, and the addition of whole numbers. The results of these assessments are presented in Table I. The evaluation data for the behavioral objectives sessions are interesting in that almost 80% of the teachers did acquire the behavior of being able to distinguish between behavioral and non-behavioral objectives, but the 20% level of acquisition suggests most teachers did not acquire the behavior of being able to construct a behavioral objective. Revisions have been made in the second experimental edition of these sessions so as to provide additional experiences in helping the teacher to acquire these competencies.

The three objectives associated with the game rules sessions are (1) being able to identify and name examples of each of the game rules given the game, (2) being able to demonstrate each of the game rules by moves from a given game, and (3) being able to construct data which support the presence or absence of a given game rule for a particular game. The results of the behavioral assessment for the first objective are reasonably encouraging for a first trial with a 65% level of acquisition. The results on the assessment for the second and third objectives are not as encouraging with approximately 5% and 30% acquisition levels observed for Objectives 2 and 3 respectively. These data led to the development of a second game rule assessment in order to determine whether the tasks were not clear on the first measure or the low level of acquisition was actually attributable to the failure of the teachers to acquire the desired behaviors. This second game rule assessment is included in Appendix B after Session V. The second set of game rule data reports approximately 40% and 50% levels of acquisition for Objectives 2 and 3. The second testing would appear to support both the hypotheses that the difficulty was in the failure of acquisition and in the lack of clarity of the first measure. On the basis of this information as well as the tryout feedback, it was decided to develop a new game which should enhance the acquisition of the desired behaviors. This game is included as Session IV in Appendix B.

TABLE I

PILOT STUDY DATA ON IMMEDIATE ASSESSMENT MEASURES

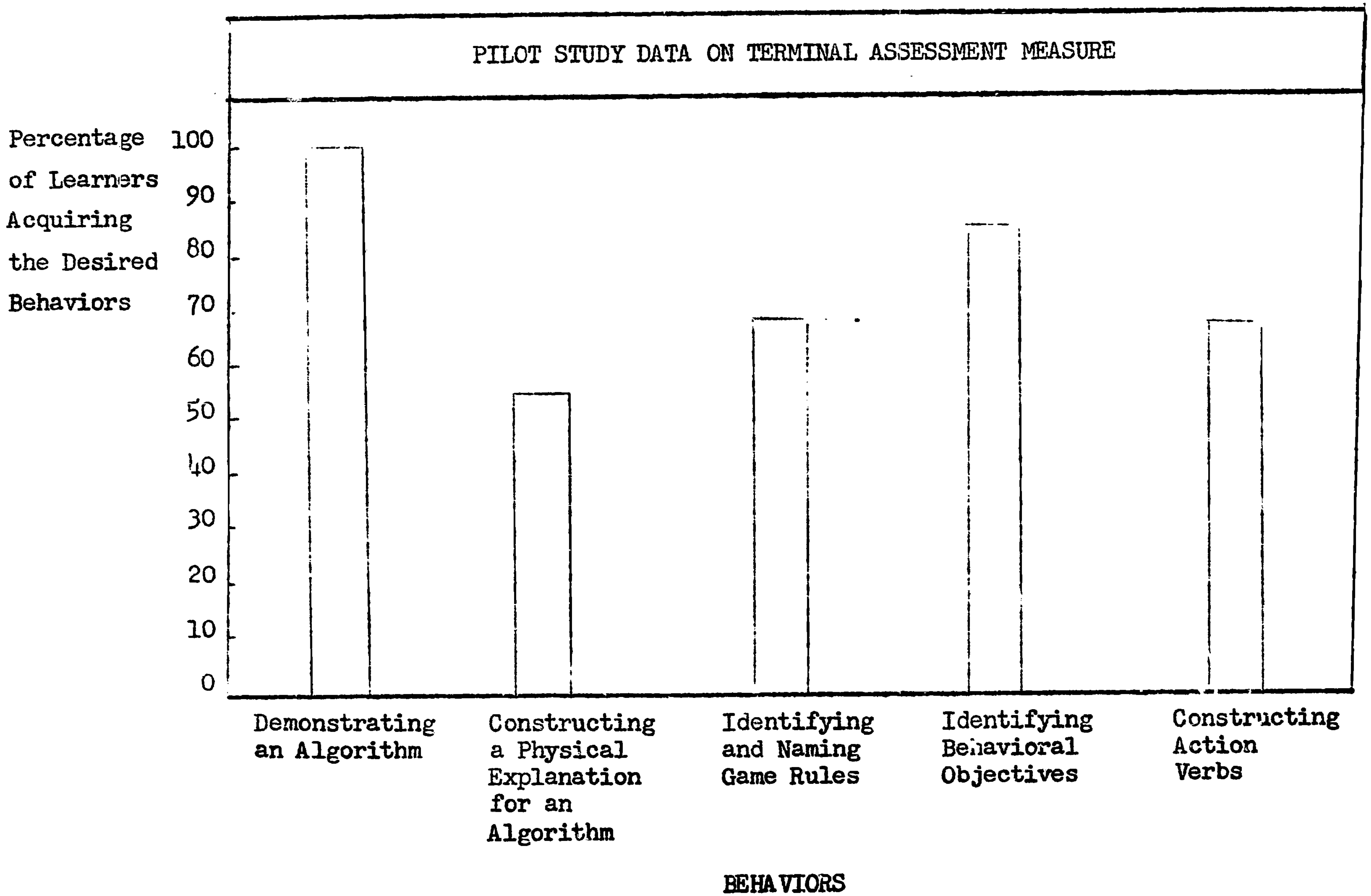


Objectives 1 and 2, the demonstrating behavior and the constructing explanations behavior with physical situations related to whole number addition, were attained by more than half of the teachers. This is not unacceptable as a level of acquisition for the first trial although revision is clearly needed. The constructing behavior related to the application of the game rules did not meet with equivalent success. The level of acquisition for Objective 3 was about 5% and such result can only be described as a disaster. As anyone who has worked with elementary teachers might have hypothesized, this is the most difficult of the behaviors for the elementary teacher to acquire and, therefore, such a result on a first trial is not unexpected. Copies of the immediate assessment instruments are included at the end of the appropriate sessions in Appendix B.

The second experimental edition makes use of these behavioral acquisition data to correct weaknesses identified in the desired behavioral acquisitions intended of this collection of instructional materials.

The progress of the pilot study teachers toward acquiring the desired behaviors of the tryout sessions was also assessed by a terminal measure administered at the final pilot study session. The presence or absence of the behaviors described as instructional objectives for the sample sessions was tested by means of tasks on this terminal measure. A copy of this instrument is included as the last item of Appendix B. The behavioral acquisition data for this measure is reported in Table II. The level of acquisition for each of the behaviors was greater than 50%. These data indicate that the teachers demonstrated substantial progress toward acquiring the desired behaviors.

TABLE II


PILOT STUDY DATA ON TERMINAL ASSESSMENT MEASURE
Percentage of Learners Acquiring the Desired Behaviors
100
90
80
70
60
50
40
30
20
10
0
Demonstrating an Algorithm
Constructing a Physical Explanation for an Algorithm
Identifying and Naming Game Rules
Identifying Behavioral Objectives
Constructing Action Verbs
BEHAVIORS

APPENDIX A

Maryland Elementary Mathematics Inservice Program (MEMIP) Staff

Director	- Dr. James Henkelman, Asst. Professor Mathematics and Education, University of Maryland
Associate Directors	- Mr. Thomas Rowan, State Supervisor of Mathematics, State of Maryland
	Dr. Henry Walbesser, Asst. Professor Mathematics and Education, University of Maryland
Research Associate	- Dr. Robert Ashlock, Asst. Professor Early Childhood-Elementary Education, University of Maryland
Research Assistants	- Mrs. Sandra Shockley, Teacher, Frederick County, Maryland
	Mrs. Carol Young, Teacher, Frederick County, Maryland
Research Assistants	- Mr. Thomas Bennett, Graduate Assistant Mathematics Education, University of Maryland
	Mrs. Sada Chernick, Graduate Assistant Mathematics Education, University of Maryland
	Miss Arline Engel, Graduate Assistant Mathematics Education, University of Maryland
	Miss Roberta Engel, Graduate Assistant Mathematics Education, University of Maryland
Consultants	- Dr. John R. Mayor, Professor Mathematics and Education, University of Maryland
Mathematics Education Seminar	- Mildred Cole, Marvin Cook, William Gray, Rufus Jones, Genevieve Knight, Ilene Lasher, Ronald McKeen, William Moody, Neil Seidl

Second Experimental Edition

APPENDIX B
SESSION I

ORIGINATING THE PROBLEM

Note: On this page you may respond by writing on the blanks provided.

Do you recall the word association game? You know, the game is played by someone saying one word, and then you respond by saying the first word which occurs to you.

For example, someone says table, and you would say______________________.

Write something! You must participate to derive maximum benefit from this activity. Now read each of the following words and write down the first word which occurs to you:

SUN:______________________

KNIFE:______________________

RED:______________________

FREEDOM:______________________

OBJECTIVES:______________________

Did you say "useless" or "ambiguous" or "unimportant" in response to objectives? These are common responses to the word objective.

ACTIVITY ONE

Just what purpose do statements of objectives serve? Do you use the statements of objectives found in text books or courses of study to plan your instructional program?

Yes or no?______________

Be honest now, no one is going to collect your responses. Suppose you plan an instructional session from a teacher's commentary which contains the usual statements of objectives. Now suppose all the statements of objectives in your book were eliminated, for example, by covering them with tape. How could your instructional planning be observably affected?

Would your planning be different? Yes or no?__________ From your responses to the last two questions, it would appear that the description of curriculum objectives does not usually serve an instructional purpose.

There are few teachers from the inexperienced to the experienced who would say that statements of curriculum objectives (as they are usually constructed) actually contribute to their planning for, or execution of, instruction. Why is this? Is it simply that objectives can serve no useful instructional purpose? Must curriculum objectives remain an instructional window dressing or can they be translated into a functional purpose?

You are about to participate in an instructional program which identifies certain of the critical decisions needed in order to accomplish the transformation from vague, ambiguous descriptions of objectives to instructionally functional descriptions of objectives.

For the remainder of this session you will be asked to respond at various intervals by writing on the response sheet which you have been provided. If you are to acquire the competency expected from this portion of the exercise, you must respond when asked to do so in the program. Plan on responding quickly; for most of the tasks, plan on making a response in about fifteen seconds. It is important that you be an active participant. Be certain that you have your response sheet and a pencil.

The objectives of instructional materials should be stated in a clear, unambiguous manner. Certainly there are few who would refuse to acknowledge this as an important characteristic, applicable to all statements of curriculum objectives. Do the statements of objectives for mathematics programs satisfy this requirement of specificity and clarity? What if the statements of mathematics objectives don't meet these criteria--what is lost? For one thing, the intended meaning of an objective may be jeopardized by a variety of interpretations.

Consider the following illustration with the statement of a familiar objective--one common to many experimental and commercial modern mathematics curricula.

2

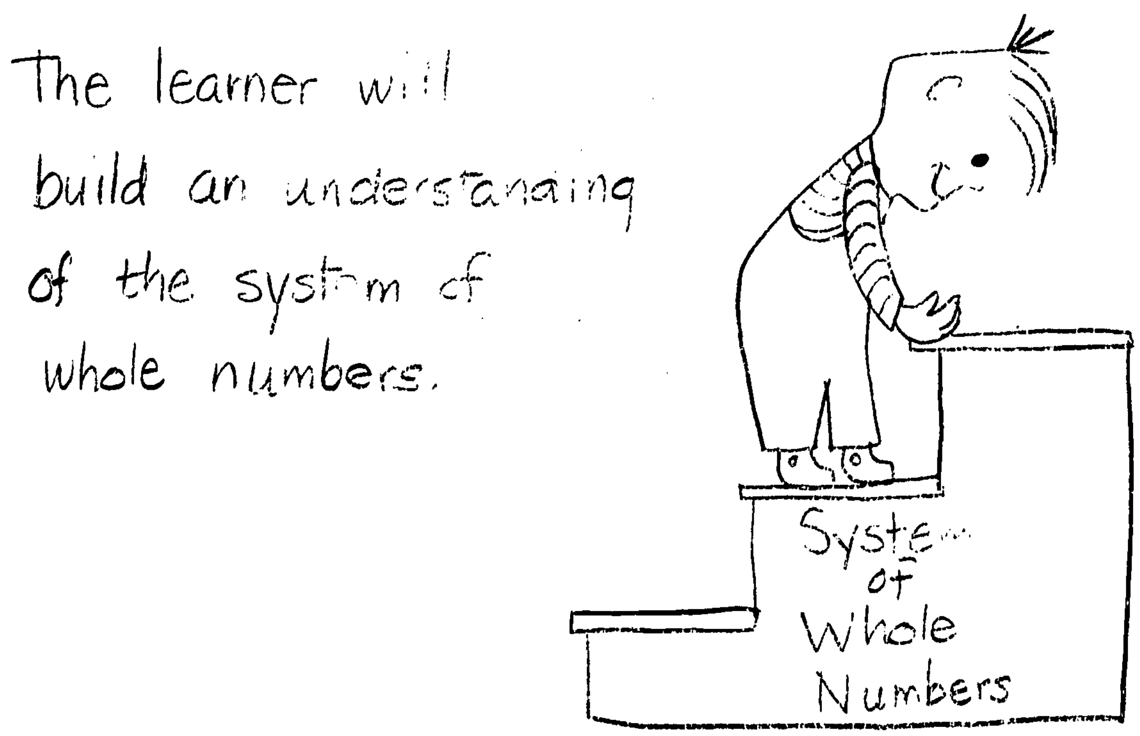


Illustration I

What characteristics would you ascribe to an instructional program in mathematics if it is attempting to achieve this objective? Is the statement of the objective phrased in such a manner that several other mathematics teachers, working independently, would arrive at the same interpretation of the meaning of this objective? Yes or no? (1)____________________

Have you made your selection? Go ahead write down a choice! Good! You will find the correct selection on page 4, line 4, word 4.

Illustration II identifies a stated objective of numerous modern mathematics programs which you are almost certain to recognize.

The learner will acquire an appreciation of the STRUCTURE of mathematics.

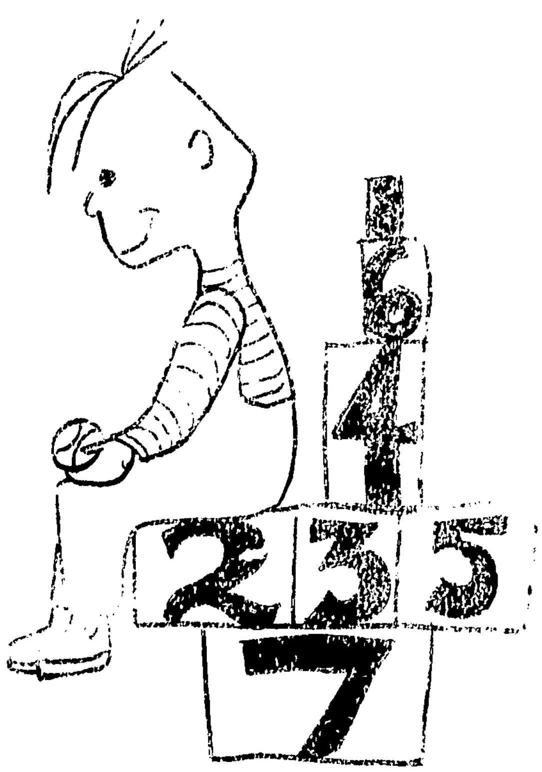


Illustration II

What activities would be necessary to achieve this objective in a mathematics curriculum? Do you suppose other mathematics educators would identify the same components as necessary to achieving this objective?

Respond yes or no? (2)________________

Have you made a written response? Fine! To see if your response is acceptable turn to page 5, line 4, word 3. Are the variety of possible interpretations for these two objectives surprising? Not at all, when one considers their lack of specificity. In fact, that which is truly remarkable is the skill which curriculum innovators and textbook authors have demonstrated in constructing ambiguous statements of objectives! Perhaps the most startling observation, however, is not the wide-spread use of these statements, but rather that most teachers so complacently accept these statements! As teachers, we acknowledge, or at least tacitly accept, these statements as reasonable descriptions of our goals and use them as the basis for justifying the selection of certain instructional materials or the performance of particular instructional acts. Each of these decisions is made or actions initiated even though there is this diverse "agreement" as to the meaning of these objectives. Consider the following statements of an often cited objective of modern mathematics curriculum.

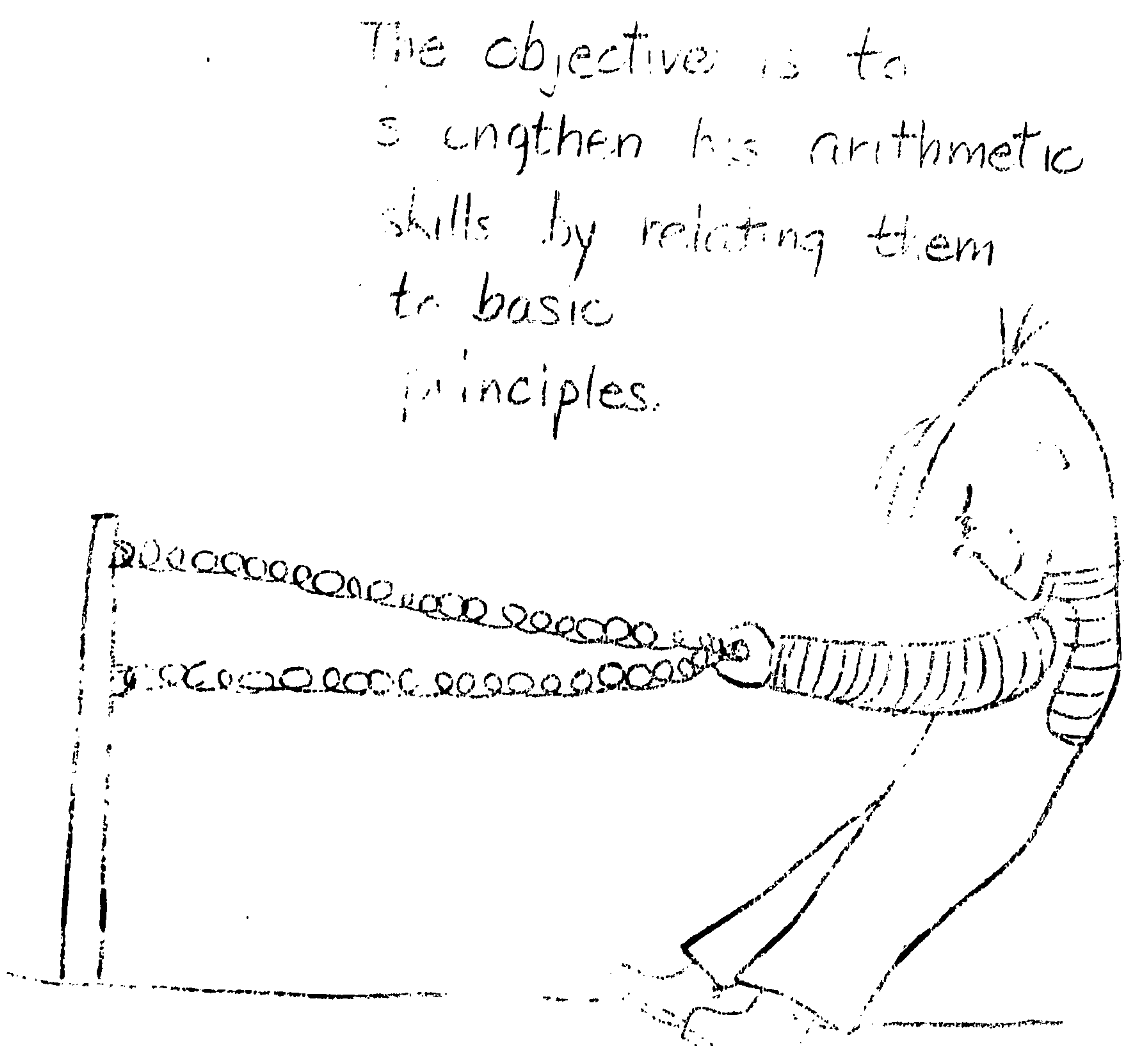


Illustration III

What specific instructional activities in mathematics would you design to achieve this objective? Do you suppose other mathematics instructors would reach a similar decision as to the meaning of this objective?

Yes or no? (3)________________________

Come on now, make a choice. All right. When you have made your choice and you have written it down, look on page 6, line 8, word 3 to find the acceptable response.

Certainly this third objective is unlike the first two in that it names a particular field of study in mathematics, namely arithmetic skills. However, narrowing the content from all of mathematics to arithmetic skills is obviously not an adequate solution to the interpretation dilemma. This is so because of the large number of varied interpretations which still can be made for the meaning of the objective. Such specification is useful but is not sufficient.

The three previous illustrations suggest that the description of an objective needs to be specific if there is to be any hope of attaining uniform interpretation. The need for each mathematics objective to be uniformly interpretable is especially important for those charged with the construction and/or implementation of an instructional program's objectives.

However, implementors of mathematics curricula are not alone in the acceptance of ambiguous objectives. Consider this statement of a favorite objective of contemporary mathematics curriculum developers.

Illustration IV

Now suppose you are one of four committee members charged with the task of independently observing students who have been exposed to instructional materials designed to aid the learners in acquiring this objective. Further, let's suppose that based upon these observations, you are to make a decision as to whether each student you observed had or had not been successful. Does the description of the objective in Illustration IV identify the specific performances which you would look for in your observations?

Yes or no? (4)____________________

Just what performances one would be expected to observe in learners who had acquired a familarity with these properties is certainly not contained in the statement of the previous objective. Therefore, the appropriate response to the question concerning what performances you are directed to observe is an emphatic NO.

The description of an objective must identify the observable behavior which a learner, who has successfully achieved the objective, is expected to have acquired. Read the objective stated in Illustration V with the purpose of identifying the observable behaviors a student should be able to exhibit if he has achieved the competency described by the objective.

The purpose is to help the learner gain an appreciation of the structure of positive fractions.

Illustration V

Are there observable behaviors identified in the statement of this objective? Are there behaviors described in a way that would enable you to separate the successful from the unsuccessful ones?

Yes or no? (5)________________________

Since you decided yes, the description does identify the desired observable behaviors, and you can, of course, name them. Oh?! You say you decided no. Good for you! The statement does not contain any such performance specification and therefore, the appropriate response is no.

The statement of an objective should describe desired learner behaviors. In order to be able to interpret an objective, these behaviors should be clearly described. The intent of an objective is reliably communicated by descriptions of observable behavior. Consider this next objective in the context of how effectively the statement communicates the desired behavior of the objective.

The purpose of this material is to describe long division.

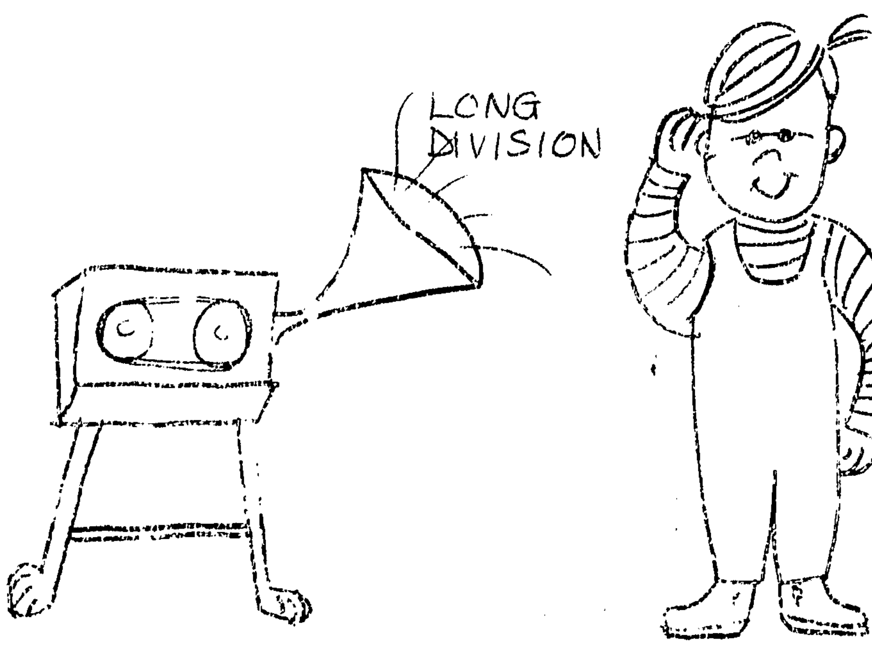


Illustration VI

Does this objective describe the behavior to be acquired by the learner?

Yes or no? (6)________________________

Should you need confirmation of the correctness of your response to this question for such an objective at this point, your best course of action would be to omit the remainder of the material in Session I.

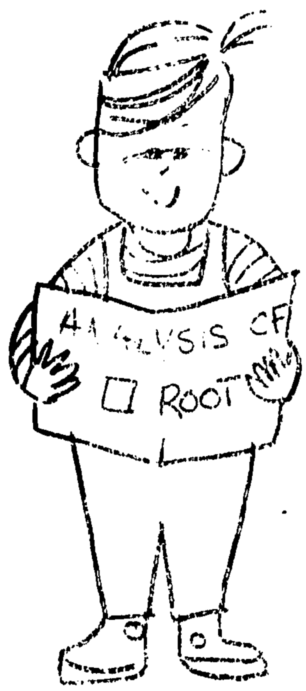
Does the statement in the previous illustration describe an environment which requires the presence of a learner?

Yes or no? (7)________________________

Have you made a written response? Don't read on until you have.

As you likely concluded already, this objective's description does not identify the behaviors the learner is to acquire. Nor, peculiarly enough, is the learner even necessary, since one might describe without any learner being present.

Read the description of the objective contained in Illustration VII and decide whether the learner is necessary to the achievement of this objective.



Present a detailed analysis of finding the square root of a number.

Illustration VII

What did you decide about the necessity of a learner in the achievement of this objective? Is he necessary or is he not necessary?

(8)____

Cleraly, a learner is necessary to the acquisition of the behaviors described in Illustration VII. You don't agree? Good for you. Obviously a detailed analysis might be given even though no learner is present.

Objectives must be constructed so as to be specific descriptions of what a learner is to do or say. Only fulfilling this descriptive requirement of learner performance can objectives become functional for the innovator, planner, developer, teacher, and learner. Objectives must be constructed so as not to allow for the exclusión of a learner under any interpretation.

Ambiguity is often cloaked in the garment of prestigous words. The next illustration contains statements of objectives for modern mathematics curricula which reflect examples of the "in" words of this decade. The fund of ambisuous words which frequent the pages of newer and older organizations of instructional materials for mathematics are legion. A few of the most common of these phrases are identified in Illustration VIII together with a ringer--one phrase which does not belong because it conveys specifically a desired behavior.

builds an understanding
appreciating
developing a feeling for
pointing to
having an awareness of
conveys the concept

Illustration VIII

Did you identify the phrase which does describe an observable performance? Which one was it? (9)____

Select one. Don't hesitate, write it down. Now!

Did you select "builds an understanding"? No. Good for you. Perhaps you picked out the "appreciating" phrase, or the "feeling" phrase, or the "awareness" phrase, or the "conveys" phrase. No. Good! "Pointing to" is the appropriate choice and should have been identified without difficulty.

Suppose a variety of three dimensional objects such as those in the next illustration were placed in front of you.

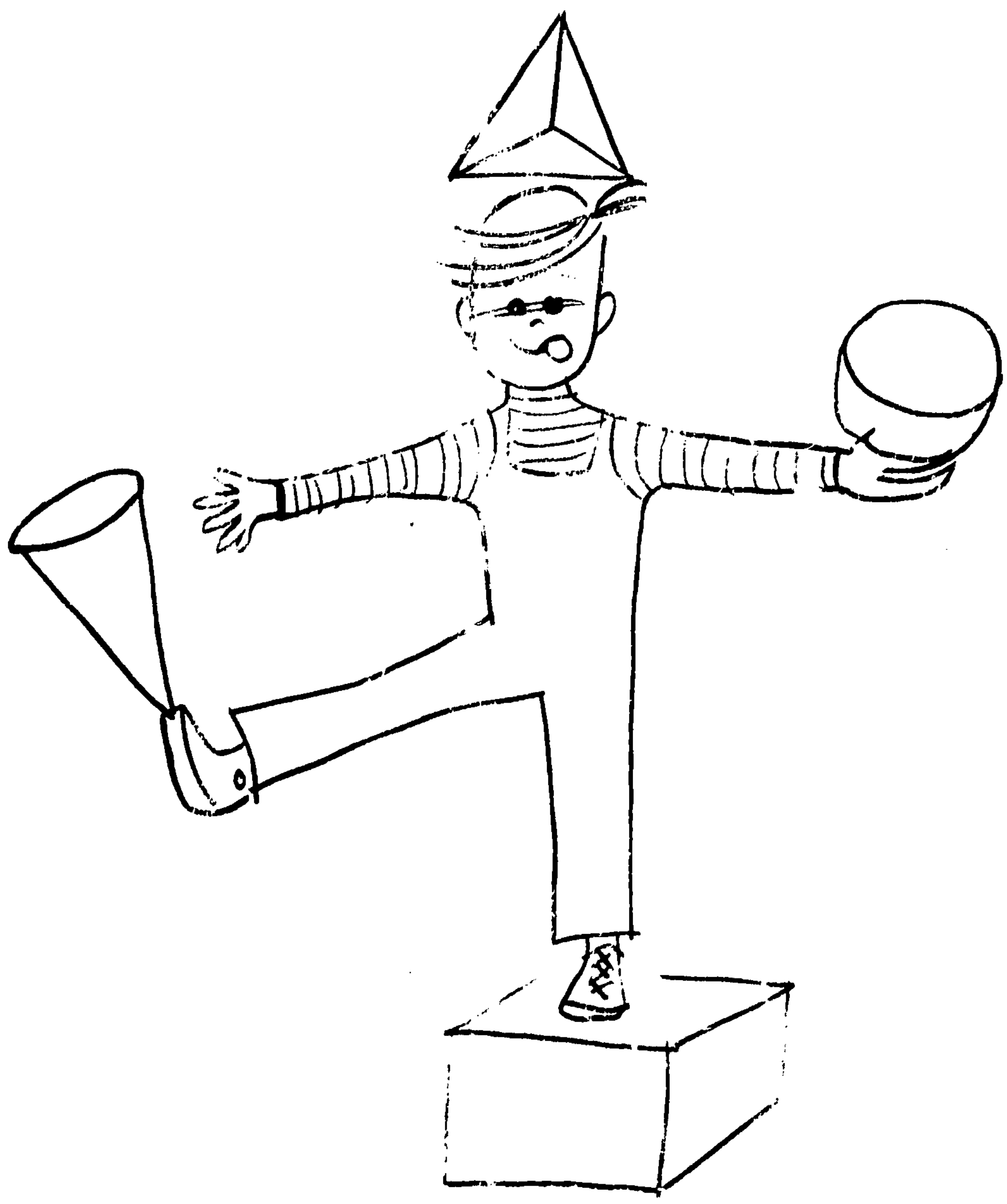


Illustration IX

Let's suppose you have been asked to identify the cone. Would identifying be interpreted as demanding some sort of an observable or a vague action on your part? Observable or vague? (10)____________________________________

It is an observable action of course. You might carry out the identifying by pointing to an object, or by placing your finger on an object, or by actually picking up an object.

Up to now we have merely examined statements of objectives as they are usually written for mathematics curricula. The descriptions are usually ambiguous and tend to have a large number of possible interpretations. We also have seen that a description of an objective which is more specific must specify the performances which the learner is expected to exhibit.

In Illustration X two objectives are described. Read them carefully and select the description of the objective which is behavioral.

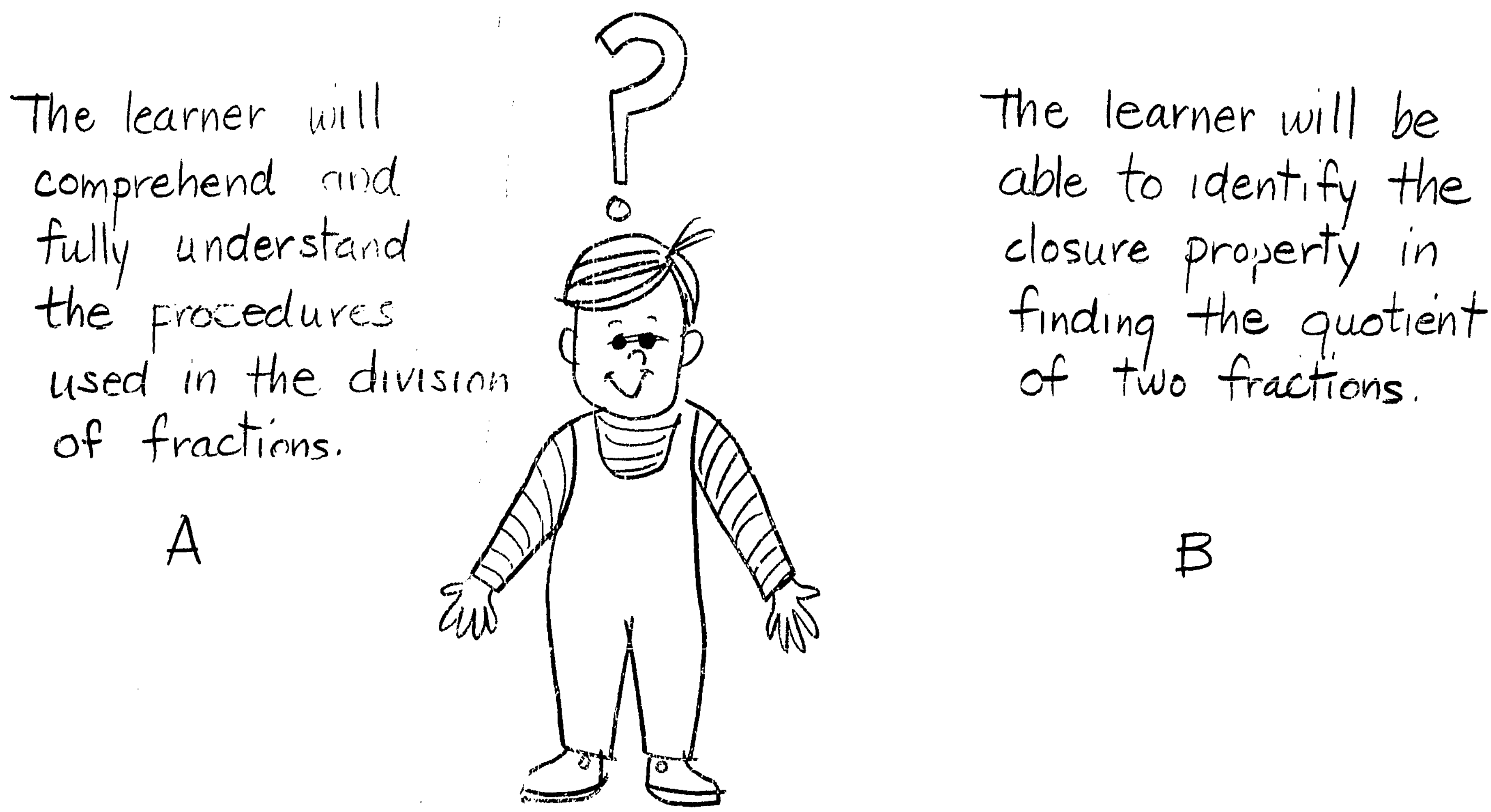


Illustration X

Did you select statement A or statement B? (11)____________________

Should you have any doubt about which of these descriptions is behavioral you will find the acceptable response on page 14, line 2, word 2 , 1st letter of the word.

Illustration XI suggests four verbs which might be used in the description of behavioral objectives. Two of the verbs are action verbs (which describe learner performances) and two of them are verbs which do not describe reliably observable performances. Select the two verbs which describe reliably observable performances.

(12)____________________, (13)____________________.

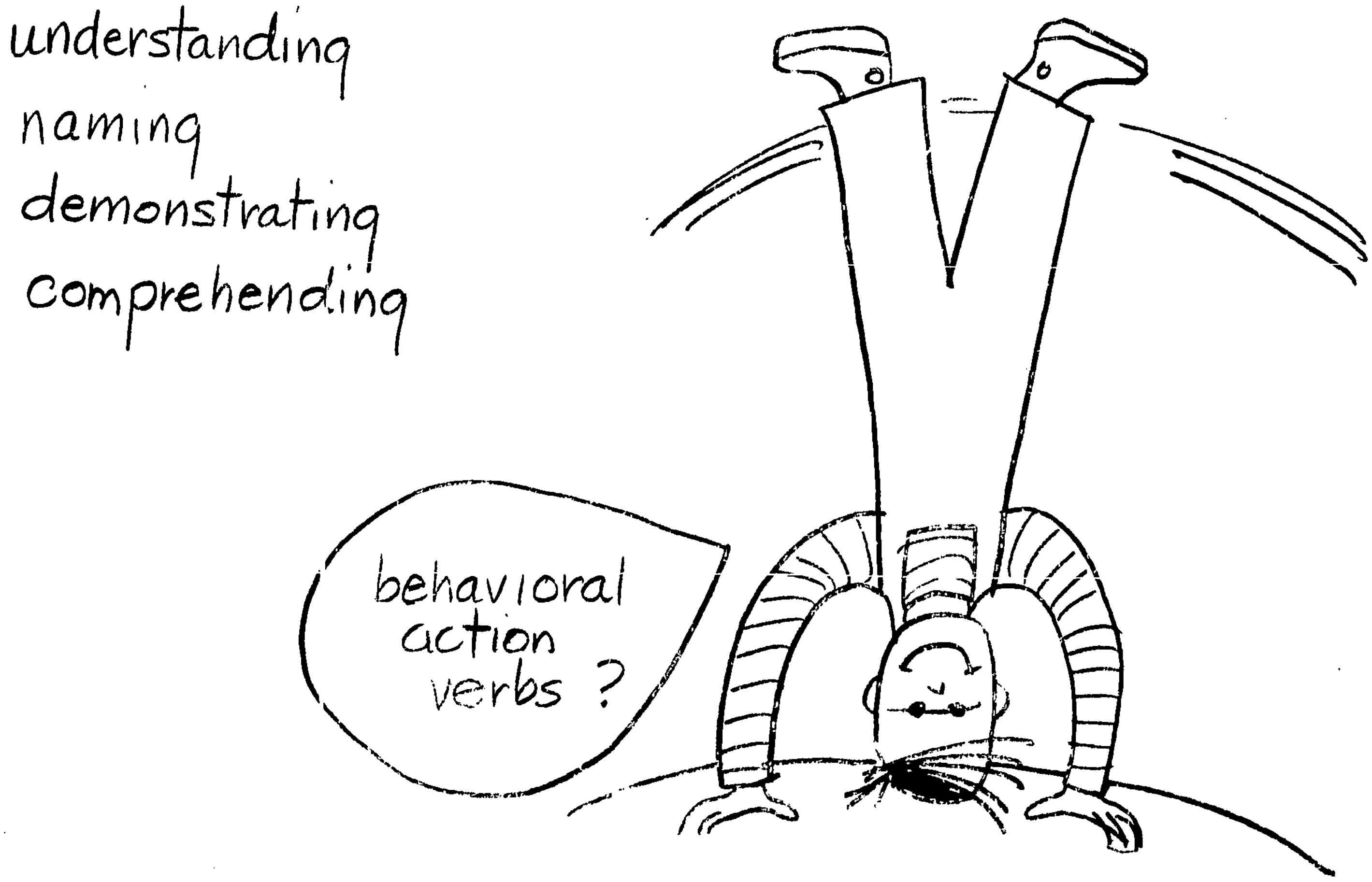


Illustration XI

If you selected the words "understand" and "comprehend" you are just not with it today! The two action verbs which describe reliably observable performances are clearly "naming" and "demonstrating".

Read Illustration XII and identify the verbs which are action verbs. That is identify those verbs which could be used in a description of reliably observable performance.

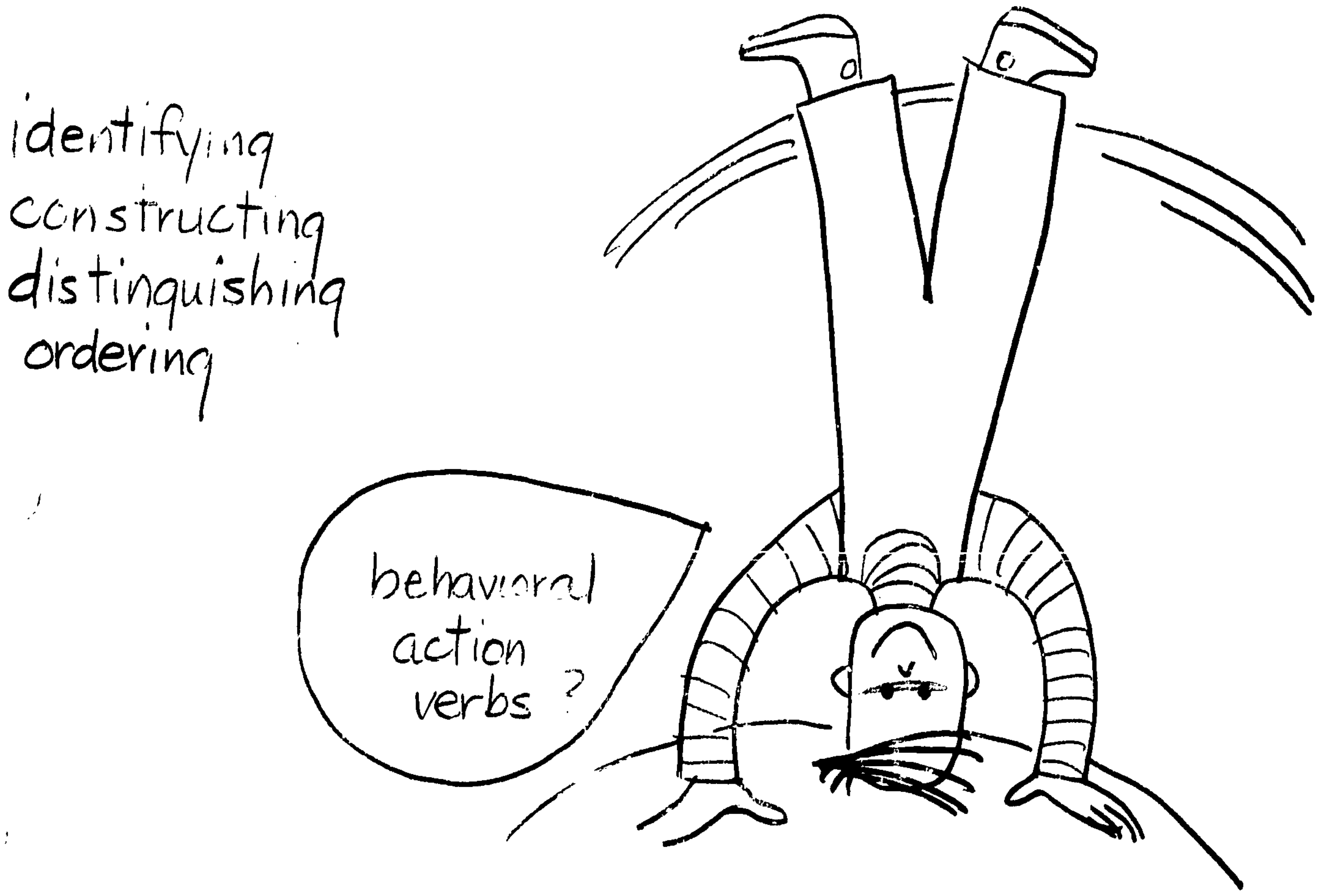


Illustration XII

Which of the words did you select? (14)____________________________________

The correct choices are the last word on page 18 and the 7th word on page 6.

Illustration XIII contains a description of two objectives--one of the objectives is behavioral. Identify the behavioral objective by selecting A or B.

(15)________________________

The learner will be able to demonstrate examples of the commutative property with sums of whole numbers.

A

The learner will acquire appreciation of the discovery method of teaching mathematics.

B

Illustration XIII

Did you identify the appreciating objective as the behavioral one? No. Good! Obviously the demonstrating objective is the behavioral one.

Examine Illustration XIV which contains two descriptions of objectives. Which of these objectives is a behavioral description of desired learner performance? A or B? (16)________________________________

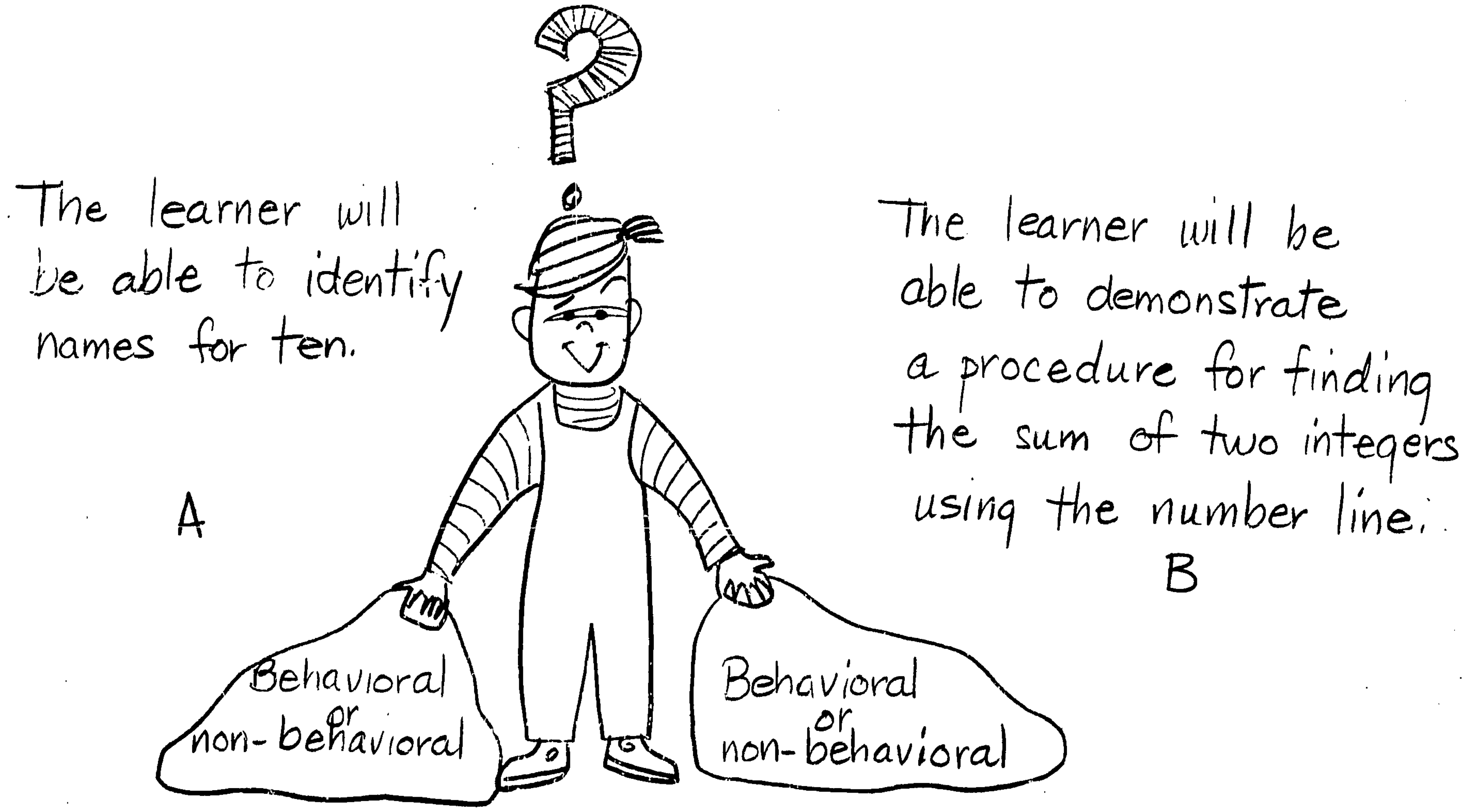


Illustration XIV

I suppose that you decided that neither of these were behavioral objectives. No? Oh! You made a choice that only statement A was behavioral description or that only statement B was behavioral description. No? Good! Did you decide both A and B are behavioral descriptions of desired student performance? If so, you have acquired the behavior of being able to identify behavioral objectives.

ACTIVITY TWO

If you are to acquire the competency expected from this portion of the exercise, you must again respond when asked to do so. Plan on responding quickly; for most of the tasks plan on making a response in about 15 seconds. It is important that you continue to be an active participant.

What functions do behavioral objectives fulfill that non-behavioral objectives do not?

(17)____

____

____

Certainly the previous collection of illustrations make the point that non-behavioral objectives tend to be ambiguous and general, while behavioral objectives seek clarity by specificity. But of what use is this specificity to the teacher? One important instructional benefit of this specificity is that the teacher would possess a description of the observable behaviors all students should minimally be able to exhibit after instruction.

Let's suppose for the sake of argument, you are convinced of the need to communicate specific instructional purposes, and further that behavioral objectives are means you have settled on for accomplishing this communication. But how do you construct behavioral objectives? What makes the description of an objective behavioral? Consider the tasks you have just performed in Activity One. What are the characteristics of a behavioral objective as they were described in the program? Name as many as you can. (18)________________________________

____

Is there an action identified in the description of a behavioral objective? Yes or no? (19)__________________. It would seem after even the most cursory examination that each description of a behavioral objective does contain some action word or phrase. What class of words most often describes action in English--nouns, verbs, adjectives, or what? (20)________________. Of course, most often action is communicated by verbs.

It seems rather obvious, then, that one necessary component in the description of a behavioral objective is an action verb. But now just wait a minute! How many possible action verbs are there in English--a few or a great many? (21)___ _________.
You could decide to use any of this large variety of action verbs. The variety itself, however, contributes more to maintaining the ambiguity than to facilitating clarity.

The problem would now appear to be one of reducing the number of possible action verbs used in the description of objectives without reducing the variety of learner performances being called for by the objectives.

Each individual has been given a number of packets of material. Spread out the materials in packet A. You will be asked to make several performances. Carry out each task as best you can:

(1) Pick up a triangle.

Go ahead, don't be bashful, pick it up. That's better!

(2) Now select a square.

Have you made a selection? Good!

(3) Identify the ellipse.

Notice that three different action verbs were used in initiating the three performances you made. One of the action verbs was selecting. What were the other two action verbs? (22)________________________ and (23)________________________.
Did you write picking up and identifying? Wonderful! Do the three performances you were asked to make have some common action characteristic? Yes or no? (24)______ ____________. Of course, they do!

Why use all three of these action verbs? If behavioral objectives are to be specific and describe observable behavior, would it seem sensible or not sensible to use one verb in place of all three? (25)____
The sensible thing to do is to have many different verbs describe the same action. No, of course it's not!! The sensible thing is to agree upon one action verb and use it. Which one of the three verbs shall we agree to use? Since it does not seem to make much difference, let's agree to use <u>identifying</u>.

Identify all of the triangular regions from Packet A. Do you have them all? Arrange the triangular regions from the one with the least area to the one with the greatest area.

As soon as you have completed this task, identify all of the square regions. Order the square regions from the one with the longest side to the one with the shortest side.

After identifying the two sets of objects, you performed two tasks. The instruction for each involved an action verb.

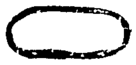
One of the action verbs was arranging. Name the other action verb. (26)__________

__________. Were the performances you exhibited alike or different? (27)____

__________. What do you conclude about the actions called for by these two action verbs arranging and ordering? (28)______________________________

____

If you are reading this before you have written a response to the last question, you are not playing the game. Go back and try to write a response to the question. The conclusion which seems justified is that these two action verbs are behavioral synonyms (call for a similar action). Let's agree to use ordering whenever such a behavior is called for in the description of a behavioral objective.

What do you call an object shaped like this  ? (29)______________

What is the name of a three dimensional object shaped like this  ?

(30)________________.

Tell the number of triangles pictured here.  (31)______________

Are the performances required by these three tasks similar or different?

(32)________________.

Similar, of course. What action verb would you use to describe these behaviors?

(33)________________. Any number of different action verbs are possible candidates. A few of these behavioral synonyms are telling, starting, calling for, and naming. Let's agree to use naming.

Return the shapes to packet A.

The agreements about action verbs made up to now would mean that when you describe a behavioral objective and the performance is

1. "choosing the rectangles" you would write

   "(34)____________________ the rectangles"

2. "classifying the objects from heaviest to lightest" you would say

   "(35)____________________ the objects from heaviest to lightest"

3. "telling the colors in this painting" you would write

   "(36)____________________ the colors in this painting"

If you're reading this before you have responded to the previous four tasks, go back and respond. Did you write identifying, ordering, and naming? That's a collection of acceptable responses. Now you're really catching on!

That's all.

Second Experimental Edition

MATERIALS FOR SESSION I

Packet A

11 felt pieces:

2 rectangles, 4 squares, 1 circle, 1 ellipse,
3 triangles

Second Experimental Edition

SESSION II

At the end of our first session, we were able to identify behavioral objectives which describe the specific action that is desired of the learner. We also agreed on some of the action verbs we are to use to describe a desired action. The procedure today will be similar to the procedure used in our last session. Be sure to write each response on the response sheet you have been given. Let's see how many of the action verbs you remember from Session I.

If I asked you to pick out a pencil from a collection of different objects, you would be (1)____________________ the pencil. Did you say identifying? Good!

If I required you to tell me the color of the pencil you are using, you would be, (2) ____________________ the color. If you said naming, you are remembering correctly from the last session. Good! Keep going.

When I arrange a set of objects according to size, I am (3) ____________________ the objects. Did you use the word arranging? No? Good for you. We agreed to use the word ordering.

Today we will learn the remaining action words.

Take out the materials in packet B and place them on the table. Show how you would decide which of the line segments - a or b - is longer using the rectangular felt shape. Go ahead and do something. Now demonstrate how you would decide whether the sheet of paper is a square by some folding procedure. Name the action verbs which initiate each of these performances.

(4) ____________________ and (5) ____________________ . Are these two action verbs behavioral synonyms? Yes or no?

(6) ____________________ . Of course they are. Let's agree to use <u>demonstrating</u> as the action verb for this set of behaviors.

Return the materials to packet B; then take out the graph from packet C and place it on the table. The graph records data obtained on the number of ice cream cones sold at various air temperatures. Make a prediction concerning the number of ice cream cones which will be sold if the temperature is 110°.

(7) ____________________ . Go ahead, make some prediction.

Now examine the two vials with spheres in them. Invert the two vials and watch what happens. Construct an explanation which accounts for the difference in the behavior of the spheres. (8) ____________________ . Now name the action verbs which initiated each of these performances. (9) ____________________

and (10) ________________ . Are these two action verbs behavioral synonyms? Yes or no? (11) ________________ . Let's agree to use the action verb constructing in descriptions of objectives involving such behaviors. Return the graph to packet C.

Consider the object which is in packet D. Suppose someone has a group of objects in front of him, one of which is similar to the object you have taken from packet D. This person is able to hear you, but cannot see you. Your task is to identify and name as many characteristics of the object as you can. The description should enable the second person to identify a similar object in his collection. Start naming: (12) ________________________________

____

If you said red and round, your description is not adequate for it fits most all of the objects which the second person has in front of him. Add a few more characteristics. If you added mass, volume, diameter, and thickness you would be much closer to a satisfactory description.

Take the tablet in packet E and drop it in a glass of water. Observe what happens. Describe what happened so that another individual would be able to pick out (identify) the similar event if he were confronted with the various situations shown in Illustration XV.

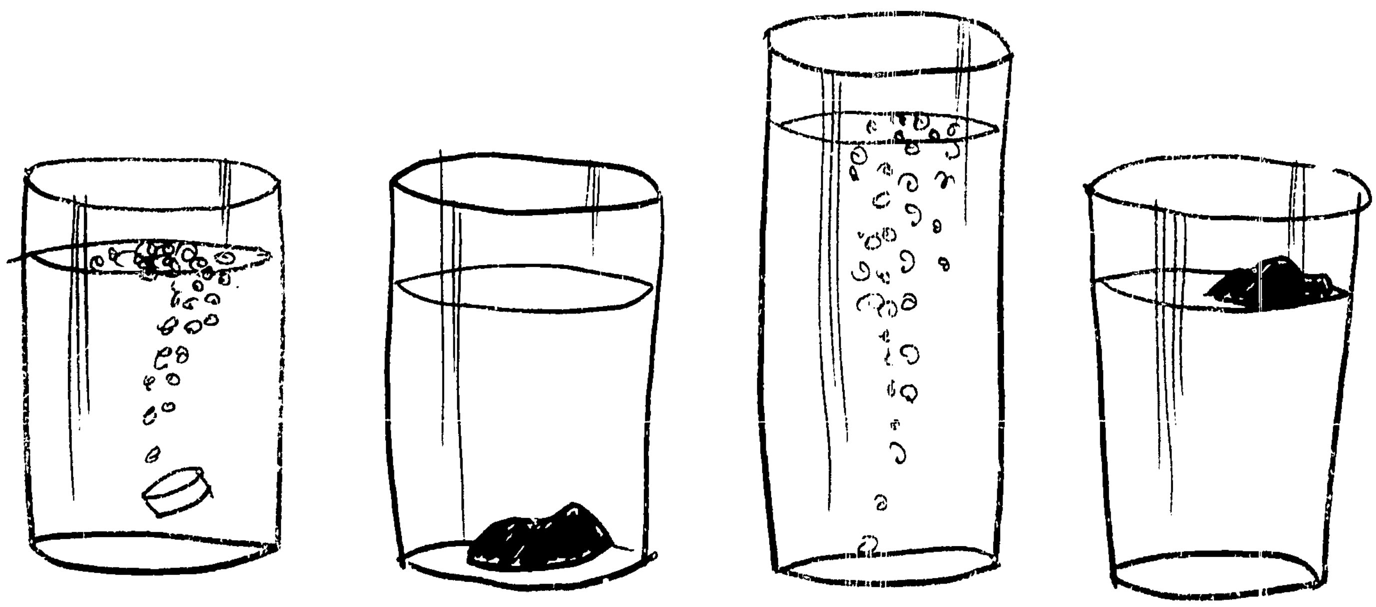


Illustration XV

Descriptions (13) ____________________

____

____

What shall we call such behavior? What action verb should we use? Could we use identifying? Yes or no? (14) __________ . Or naming? Yes or no? (15) __________ .

Since identifying requires the individual to select an object which has been named for him, this action verb does not seem satisfactory. In the same way, naming does not seem an appropriate choice since the name of the object or action which is used has been previously supplied by someone other than the learner. The distinctive characteristic of this new behavioral class is that the learner identifies and names the characteristics or properties. More than one characteristic is usually included, and there must be a sufficient number of these characteristics so that a second individual will be able to identify what is being discussed.

How then shall we name this class of behaviors? Suggest a possibility? (16) ____________________ . Many choices could have been made. The particular action verb which seems most appropriate is describing. This describing behavior involves the individual identifying a sufficient number of characteristics of an object or action so that a second person would be able to identify it without having it pointed out to him.

Sometimes the behavior is the description of a particular procedure. For example, a procedure for finding the speed of an object might be stated as follows: "How fast an object changes position is distance moved per unit of time which can be found by dividing the distance by the time."

Another procedure might be one dealing with fractions. For example, in order to find the sum of two fractions which have the same denominator you add the numerators for the new numerator and keep the common denominator.

The two previous paragraphs describe a procedure or rule for doing something. Could the behavior of stating procedures such as these be described by naming or identifying? If so, which one? (17) ____________________ .

Naming could be used, but the statements of rules, such as given by the two examples, are special. For this reason it is convenient to describe this class of behavior by calling the category, stating the rule.

Suppose a boy walks 140 meters in seven minutes. How fast is the object changing position if you use the speed rule previously stated?

(18) ____________________ . Just what did you do to obtain the result?

You were (19) ____________________ the speed rule. Using or applying would be the most commonly acceptable response.

Given the fractional names 3/17 and 5/17 find the sum. The sum is (20) ________________. For your own information, the acceptable response is $\frac{3+5}{17}$ or $\frac{8}{17}$.

In each of the last two tasks, would you describe your performance as naming or demonstrating? (21) ____________. The correct response is, of course, demonstrating. However, since the demonstration is special, in that it is based upon a stated rule, this behavior might warrant a separate name. Let's agree to call these applications of stated rules, applying a rule behavior.

Suppose you were asked to find the density of an object such as a marble. Let's also suppose that you were told that the density of an object is found by determining the mass of the object, the volume of the object, and finally the quotient of mass divided by volume. This description of how the density of an object is obtained is an example of which of the following behaviors: identifying, constructing, stating, or applying a rule?

(22) ______________. Since the description deals with using a procedure, the most acceptable choice is applying a rule. Was a single rule applied, or was it necessary to apply more than one? (23) ____________________.

Yes, more than one rule was used--in fact, three rules are stated, one each for mass, volume, and quotient. Now if you are asked to use the three procedures to arrive at the quotient (density), what kind of behavior would it be? Applying a rule would be too simple, since this is a sequence of three rules, interrelated in the process of finding density. The entire behavior might be described as a series of related applying a rule behaviors and for this serial task performance, let's adopt the action verb interpreting.

The agreements about action verbs made up to now mean that whenever you describe a behavioral objective you will use one or more of the action verbs we agreed upon, but no other.

Rewrite the following performances using our list of action verbs:

1. Telling how to get to your house.

   (24) ________________ how to get to your house.

   (Check your response with the 1st underlined word on page 3 or the 2nd underlined word on page 3.)

2. Showing how you could decide that the rock is limestone.

   (25) ________________ that the rock is limestone.

   (Check your response with the underlined word on page 1.)

3. Making a definition for an action verb.

(26) ____________________ a definition for an action verb.

(Check your response with the underlined word on page 2.)

4. Following a procedure for finding the product of two fractions.

(27) ____________________ finding the product of two fractions.

(Check your response with the 1st underlined word on page 4 or the underlined word on page 1.)

5. Defining a prime number as a whole number which has exactly two different whole number factors.

(28) ____________________ a prime number.

(Check your response with the 2nd underlined word on page 3 or the 1st underlined word on page 3.)

6. Building the graph of a relation.

(29) ____________________ the graph of a relation.

(Check your response with the underlined word on page 2.)

7. Identifying and naming the reasons which justify the steps in the long division process.

(30) ____________________ the long division process.

(Check your response with the 2nd underlined word on page 4.)

In performances 1, 4 and 5 two possible answers were given because the action verb depended upon what the writer had in mind. In 1, for example, if you were telling how you, yourself, get home, you would be describing your journey; on the other hand, if you are giving someone a route to follow in getting to your house, you would be stating a rule.

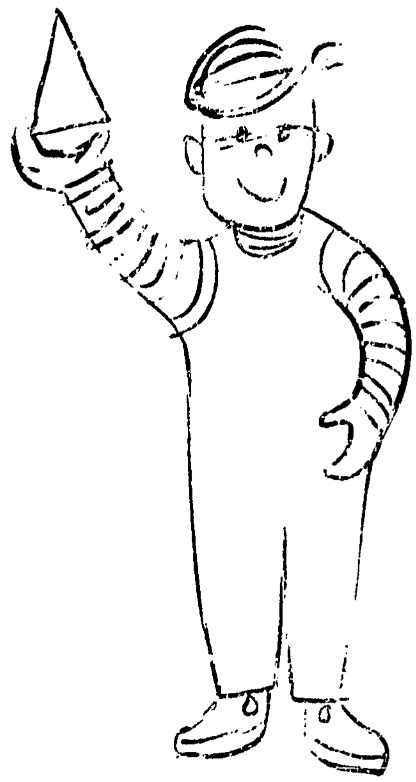
Occasionally when the identifying behavior is called for and the stimuli are highly confusable, it is useful to have a special action verb. Whenever this situation arises, the action verb <u>distinguishing</u> is used in the behavioral description in the place of identifying.

Each of the action verbs we have agreed upon is described on the following three sheets. This may prove useful as reference material for the next activity.

DEFINITIONS OF ACTION WORDS

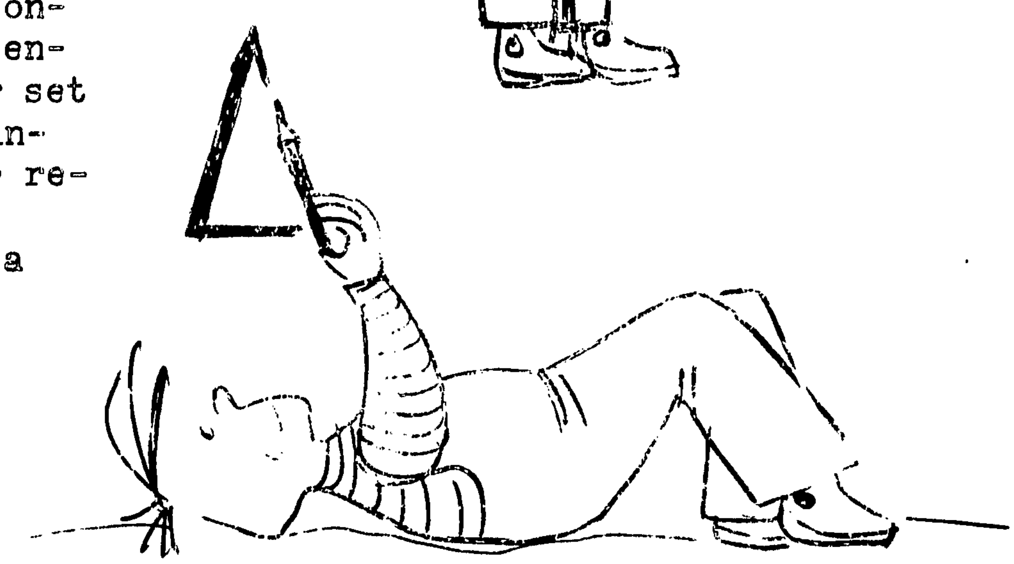
The action words which are used in the construction of behavioral instructional objectives are:

1. IDENTIFYING. The learner selects (by pointing to, touching, or picking up) the correct object of a class name. For example: Upon being asked, "Which animal is the frog?" when presented with a set of small animals, the learner is expected to respond by picking up or clearly pointing to or touching the frog; if the learner is asked to "pick up the red triangle" when presented with a set of paper cutouts representing different shapes, he is expected to pick up the red triangles. This class of performances also includes identifying object properties (such as rough, smooth, straight, curved) and, in addition, kinds of changes such as an increase or decrease in size.



2. DISTINGUISHING. Identifying objects or events which are potentially confusable (square, rectangle), or when two contrasting identifications (such as right, left) are involved.

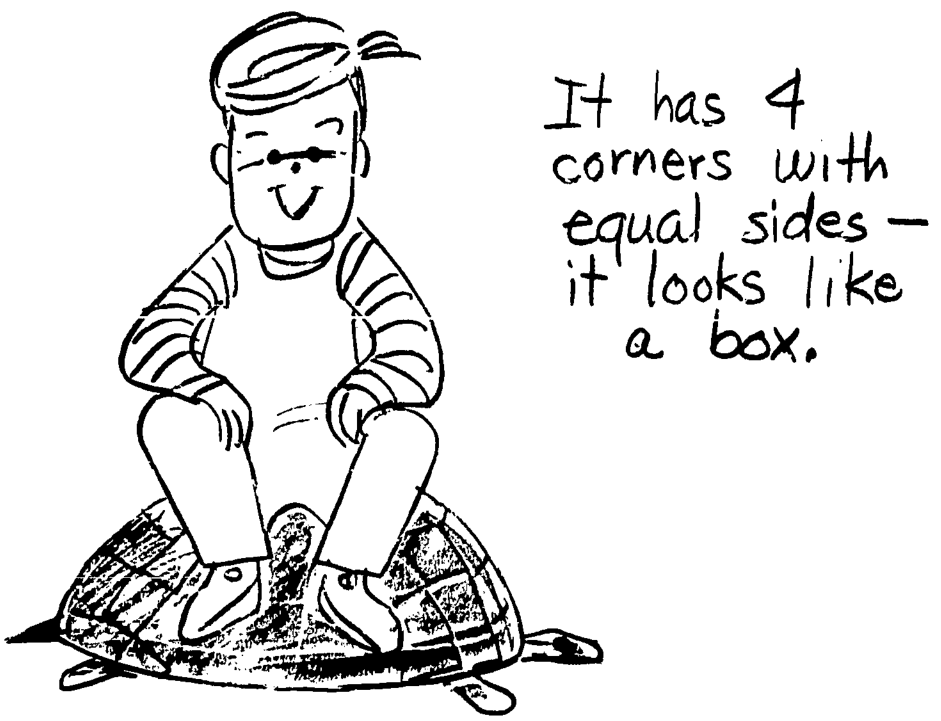
3. CONSTRUCTING. Generating a construction or drawing which identifies a designated object or set of conditions. Example: Beginning with a line segment, the request is made, "Complete this figure so that it represents a triangle."



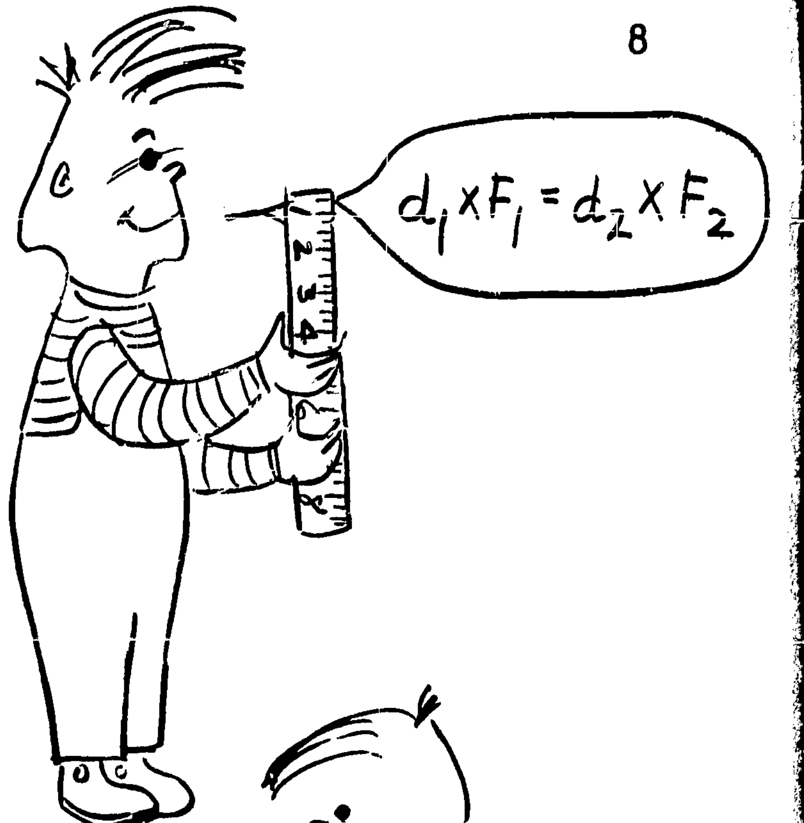
4. NAMING. Supplying the correct name (orally or in written form) for a class of objects or events. Example: "What is this three-dimensional object called?" Response: "A cone."

5. ORDERING. Arranging two or more objects or events in proper order in accordance with a stated category. For example: "Arrange these moving objects in order of their speeds."

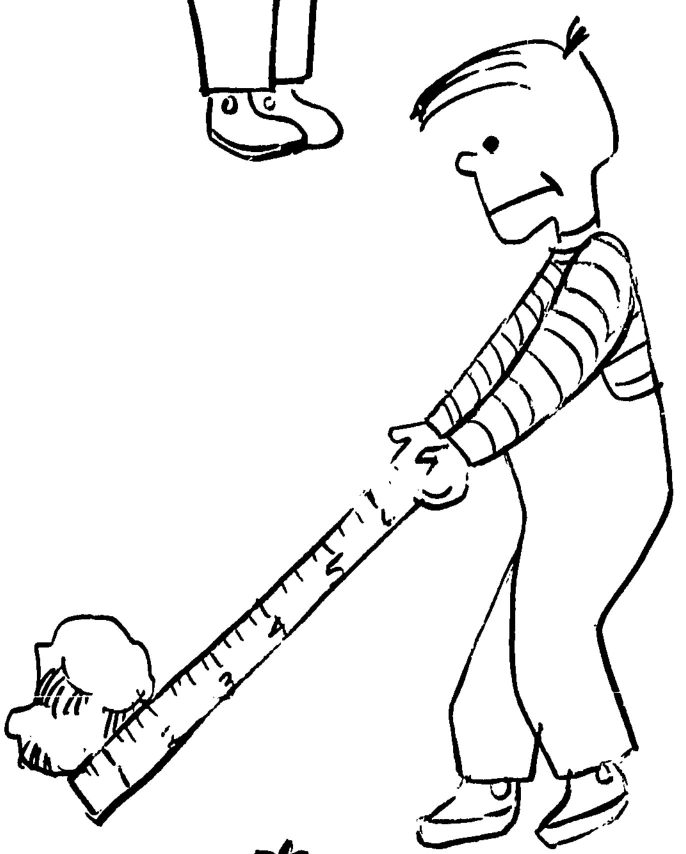
6. DESCRIBING. Generating and naming all of the necessary categories of objects, object properties, or event properties, that are relevant to the description of a designated situation. Example: "Describe this object," and the observer does not limit the categories which may be generated by mentioning them, as in the question "Describe the color and shape of this object." The learner's description is considered sufficiently complete when there is a probability of approximately one that any other individual is able to use the description to identify the object or event.



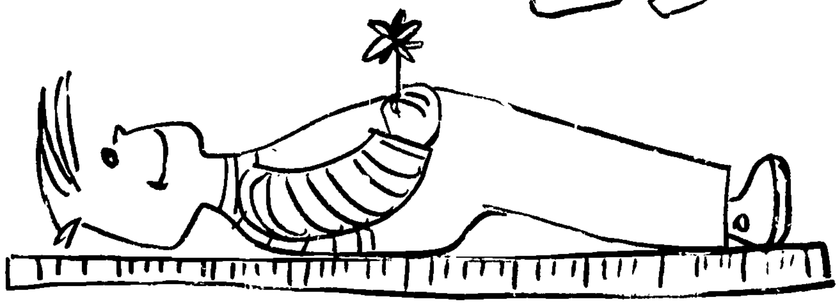
7. STATING A RULE. Makes a verbal statement (not necessarily in technical terms) which conveys a rule or a principle, including the names of the proper classes of objects or events in their correct order. Example: "What is the test for determining whether this surface is flat?" The acceptable response requires the mention of the application of a straightedge, in various directions, to determine touching all along the edge for each position.



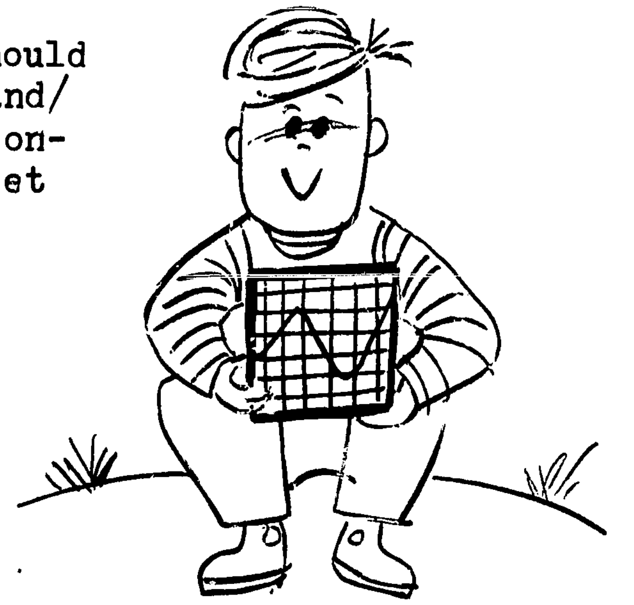
8. APPLYING A RULE. Using a learned principle or rule to derive an answer to a question. The answer may be correct identification, the supplying of a name, or some other kind of response. The question is stated in such a way that the individual must employ a rational process to arrive at the answer. Such a process may be simple, as "Property A is true, property B is true, therefore property C must be true."



9. DEMONSTRATING. Performing the operations necessary to the application of a rule or principle. Example: "Show how you would tell whether this surface is flat." The answer requires that the individual use a straightedge to determine touching of the edge to the surface at all points, and in various directions.



10. INTERPRETING. The learner should be able to identify objects and/or events in terms of their consequences. There will be a set of rules or principles always connected with this behavior.



Now that we have agreed upon a set of operational definitions for some action words in the construction of behavioral objectives, let's turn our attention to the problem of constructing a few behavioral objectives. Read the objective in Illustration XVI.

The learner will understand place value.

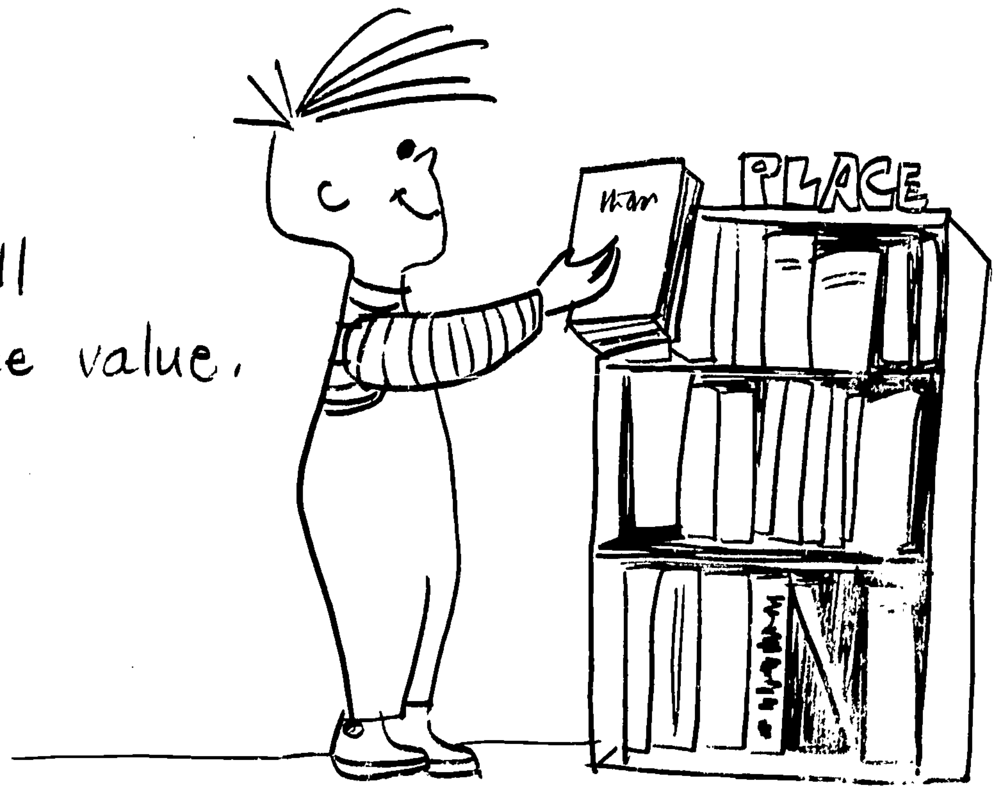


Illustration XVI

Is the objective described in Illustration XVI behavioral? Yes or no?

(31)__________________ No, of course it is not. With the use of our agreed upon set of verbs rewrite the non-behavioral objectives described in Illustration XVI and make it a behavioral objective. (32)____

____

____

If you have not completed rewriting the objective, do not read this section. Go back and do it now. When you have completed the task of rewriting the objective, read it over to see whether you have: 1) used one of the action verbs, 2) described the situation in which the learner should exhibit this particular behavior, and 3) indicated the nature of the product the learner is to produce. Learner products may be quite varied: a sentence, a word, a drawing, a series of check marks, etc.

Illustration XVII contains a few of the possible descriptions of behavioral objectives which could have been constructed from the non-behavioral objective.

The learner will identify the units, tens, and hundreds place, given a numeral.

The learner will identify the position of the 512's place for base eight numerals.

Illustration XVII

Now examine the non-behavioral objective described in Illustration XVIII.

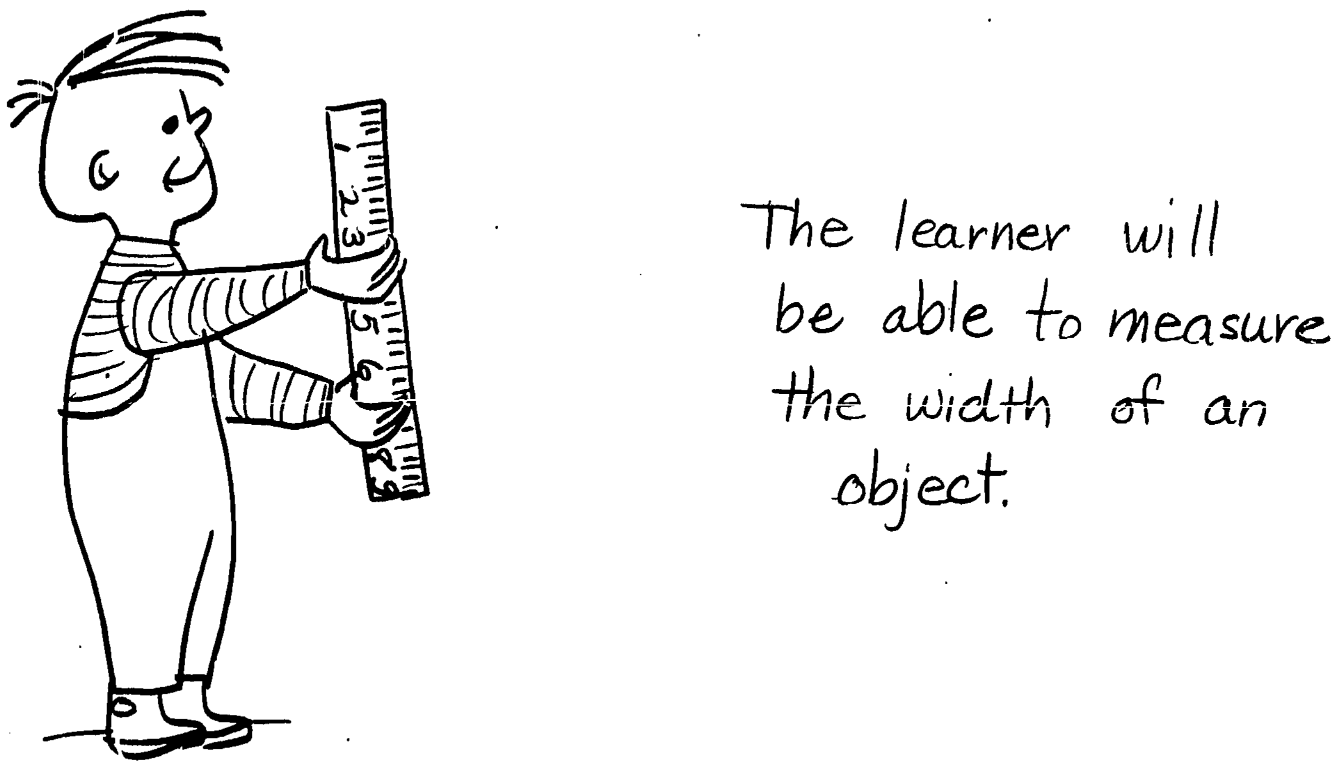


Illustration XVIII

Notice that the non-behavioral objective appeals to a word which in general is supposedly "well understood" by all teachers. When one says "measure the width of a desk," the statement does not suffer from the same lack of specificity as did the non-behavioral objective in the previous example. However, the objective does suffer from the fact that it is not a description of what the learner is asked to do. How would you recognize whether a learner had acquired this behavior?

(33)____

Notice that it is not a difficult task to translate this particular objective into a behavioral objective. The translation could be accomplished merely by stating what one would look for in terms of learner performance rather than in terms of the non-performance description which the word measure conveys.

Examine the objective described in Illustration XIX.

The learner will acquire a familiarity with the commutative property.

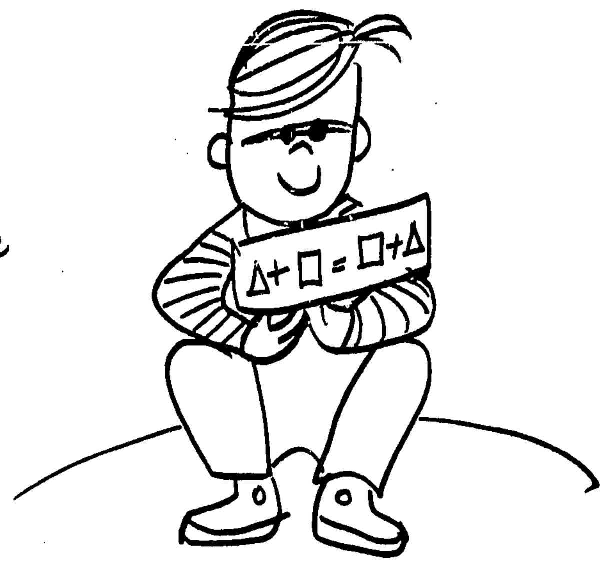


Illustration XIX

Rewrite this objective so that it is behavioral.

(34)____

When you have completed the task of rewriting the objective and making it a behavioral objective, read these next statements.

1. Did you use one of the action verbs we have agreed upon?

   Yes or no? (35)____________________

2. Is the situation in which the learner is to exhibit this

   performance clearly specified? Yes or no? (36)____________________

3. Is the nature of the output which the learner is to provide clearly specified as well as any restrictions on that

   particular output? Yes or no? (37)____________________

If you were not able to respond "Yes" to each of the three questions, go back and correct whatever difficulties you have identified. Does the description of the objective in Illustration XIX identify the specific performances which you would

look for in your observations? Yes or no? (38)____________________

Just what performances would one be expected to observe in learners who had acquired a familiarity with commutativity? It is certainly not contained in the statement of the previous objective. Therefore, the appropriate response to the question concerning observable performances in the objective is an emphatic "No, it does not identify them."

MATERIALS FOR SESSION II

Packet B

folder containing:

felt rectangle
paper square
sheet with lines a and b.

Packet C

folder containing:

graph - "Number of Ice Cream Cones sold in one day"

Packet D

1 red chip

Packet E

1 alka-seltzer

2 vials -

1 containing marble and water
1 containing marble and Karo syrup

SESSION III

ACTIVITY THREE

Here is an example of an instructional activity. Read it over carefully and decide what actions describe the desired instructional outcomes.

> Select two or three objects that contain the various two-dimensional shapes. Have the children point out and name the circles, ellipses, triangles, rectangles, and squares for one object at a time.
>
> Hold up the pyramid in various positions. Ask: "What two-dimensional shapes can be seen in the pyramid?" (Triangles, square on the base.) Have them trace the shapes with their fingers.
>
> Pick up the cone and let the children pick out the shapes they see. If the children have difficulty selecting the triangles, hold the cone next to the chalkboard and trace its edges. When the cone is removed, ask: "What shape is drawn on the board?" This same procedure may be helpful in identifying the rectangle that may be associated with a cylinder.

Use our action verbs to name at least two actions which are part of this instructional activity. (1)________________ and (2)________________.

Write one behavioral objective for this activity. Remember to use one of the ten action verbs. At the end of this instructional activity the learner will be

(3)____

____

Have you written a behavioral objective? If not, don't read beyond this sentence and go back and try! If your description of a behavioral objective resembles one of the following statements, you're on the right track.

1. Identifying and naming the following three-dimensional shapes: sphere, cube, cylinder, pyramid, and cone.

2. Identifying and naming two dimensional shapes that are part of regular three dimensional shapes.

Now let's suppose you are given a description of a behavioral objective such as:

"The child should be able to name the primary colors."

What does such an objective communicate about instruction and how will you know when you have been successful? Let's take the instructional question first, but in the context of comparing it to another objective which said "identifying each of the primary colors." What would be different in the instructional activity trying to help children acquire the naming behavior as opposed to acquiring the identifying

behavior? What do you think? (4)____

Write something. The important thing is to commit yourself. One could conclude that the "naming objective" would have children saying the names of primary colors when shown an object, while the "identifying objective" would probably see the children pointing to or picking up objects having been asked something such as: "Find a red object in the room."

In which of the following behavioral objectives would you expect to see small groups or individual children doing things?

1. Constructing a bar graph.
2. Ordering objects on the basis of similarity; for example, most like a circle, somewhat like a circle, least like a circle.
3. Demonstrating the comparison of volume of containers by determining how many unit volumes are required to fill each of the containers.

(5)____

In which ones would you expect only a teacher demonstration? (6)________________

If you responded that all objectives suggest small groups or individual instructional activities, you're with it. None of the descriptions of the behavioral objectives are suggestive of only teacher demonstrations.

Occasionally, behavioral objectives are related to one another in that they can be sequenced so as to construct an ordering from less complex behaviors to more complex related behaviors. For example, consider the three behavioral objectives:

(1) identifying and naming the primary and secondary colors

(2) describing an object in terms of characteristics such as color and two dimensional shape

(3) identifying and naming common two dimensional shapes.

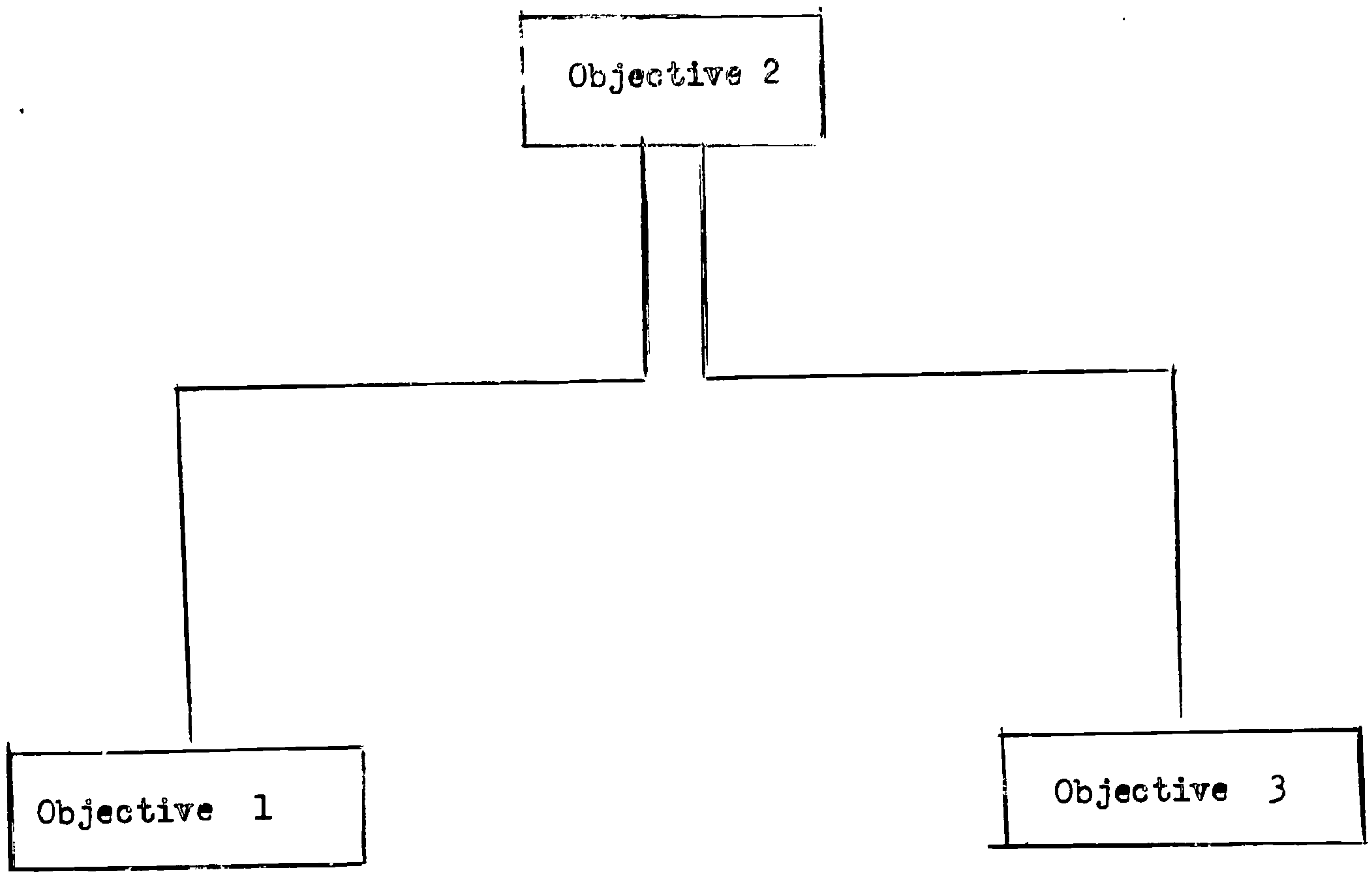
Which of these three behavioral objectives do you think describes the most complex

behavior? (7)____

Make a choice. The second one is correct. Are the other two behaviors related to this more complex behavior?

Yes or no? (8)____

And are the other two behaviors subordinate, less complex behaviors? Yes or no? (9)____________________ Is either of these subordinate to the other? Yes or no? (10)________________. You might imagine constructing a diagram to show this relationship and it would look something like



Remove the contents of packet F and place them on the table. The five behavioral objectives represent a collection that can also be ordered into several levels. Arrange the statements of the behavioral objectives in an ordering from most complex on top to least complex on the bottom. Your instructional sequence should look like the one in Illustration XX.

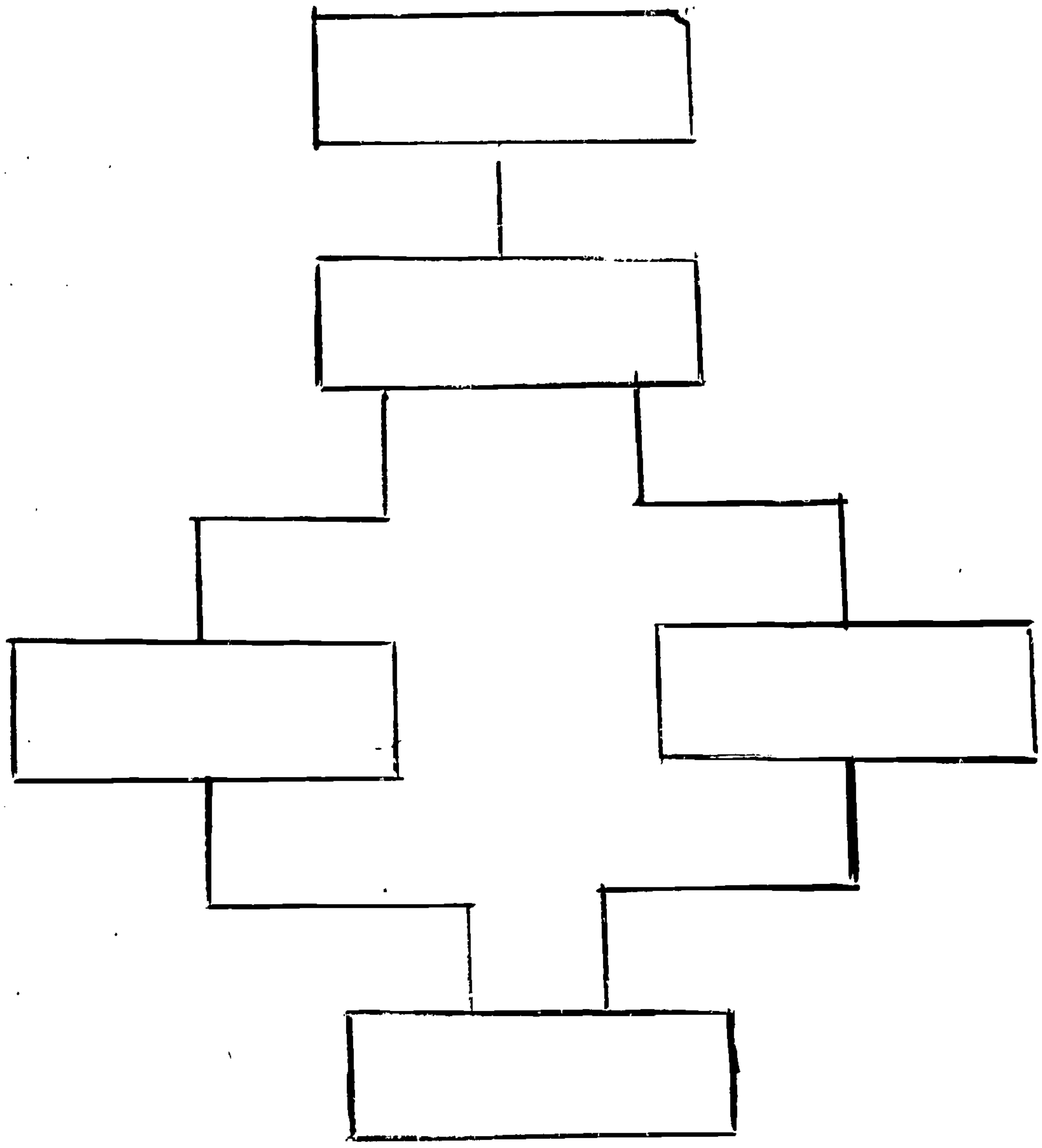


ILLUSTRATION XX

Just how did you proceed with your analysis? Did you attempt to identify the least complex behavior? Yes or no? (11)____________. You say you did! Well, most people do, the first time they try. Why not? Well, recall that one is interested in identifying those behaviors which may need to be acquired before attempting to acquire a complex behavior. That is, the procedure is one of trying to identify the subordinate behaviors for a given learning task. How is one to hunt for subordinate behaviors when the terminal task is not identified? So, it would seem, that the most complex task should be identified first. Which of these five tasksis the most complex? 1, 2, 3, 4, or 5? (12)____________________________.

One looks like a good candidate, but let's examine the others. Certainly two, three and five are not as complicated as one since only correspondence or matching of some kind is involved. I guess that leaves only four.

Considering that one demands at most 99 names and four demands at most nine names which one must be the most complex? 1 or 4? (13) ________. Naturally the acceptable response is one. What must the learner be able to do before he can acquire the behavior labeled 1? What is the next most complex behavior? 2, 3, 4, or 5? (14) ____________. Of course, four is the next most complex behavior and a reasonable subordinate behavior to one. What is the next most complex pair of behaviors? An examination of the three remaining possibilities suggests which ones as the next most complex pair? 2, 3, or 5?

(15) ______________ and (16) __________________. Since ordering three sets and identifying sets with the same number of objects both require one-to-one matching, two and five appear to be the most likely candidates. That leaves three as the least complex behavior.

Just making these decisions, does not make the instructional sequence valid. The sequence is a series of hypotheses. One now tries out the sequence of instruction and determines if it works--does the learner acquire the desired behaviors if one follows the instructional sequence? Modifications are then made in the instructional sequence on the basis of observations.

Thus, whenever all of the behaviors described in the various objectives for a task are arranged in such an ordering and the relationships between levels are shown, the ordering is called a behavioral hierarchy. Behavioral hierarchies are useful in that they describe the behavioral development within a process. Packet G contains the behavioral hierarchy which follows from the analysis for the tasks described in packet F.

Suppose we were interested in providing instruction that would help a learner acquire the most complex behavior among these five--identifying and naming the number of objects in any set with zero to 99 members. How might we proceed instructionally? Oh! That's one of the fascinating applications of behavioral hierarchies. The behavioral hierarchies suggest one possible instructional path. For those learners who did not already possess the most complex behaviors, would we begin instruction with the least or most complex task which the learner does not exhibit? Least or most? (17) ______________. Naturally instruction would begin by helping those learners acquire the simplest behaviors, the least complex which they have not already acquired. Then instructions would proceed to the next level of complexity and so on through each of the behaviors considered prerequisite to the final task. A rather delightful consequence of having stated each instructional task as a behavioral objective is that you can then identify when your instruction has been successful. All that you as an instructor need to do is to give the learner a task representative of the described behavior and then observe whether or not the learner exhibits the desired behavior.

Second Experimental Edition

MATERIALS FOR SESSION III

Packet F

Hierarchy in 5 pieces

Packet G

Hierarchy

Packet H

List of Objectives

Appraisal

Construct a description of the behavioral objectives suggested by the instructional activity in the first three sessions. In writing these behavioral objectives, consideration should be given to the action words which were defined. When you have completed this task, look at the statement of behavioral objectives which were used in order to write the first three sessions. The statement of these behavioral objectives is in packet H.

First Experimental Edition

SESSION IV

THE GAME OF SUMS

OBJECTIVES:

At the end of this session the learner should be able to:

(1) identify and name examples of each of the game rules given the game, elements, and operations of the game.

(2) demonstrate each of the game rules using the elements and operations from a given game.

(3) construct data which support the presence or absence of a given game rule for a particular game.

This is a game called Sums. Any number of players could play, but we'll start with two players. The play begins with each player spinning the spinner. The player spinning the largest number is first. In the event of a tie, each player spins again. The first player then draws a card from the blue pack. Play alternates until one player has reached the end.

If the player draws a one spin card, he spins the spinner. The player then moves his piece the indicated number of spaces on the board.

If the player draws a one spin card, he spins the spinner, records the result, spins the spinner again, records the second result, and combines the two results. To combine the two results construct their sum. If their sum is zero to nine, make the move; if the sum is ten or greater, the player moves the number of spaces named by the units digit. For example, suppose the first spin was 6 and the second spin was 8. Their sum is 6 + 8 or 14. Now since 14 is greater than 9, the number of spaces to move is given by the units digit or 4. So the player who spins a 6 and then an 8 would move 4 spaces.

If the player draws a three spin card, he spins the spinner, records the result, spins the spinner a second time, records the second result, spins the spinner a third time, records the third result, and combines the three results. To combine the results construct their sums. If their sum is zero to nine, make the move; if the sum is ten or greater, the player moves the number of spaces named by the units digit, which will be zero to nine. For example, suppose the results of the three spins were 6, 9, and 7. Their sum is 6 + 9 + 7 or 22. Now since 22 is greater than 9, the number of spaces to move is given by the units digit or 2. So the player who spins 6, 9, and 7 would move 2 spaces.

If the player draws one of the special cards from the blue pack which is labelled "1 spin repeated twice", he spins the spinner, records the result twice, and then combines the two results in the same way they were combined when a two spin card was drawn. For example, suppose the spin was 8. Now since the sum of 8 and 8 is 16, the number of spaces to move is 6.

Another special kind of card in the blue pack is labelled "1 spin repeated twice followed by another spin repeated twice." If a player draws this card, he spins the spinner and gets his result in the same way he does when directed to take one spin repeated twice. Then he takes a second spin and does the same thing. Finally he combines the two results. Perhaps an example here would help. Suppose the first spin was a 8. Since the sum of 8 and 8 is 16, the player would record a 6. Now suppose the second spin was a 9. Since the sum of 9 and 9 is 18, the player would record an 8. He would then combine the 6 and 8 for a move of 4 spaces.

The last kind of card in the blue pack is one labelled "2 spins repeated twice." If the player draws such a card, he spins the spinner, records the result, spins the spinner again, records the second result and combines the two results in the same way they were combined when a two spin card was drawn. Then this combined result is recorded twice, and the two combined results are combined again in the same way they were combined when a two spin card was drawn. For example, suppose the two spins were 4 and 3. Now since the sum of 4 and 3 is 7, the 7 is recorded twice. But combining 7 and 7 we find that the number of spaces to move is 4.

NOW PLAY THE GAME FOR AWHILE. When you land on a space on which directions for you are written, be sure and do as you are told. If you encounter the word identity or inverse, be sure and read the appropriate section which follows.

THIS SECTION IS NOT TO BE READ UNTIL ONE OF THE PLAYERS MUST ADVANCE THE IDENTITY OR A PLAYER IS INSTRUCTED TO READ THIS SECTION.

The identity is easy to find in the game of Sums. In order to find it we look at two spin moves. Let's say the first spin was a 4. Now is there a second spin that is possible so that the combined number of spaces to move is still 4?

Yes or no? (1)________________. What would the second spin be? (2)________________

________________. If you wrote 9, you are close since the combined number of spaces to move would be 3 since the sum of 4 and 9 is 13. The only number which is possible for the second move is 0 since the sum of 4 and 0 is 4.

Fill in the following table and watch the way the number 0 acts.

Table I

First Spin	Second Spin	Result
5	0	______
8	0	______
0	0	______
9	0	______
3	______	3
7	______	7
2	______	2
1	______	1

Among all the elements - 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 - there is a second spin which will always make the result of a two spin move the same as if the first spin was a one spin move. We call such an element the identity. Which is the element that appears to work out as an identity? (3)____________________

The correct response is on page 2 , last line, word 8 . Since we can find an identity in this game, we say that the identity game rule holds for the game of Sums.

Now you can continue your play.

THIS SECTION IS NOT TO BE READ UNTIL ONE OF THE PLAYERS LANDS ON A SPACE OR DRAWS A CARD WHICH MENTIONS THE WORD INVERSE.

Read the identity section first. Now that you have found that zero is the identity, let's fill in some of the blanks in Table II.

Read the identity section first. Now that you have found that zero is the identity, let's fill in some of the blanks in Table II.

Table II

First Spin	Second Spin	Result
3	7	______
6	4	______
0	0	______
9	1	______
2	______	0
0	______	0
5	______	0
______	8	0
______	6	0
______	3	0

The responses in order should be 0, 0, 0, 0, 8, 0, 5, 2, 4, and 7. Notice in the first four examples in Table II that the result of two spins was a move of zero spaces. Recall that zero is the identity. Now from the other examples were we able to find two spins such that the result of the two spins is the identity?

Yes or no? (4)________________. Let's try another example. If the first spin was a 1, what would the second spin have to be in order to have the result the identity? (5)________________. Since the sum of 1 and 9 is 10, the number of spaces to move is 0 - the identity. We call 9 the inverse of 1. Can we find an inverse for each of the elements - 0, 1, 2, 3, 4, 5, 6, 7, 8, 9? Yes or no? (6)____________. Let's systematically try them all.

The inverse of: 0 is (7)__________, 1 is (8)__________, 2 is (9)__________, 3 is (10)__________, 4 is (11)__________, 5 is (12)__________, 6 is (13)__________, 7 is (14)__________, 8 is (15)__________, and 9 is (16)______________.

When each element has an inverse in this game, we say that the inverse game rule holds for the game of Sums. Now go back to the game and continue your play.

AFTER YOU HAVE PLAYED FOR A FEW MINUTES, BEGIN TO SEARCH FOR THE ANSWERS TO THE FOLLOWING QUESTIONS.

There are many interesting patterns which we identify when we have played the game awhile. Let's look at one.

When you make a two spin move with 7 the first spin and 5 the second spin, you recorded the numbers and combined. Since the sum of 7 and 5 is 12, the number of spaces moved was 2. How many spaces would you move if the first spin was 5 and the second spin 7? (17)____________. Surprising, isn't it. Do you suppose a similar observation can be made for any other pair of spins? Try a few.

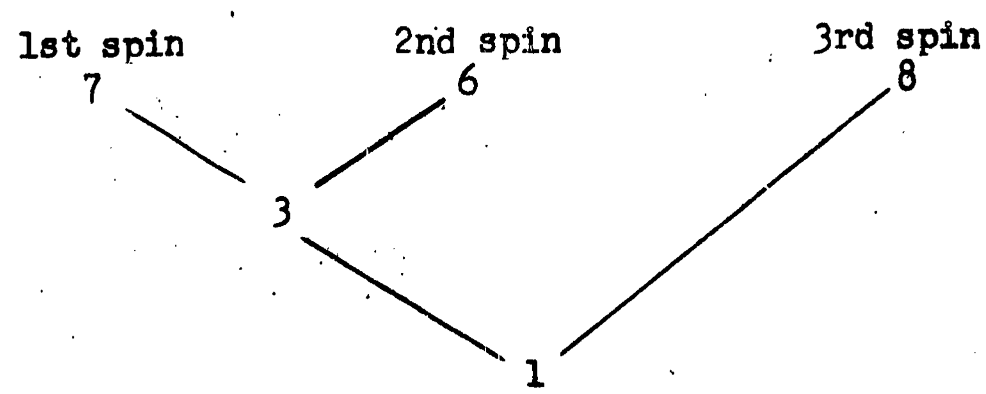
Did you find that reversing the spins always seems to give you the same result?

Yes or no? (18)_______________. The answer is on page 2, line 27, word 1.

When such a characteristic holds for all possible pairs, we say that reversibility game rule holds for the game of Sums. Some people describe this characteristic as the commutative characteristic, but we will use the name reversible to remind us that we reversed the order of the two spins.

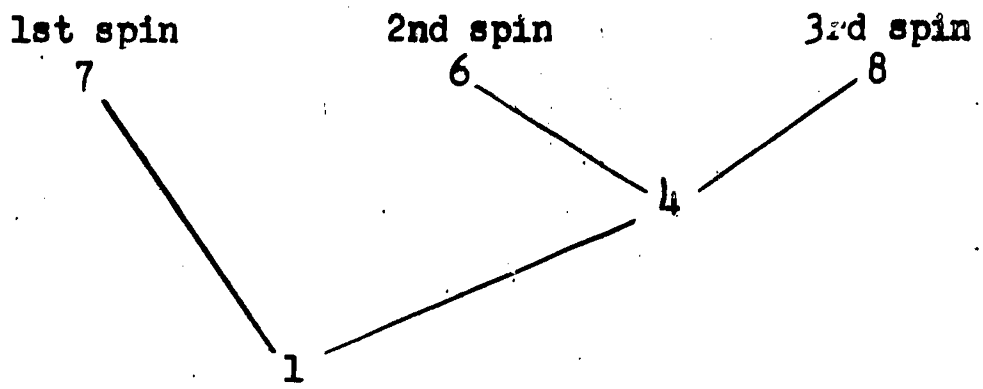
There is another characteristic of the two spin move which warrants a look. Was the result of a two spin move ever a number which was not the result of a one spin move? Yes or no? (19)____________________. The answ is on page 2, line 27, word 3. This is thought provoking. Even tho : we take two spins, the result is always a move of 0. 1, 2, 3, 4, 5, 6, 7, 8, or 9 spaces. When a game has this characteristic, we say that closure holds. Saying that the game has closure would mean that combining numbers for two spin moves does not introduce any elements which we didn't already have for one spin moves.

Let's take a look at three spin moves. Suppose the results of three spins were 7, 6, and 8. The result could be found this way, thinking about the first two spins as a two spin move.



The player would make a move of 1 space.

The result could be found by thinking about the second and third spins as a two spin move.

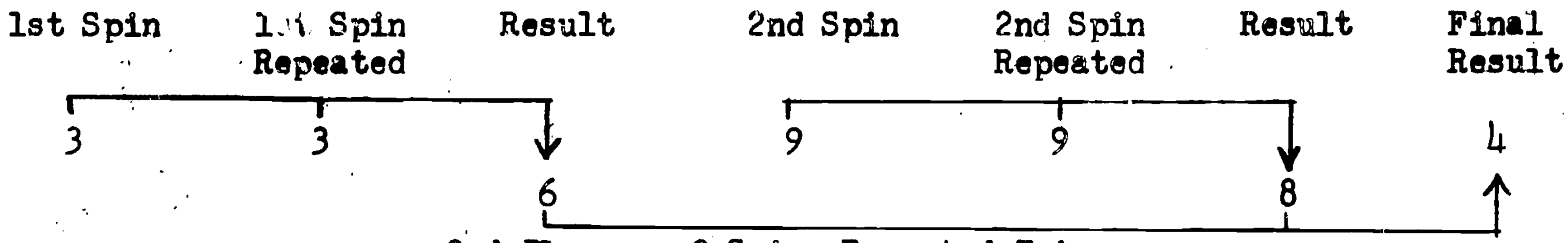


The player would again make a move of 1 space. That's interesting! The result is the same. Do you suppose that usually happens? Try some three spins.

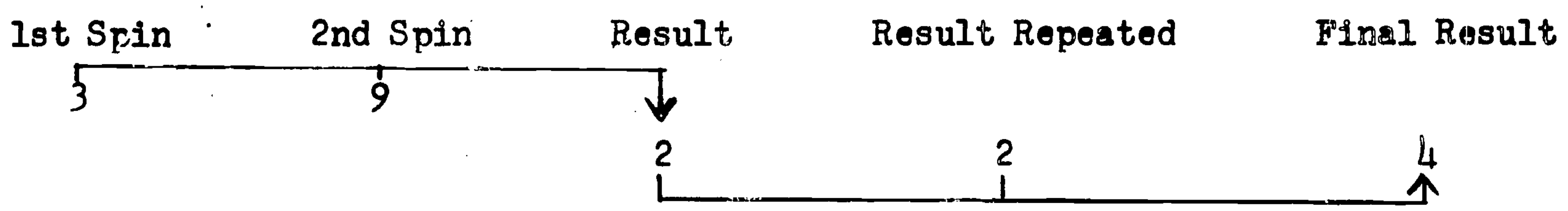
Some people describe this particular characteristic of three spin moves as the associative characteristic since it seems to associate two of the three in some sort of an arrangement. Others simply call it an arranging game rule. Either of these is a perfectly good name. For the sake of consistency among the various games which we discuss and describe, it would be useful to settle on one name. Let us agree to call it the <u>arranging</u> game rule.

Now recall those special cards with the unusual directions in the blue pack which say - "1 spin repeated twice followed by another spin repeated twice" and "2 spins repeated twice." Notice that there seems to be a curious characteristic of these moves. Look at the following example where the first player drew a card which instructed him to take a spin repeated twice followed by another spin repeated twice and he spun a 3 and a 9. A second player drew a card which said to take two spins repeated twice, and he spun a 3 and a 9.

1st Player - 1 Spin Repeated Twice Followed by Another Spin Repeated Twice



2nd Player - 2 Spins Repeated Twice



Both players end up with the same move. It doesn't seem to make any difference which card is chosen. Does this characteristic work for other spins? Try a few.

Did you find that this characteristic was always true? Yes or no? (20)__________

__________. The answer is on page 2, line 27, word 1. Either blue card always seems to result in the same number of moves when the two spins are the same. When such a characteristic holds in a game, we say that repeating is distributive over moves. This game rule involves more than combining numbers; it also involves repeating. Thus two operations are always involved with the distributive game rule.

We have found that there are six game rules which appear to hold for the game of Sums. List them.

(21)______________________________

(22)______________________________

(23)______________________________

(24)______________________________

(25)______________________________

(26)______________________________

If you can't remember the names, look back through the session for the underlined words.

MATERIALS FOR SESSION IV

Packet A Game of Sums

1 game board
spinner (0-9) and pieces (2)

Pack of 15 cards in "Number of Spins" pack labelled as follows:

Blue	3 each	"1 spin"
	3 each	"2 spins"
	3 each	"3 spins"
	2 each	"1 spin repeated twice"
	2 each	"1 spin repeated twice followed by another spin repeated twice"
	2 each	"2 spins repeated twice"

Pack of 15 cards in "Special" pack labelled as follows:

Yellow

Hang from the Tree of Ambiguity
Enjoy Swimming in the Land of Clear Water
Get Lost in the Castle of Confusion
Advance the Inverse of 9
Retreat the Inverse of 8
Advance the Inverse of 5
Retreat the Inverse of 4
Advance the Inverse of 3
Advance the Result of 7 and 8
Retreat the Result of 6 and 9
Advance the Result of 4, 7, and 9
Retreat the Result of 2, 5, and 8
Advance the Inverse of 1
Advance the Identity
Advance the Inverse of the move you have just made

Second Experimental Edition

SESSION V

THE GAME OF FLIP THE CHIP

OBJECTIVES:

At the end of this session the learner should be able to:

(1) identify and name examples of each of the game rules given the game, elements, and operations of the game.

(2) demonstrate each of the game rules using the elements and operations from a given game.

(3) construct data which support the presence or absence of a given game rule for a particular game.

ACTIVITY ONE

We are going to play the game of Flip The Chip. You say you do not know how to play this game? Would it help to know the elements and operations of the game? Open packet B.

This game is played with two players; one sitting to the left of the other. Each player manipulates one piece, a chip which is white on one side and brown on the opposite side. The two players flip their chips together. The elements of the game are the flipped chips which of course will have either a brown or a white side which are showing. The operation of the game involves a consideration of the pattern formed by the two flipped chips. The object of the game is to win ten points before your opponent. Are you ready to start playing? No, what's the matter? Oh, I see. You don't know how to win a point. Here is the way you can win points. If two brown chips are showing at the end of a turn, then both players win a point. If two white chips are showing, then neither player wins a point. If the chips show different colors and the right player has a brown chip showing, then the player on the left wins a point. If the chips show different colors and the right player has a white chip showing, then the player on the right wins a point.

If you have any trouble while playing, you may refer to the table below.

Table I

Left Player	Right Player	Who Wins Point
White	White	No One
Brown	Brown	Both
White	Brown	Left Player
Brown	White	Right Player

Now go ahead and play until one player wins the game. Have you made any observations about which player had the advantage in this game? (1)______________

You're absolutely correct if you think that you have a 50/50 chance of winning a point.

We can say that at the end of each turn, a resultant chip was determined by observing the color of the chips which had just been flipped. The "resultant chip" will show the same color as the color of the chip of the player who won the point. Remember, when both players had a white chip showing, neither player won a point.

Now, can you name the color of the resultant chip? (2)______________________________

The resultant chip had to be brown! When both players had brown chips showing, both players won a point. You can easily name the color of the resultant chip.

(3)__________________________. Of course, it was brown. When the player on the right had a brown chip showing, and the player on the left had a white chip showing, we saw that the left player always won a point. Therefore, the resultant chip was white.

Now you try the fourth possibility. The player on the right has a white chip showing; the player on the left has a brown chip showing. We observed that the right player won a point. Now you name the color of the resultant chip.

(4)__________________________________. Very good! The resultant chip must be white.

Let's summarize our observations in a table.

Table II

Left Chip	Right Chip	Resultant Chip
White	White	Brown
Brown	Brown	Brown
White	Brown	White
Brown	White	White

Observations of the above patterns in Table II should enable you to identify certain generalizations. For example, if the right chip is brown, what color is the resultant chip? Brown, white, the same color as the left chip, the color different from the left chip, or can't decide? (5)____

You say, you can't decide? Notice that if the right chip is brown, the resultant chip is the same color as the left chip.

If the right chip is white, then what color is the resultant chip? Brown, white, the same color as the left chip, the color different from that of the left chip, or can't decide? (6)____

Don't be bashful. You can observe that if the right chip is white, then the color of the resultant chip is different from that of the left chip.

Now consider the task of identifying the color of the left, right, or resultant chip given information about two of the three chips. But first, let's consider certain questions related to our previous observations.

What is the color of the resultant chip when the right chip is brown? The resultant chip is the same color as the right chip, the resultant chip is the same color as the left chip, the resultant chip is not the same color as the right chip, or the resultant chip is not the same color as the left chip?

(7)____

____

You may find it helpful to review the data presented in Table II.

What is the color of the resultant chip when the right chip is white? Again, it may be helpful to review the data from Table II.

(8)____

____

Having consolidated some of your observations about the left chip, the right chip, and the resultant chip, consider Table III. Complete as many of the patterns as you can.

Table III

Color of Left Chip	Color of Right Chip	Color of Resultant Chip
brown	brown	______
white	brown	______
brown	white	______
white	white	______
______	brown	brown
______	white	white
brown	______	brown
brown	______	white

The acceptable responses to the pattern in Table III reading from bottom to top in the table, are white, brown, brown, brown, brown, white, white, brown.

Since there are only two colors for the chips, the description of what results if the left and right are identified and named can now be fully described. The characteristics of various resultant chips can be summarized by saying:

(1) If the right chip is brown, then the resultant chip is the

(same, opposite) (9)_______________ color as the left chip.

(2) If the right chip is white, then the resultant chip is the

(same, opposite) (10)_______________ color as the left chip.

The acceptable responses are opposite for (2) and same for (1).

ACTIVITY TWO

With the set of elements for the game of Flip The Chip and the method of operating with these elements identified, there is a subsequent task to set for oneself. The task is one of investigating which, if any, of the game rules hold for the game.

First, perhaps you should attempt to recall from Session IV as many of the game rules which we have identified and named as you can.

(11)_______________ (14)_______________

(12)_______________ (15)_______________

(13)_______________ (16)_______________

Your list should have included closure, reversibility, arranging, identity, inverse and distributivity.

Let's see which of these game rules do hold in the game of Flip The Chip. Recall that there are two spin moves in the game of Sums for which the player arrives at the same position he started. What did we name the game rule illustrated by these two spin moves? (17)_______________. Yes, of course, that was was our inverse game rule. Was there also a one spin move which accomplished the same thing? Was it a 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 move? (18)_______________

The 0 move fits this requirement. It is a move that keeps a player in identically the same position. This is our identity game rule. In the game of Sums when we spin a 3 and then a 0 in a two spin move, the result of the move is the same as the result of a one spin move of 3.

Is there a right chip (color) which always makes the resultant chip identical to the left chip? Yes or no? (19)______________. Let's see if we can find one. Consider your performances with the tasks of Activity One. Try working with various chips such as the patterns provided in Table IV.

Table IV

Left Chip	Right Chip	Resultant Chip
brown	_____	brown
white	_____	white

The candidate for the identity is (20)______________. The acceptable response is brown. Hence, the identity game rule holds in the game of Flip The Chip.

Is the game closed in some way? Does the game rule of closure apply? Yes or no? (21)______________. How could you investigate this question? What does it mean to say the closure game rule applies? (22)______________________________

____

____

Are there any resultant chips which are not brown or white chips? Yes or no? (23)______________. Why, of course not! The only resultant chips are brown or white. So, the closure game rule applies.

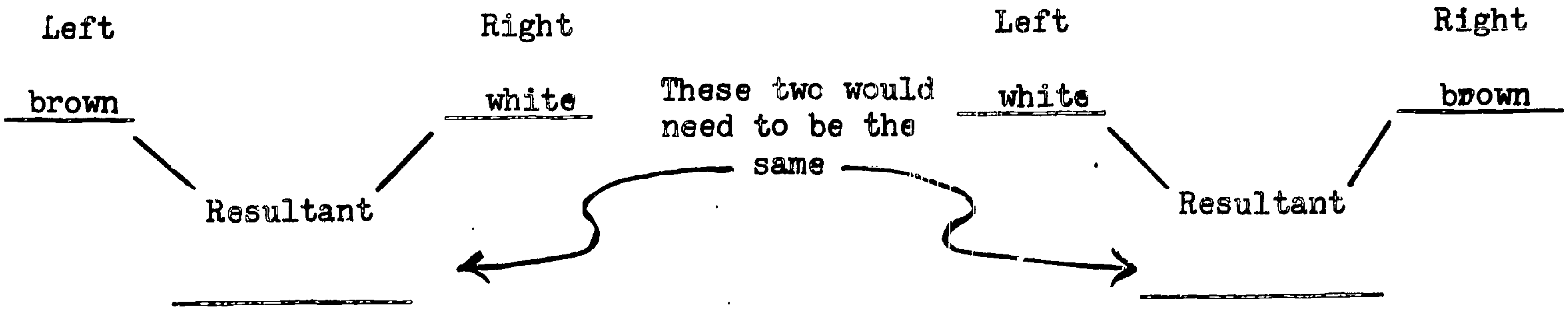
How about the reversibility game rule? What would have to be true for this game rule to hold? (24)____

____

____

If you said something like the following you are on the right track. For each color for a left chip and each color for a right chip the resultant chip's color would have to be the same if we reversed the left and right chips.

So, this would need to be the case:



Are they the same in this case? Yes or no? (25)________________. What is the resultant chip in each case? (26)_____________________. The correct answer is a white chip. Try to construct one pair with colors reversed which does <u>not have the same resultant chip</u>.

Were you able to find a pattern for which the reverse of the pattern gives a different result? Yes or no? (27)_______________. Since your answer to this question is no, what can you say about the reversibility game rule with some degree of confidence? Reversibility holds or reversibility does not hold? (28)_______________________________________. The acceptable response is reversibility holds.

There is a systematic way you could have investigated all the possibilities for various color arrangements. Construct as many of the different color arrangements as you can in Table V. The first example we tried is already included.

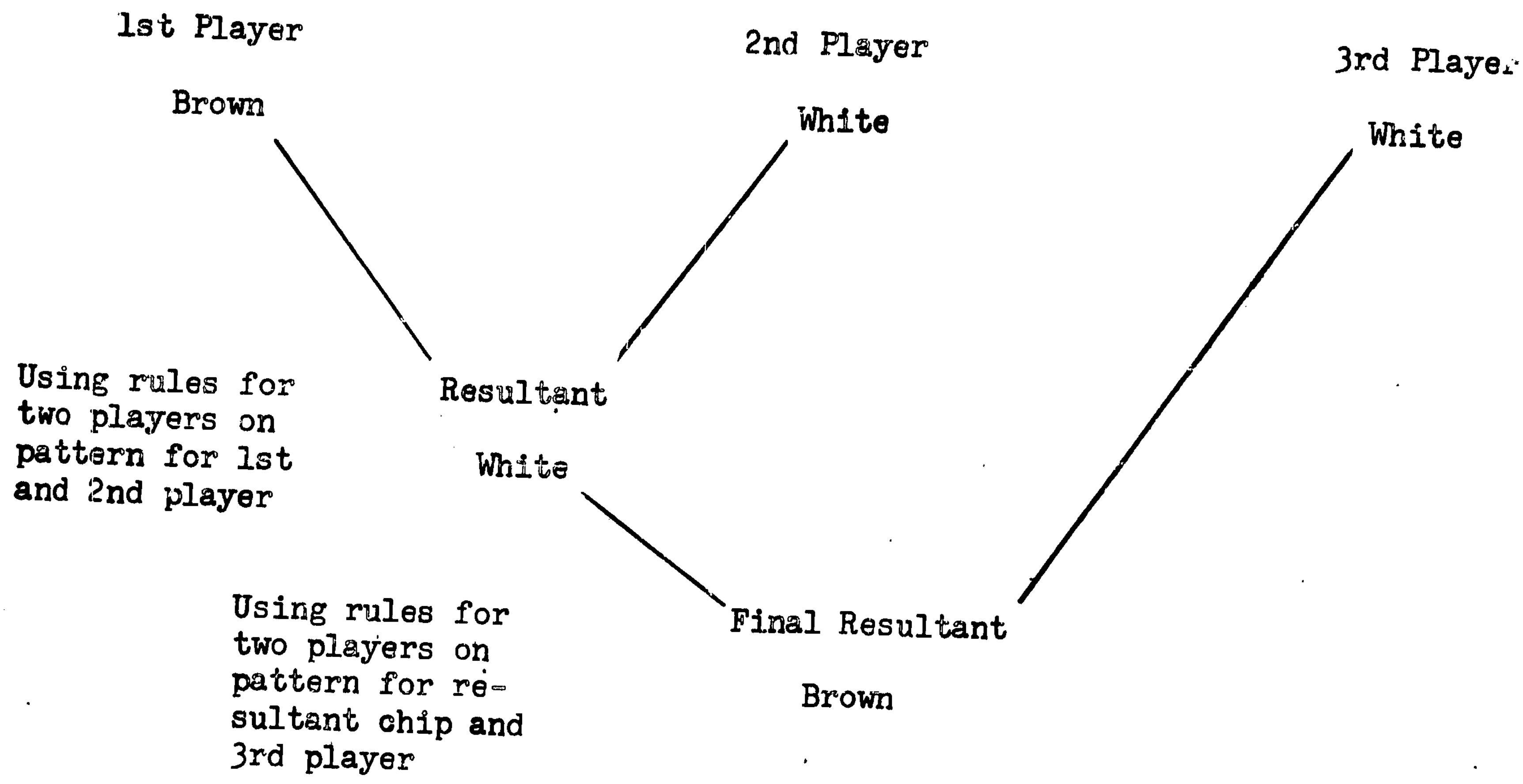
Table V

Pattern			Reverse of Pattern		
Left Color	Right Color	Resultant Color	Left Color	Right Color	Resultant Color
Brown	White	White	White	Brown	White

There are four possible arrangements which you should have examined. Now you can conclude without any reservations that the reversibility game rule does hold in the game of Flip The Chip.

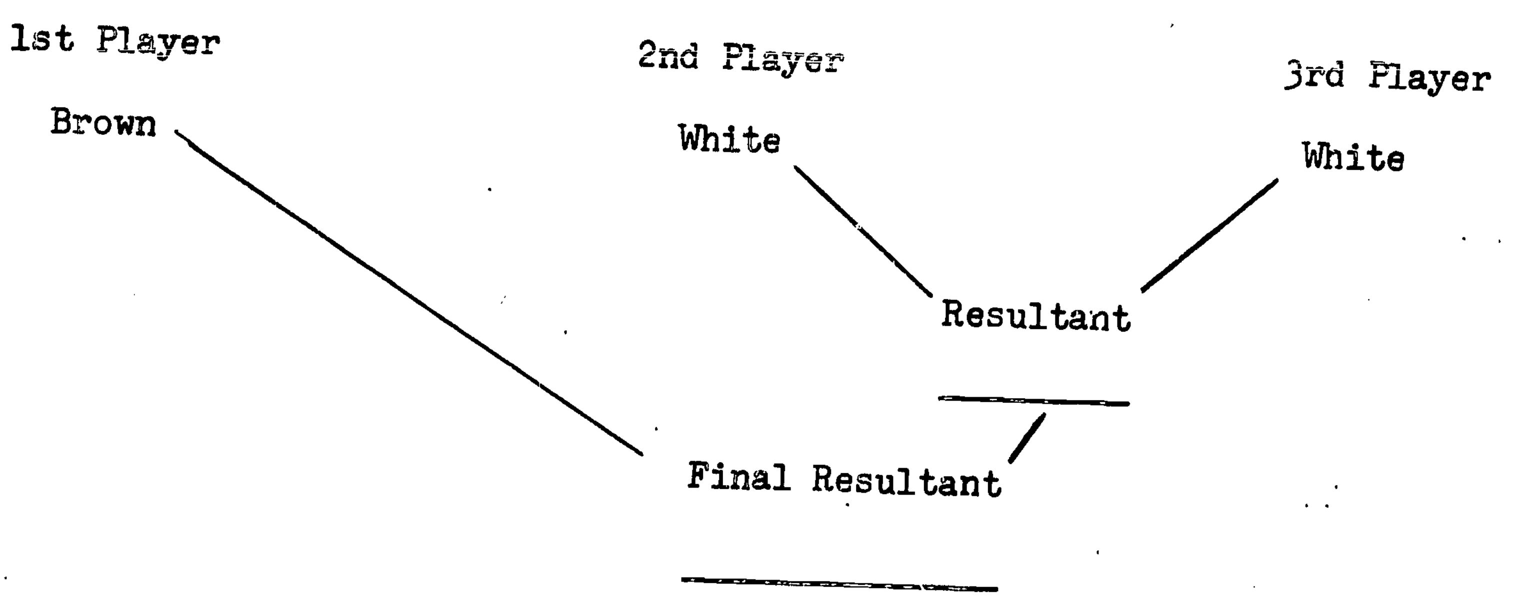
Now turn your attention to the arranging game rule. Does it hold in this game? How can you investigate it? We haven't constructed any rules for having three people play the game. Let's try it now that we are familar with the game for two.

If we have three players, they will sit in a row and all flip their chips at the same time. We will let the final resultant chip determine which players get points. A player will get a point if his chip is the same color as the final resultant chip. But how will we determine the color of the final resultant chip? We will observe the pattern of the chips for the first two players and use the resultant chip which the rules for two players prescribed. Then we will look at the pattern of the resultant chip and the third player's chip once again using the rules set forth for two players. Here is an example. Suppose the three players flipped the following pattern of chips. This is the way the final resultant chip would be determined.



Naturally, the 1st player is the only one who wins a point since he has a chip the same color as the final resultant chip.

Now does the arranging game rule hold? Could we find the result for the 2nd and 3rd player first? Try filling in the following pattern.



The final resultant is the same as when we found the result for the 1st and 2nd player first. The final resultant chip was brown. At least the arranging game rule works for this particular pattern of chips. There is a systematic way you could investigate all the color patterns in order to see if the final resultant chip is the same color regardless of whether the 1st and 2nd players or the 2nd and 3rd players are arranged together first.

Construct as many of the different color arrangements as you can in Table VI. The example we tried is already included.

Table VI

Arranged with 1st and 2nd players first				Arranged with 2nd and 3rd players first			
(1st Player color	2nd Player color)	3rd Player color	Final Resultant color	1st Player color	(2nd Player color	3rd Player color)	Final Resultant color
Brown	White	White	Brown	Brown	White	White	Brown

What decision did you reach? Does the arranging game rule hold in the game of Flip The Chip? Yes or no? (29)________________. The acceptable response is yes since you did not find any examples for whic. the arranging game rule pattern was not true. If you checked all eight possible examples and found them true, then your answer is final.

How about inverse? Is there an inverse for each of the left chips? If the right and left chips are inverses, the resultant chip will be the identity, a (30)__________________________ chip. We observed that the identity is the brown chip.

Let's try one example. Suppose that the left chip is white. What must the right chip be in order that the resultant chip be the identity, a brown chip? (31)________ _____________________. You are correct if you said that the right chip should be a white chip. Now we have an inverse when the left chip is white.

But in order for the inverse game rule to hold, each chip must have an inverse. Consider the partial data presented in Table VII and try to supply the missing data.

Table VII

Left Chip	Right Chip	Resultant Chip
brown	________	brown
white	________	brown
________	white	brown
________	brown	brown

What did you decide about the existence of inverses for every left chip? Does the inverse game rule hold? Yes or no? (32)____________________. Excellent! The acceptable response is yes.

Since the distributive game rule requires two ways of operating with or manipulating objects and there has been only one way described up to now, it does not make sense to explore this game rule. And so, we end up with which game rules holding in Flip The Chip? (33)____________________

____

And which not holding? (34)____________________

____

Second Experimental Edition

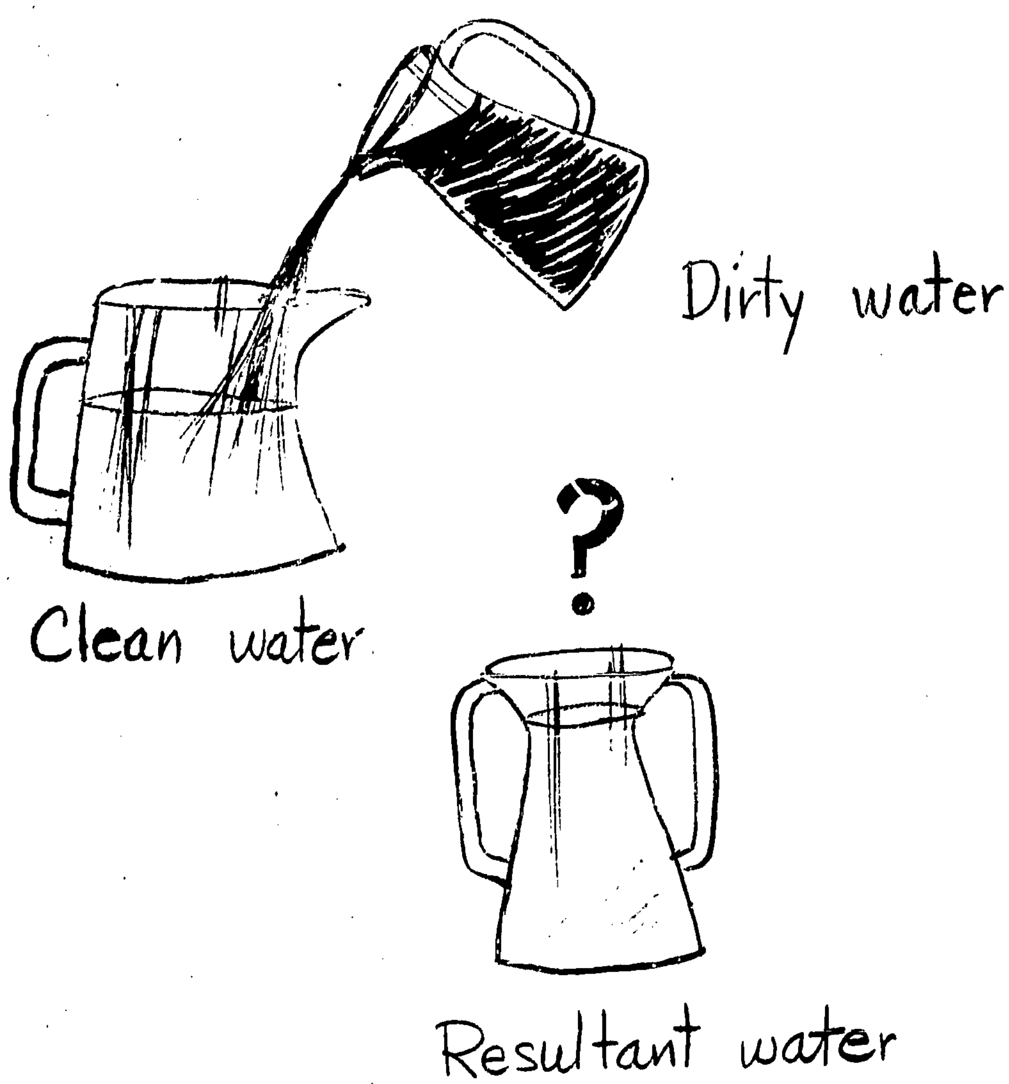
MATERIALS FOR SESSION V

Packet B

10 chips

Appraisal - Game Rules

We are going to begin today's session by talking about water. Imagine that there are four pitchers of water on the desk in front of you. Are you thirsty? Be careful! Two of the pitchers are filled with dirty water and the others are filled with clean water. We are going to investigate the process of pouring water from one pitcher into another pitcher and we are going to observe the resultant water to see if it is drinkable. We might note that there are four courses of action for us to take. We could pour clean water into clean water or we could pour clean water into dirty water. On the other hand, we could pour dirty water into dirty water or we could pour dirty water into clean water. Are we ready to consider a few of these?



Name the game rule which best describes each of the following actions:

(Objective 1) (1) The water that results from pouring dirty water into the clean water is the same as the water that results if we were to pour clean water into the dirty water.

Game rule: ______________________________

By the way, what is the resultant water like? ______________________________

You are right. Ugh! We would not want to drink this dirty water.

(Objective 1) (2) If we pour water from one pitcher into any other pitcher our resultant water is always either clean or dirty water. Notice that we can never get lemonade as a result!

Game rule: ______________________________

(Objective 1) (3) The combined actions of pouring clean water into dirty water and then pouring this result into clean water is the same as the combined actions of pouring clean water into the water that results from pouring dirty water into clean water. In both cases the resultant water is dirty!

Game rule: ______________________________

(Objective 2) (4) Using the pitchers of water, describe how you would illustrate the identity game rule.

____

____

____

(Objective 3) (5-9) **Determine which of the game rules hold for the pouring of water (clean or dirty) from one pitcher** into another. For each decision which you make, describe the data which leads you to your decision. Remember every game rule does not necessarily have to hold. You might find it helpful to look at Table I.

Table I

Pour <u>from</u> this pitcher. Water is:	Pour <u>into</u> this pitcher. Water is:	The <u>resultant</u> water is:
Clean	Clean	Clean
Dirty	Dirty	Dirty
Clean	Dirty	Dirty
Dirty	Clean	Dirty

Closure: ______________________________

____

____

____

Reversibility: ______________________________

____

____

____

Arranging: ______________________________

____

____

____

Identity: ______________________________

____

____

____

Inverse: ______________________________

____

____

____

Second Experimental Edition

SESSION VI

EXPANDED NOTATION - ADDING WHOLE NUMBERS

OBJECTIVES:

At the end of this session the learner should be able to:

(1) demonstrate each step of the expanded notation algorithm for constructing the sum of any two whole numbers of two or more digits as they would be carried out by a machine.

(2) construct a convincing explanation that appeals to observations based on a physical situation for each step in the expanded notation algorithm for constructing the sum of any two whole numbers of two or more digits.

(3) construct an explanation that appeals to agreed-upon game rules for each step in the expanded notation algorithm for constructing the sum of any two whole numbers of two or more digits.

ACTIVITY ONE

In today's session, we will investigate an old process used to construct the sum of whole numbers of two or more digits. This process is closely related to our familiar base ten number system.

In front of you, you should have packet A. Open it now and place the contents of bag 1 to your left and the contents of bag 2 to your right. How many objects, let's call them chips, are in the pile to your left?

(1)______________. How many in the pile to your right? (2)______________.

There should be 24 in the left pile and 18 in the right pile. Now, as you might expect, our process is one way that we can arrive at the answer to the question "How many chips were in packet A?"

You could easily count the twenty-four chips and then continue counting for eighteen more units to find the answer to this question, but we want a process that will be easy for combining any whole numbers, even quite large ones, and counting could be a trifle laborious. Let's agree at the start of the process to define a group of ten of our chips as one stack. This will make combining large numbers a good deal easier, as you will see.

Now that we have this new definition, let's use it. Can you express your pile of chips which has twenty-four chips in it in terms of stacks? Arrange your chips so they reflect the new term, stacks. What does your

original pile of twenty-four chips look like now? (3) __________________

____

____.

Since you all know that twenty-four has two tens in it, you should have no trouble making two stacks and then having four single chips left over. Now go on to the pile of eighteen chips. Arrange these chips using the idea of stacks. How did

this pile end up? (4)____

____

Fine! Since eighteen has only one ten in it, you could form only one stack and so had eight chips left over.

Now that we have our chips in groups reflecting our new term, we can proceed on our way to finding out how many chips there are. We said that we formed

(5)____________ stack (s) from the original pile of twenty-four chips and (6) ____

____________ stack (s) from the original pile of eighteen chips. You should have no problem remembering two stacks in twenty-four and one in eighteen. Now look at the chips in front of you. Move the stacks together. How many stacks are there?

(7)________________. You have formed, so far, a total of three stacks. How many

single chips are there that are not part of any stack? (8)______________. You can easily see that you have four single chips left from the original pile of twenty-four chips and eight single chips from the original pile of eighteen chips. None of these chips are in a stack. So we have twelve chips that are not in stacks. Can you form any more stacks out of these "extra" chips? Yes or no?

(9)____________. We're not trying to trick you, certainly you can. How many?

(10)____________________. No problem here, either, is there? One more stack

can be formed out of the twelve "extra" chips. Now are there any chips that are

still "extra"? Yes or no, if yes, how many? (11)______________________________

Since we agreed that one stack had ten chips in it and not twelve, there are two chips that are still not in any stack.

Let's now see where we stand. We had two stacks in one place and one stack in another and we combined them and had three stacks. At the same time we had twelve "extra" chips. Arrange the stacks together. How many stacks do you have in front

of you now? (12)________________. Right, one more than you had last time you

checked. You only had three last time and one more makes four. After the last time we looked at the number of stacks we had, we asked how many chips were not in stacks. It seems like a sensible thing to do here, too. How many "extra

chips did we say we say we had? (13)________________. Ah, yes! It was

only two, wasn't it. So our question is answered. There were (14)______________

stacks and (15)______________ chips in the packet A that we started with.

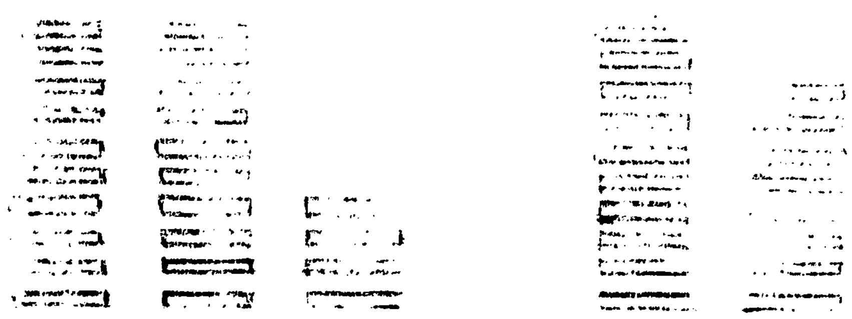
Yes, four stacks and two chips. If you were asked, by someone who didn't know our terms, what the answer to the problem was, what would you tell him? (16)______________ ____________________. Certainly, you would say forty-two chips.

Let's quickly summarize the steps we took.

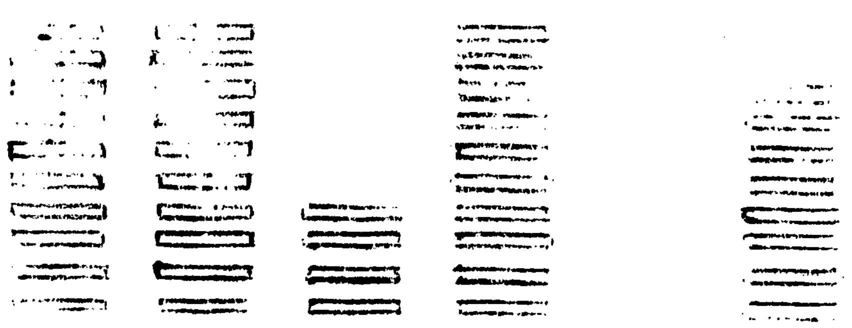
```
  2 stacks  and   4 chips
  1 stack   and   8 chips
 -------------------------
  3 stacks  and  12 chips

  3 stacks
  1 stack  and  2 chips
 -------------------------
  4 stacks and  2 chips
```

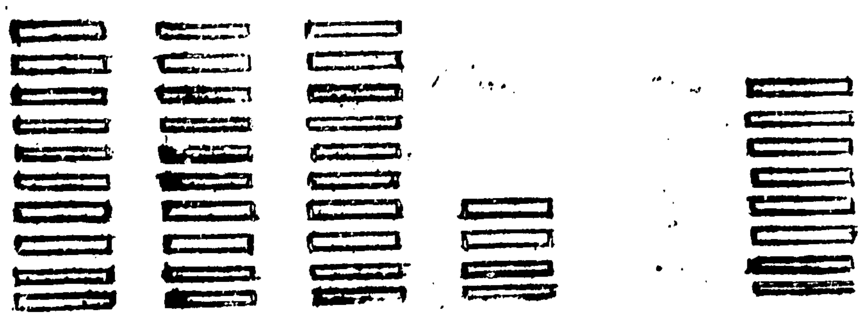
Just take a moment now to line-up your stacks and chips like this:



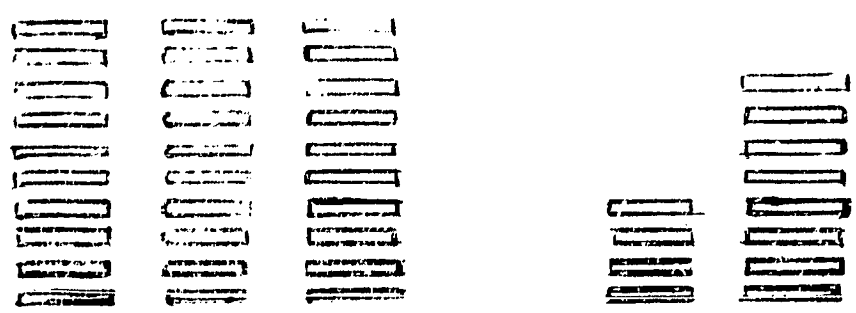
One thing we want to do is to get our stacks together, but like any game, we must follow the rules. Are there any game rules that allow us to reverse the positions of the single stack and the four chips? Yes or no, and if yes, name it? (17)________________ ______________________________. Sure, we have reversibility. But we skipped something. Remember the four chips really are arranged with the two stacks, and the one stack goes with the eight chips. What game rule may we use to take the four chips and one stack out of their arrangements and get them together before we reverse? (18)______________________________. Good! The arranging game rule will do it. Let's see what our line-up of the chips would look like after using the arranging game rule:



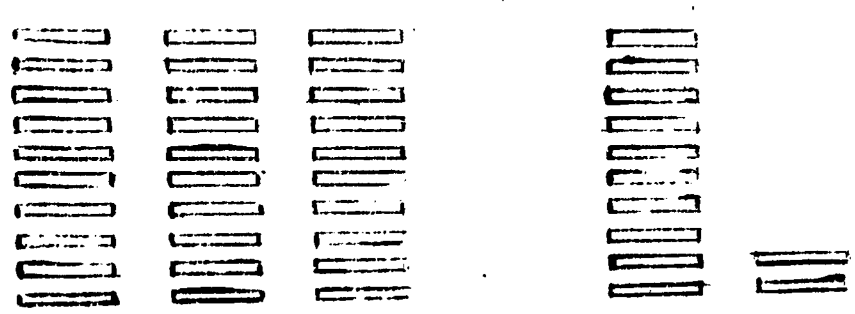
Then using the reversing game rule, the line up of chips would look something like this:



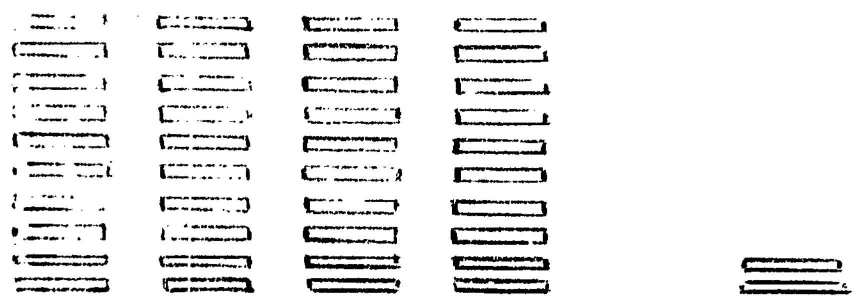
The next step involves arranging the four chips and eight chips together. What game rule are we using? (19)__________________________. Once again we are using the arranging game rule and our line-up will look like this:



But now the twelve chips can be formed into a stack and two "extra" chips. The line-up will look like this:



Once again we want to arrange all the stacks together. Using the arranging game rule we would have the following line-up:



ACTIVITY TWO

Let us now take a look at what this process looks like in regular numerical notation.

Step 1. $24 + 18 = (10 + 10 + 4) + (10 + 8)$

Step 2. $(10 + 10 + 4) + (10 + 8) = (10 + 10 + 4 + 10) + 8$

Step 3. $(10 + 10 + 4 + 10) + 8 = (10 + 10 + 10 + 4) + 8$

Step 4. $(10 + 10 + 10 + 4) + 8 = (10 + 10 + 10) + (4 + 8)$

Step 5. $(10 + 10 + 10) + (4 + 8) = (10 + 10 + 10) + 12$

Step 6. $(10 + 10 + 10) + 12 = (10 + 10 + 10) + (10 + 2)$

Step 7. $(10 + 10 + 10) + (10 + 2) = (10 + 10 + 10 + 10) + 2$

Step 8. $(10 + 10 + 10 + 10) + 2 = 40 + 2 = 42$

Where do you think our game rules were used in this process? Look at the eight steps above and ask yourself why each step was permitted.

Now, was closure used? Why or why not? (20) ____________________________

Closure means that we can find the results with our system. Here we begin with whole numbers, and our answer is a **whole number.** So closure was used.

Was the arranging rule involved in our eight steps above? Yes or no; if yes, in which step or steps? (21)______________________________

The answer to this question is quite clear since the arranging rule was used in three different steps, 2, 4, and 7.

Probably the next rule to be investigated should be the reversibility rule. Did we use it anywhere? If so, where? (22)____________________. Good for you, step three is correct.

If you look at the remaining four steps, 1, 5, 6, and 8 you will see that you can justify them merely by appealing to the renaming of numbers.

Notice that we have used our game rules as a convincing argument for the algorithm.

ACTIVITY THREE

Let's examine this expanded notation algorithm in its usual form when we don't use chips to explain it. The addition problem would usually be written like this:

$\begin{array}{r} 24 \\ \underline{18} \end{array}$; or this: 24 + 18. Since we are using the expanded notation algorithm, we will rewrite 24 in expanded notation. What will it look like? (23)____________________

____

18 would be written in the same way. The whole algorithm would look like one of these:

$$\begin{array}{l} \quad 24 \\ \quad \underline{18} \\ 2 \text{ tens} + 4 \\ \underline{1 \text{ ten} + 8} \\ 3 \text{ tens} + 12 \\ 3 \text{ tens} + 1 \text{ ten} + 2 \\ 4 \text{ tens} + 2 \\ \\ \quad 42 \end{array}$$

$$\begin{array}{l} 24 + 18 \\ 2 \text{ tens} + 4 + 1 \text{ ten} + 8 \\ 2 \text{ tens} + 1 \text{ ten} + 4 + 8 \\ 3 \text{ tens} + 12 \\ 3 \text{ tens} + 1 \text{ ten} + 2 \\ 4 \text{ tens} + 2 \\ \\ 42 \end{array}$$

The vertical procedure is probably easier to keep track of from the standpoint of bookkeeping. The horizontal procedure better fits an explanation based on the game rules.

PRACTICE EXERCISE

In case you should feel the need, at some future time, to review this session, here is a practice problem for you to work out.

Use the expanded notation algorithm to demonstrate the sum of 39 and 59. Use the idea of the stacks, or some other physical situation to explain adding 39 and 59. What game rules did you use to justify the different steps? (21) __________

______________________________________.

ANSWERS

For your Activity One, you should have something that looks like this:

39 =	3 stacks	and	9	chips
59 =	5 stacks	and	9	chips
	8 stacks	and	18	chips
	8 stacks			
	1 stack	and	8	chips
	9 stacks	and	8	chips

Your Activity Two should have the same steps as we had for our example today, but with different numbers. Since the steps are the same, your rules to justify each step will be the same as we used to justify ours. Refer to the material in Activity Two after the listing of the eight steps.

Second Experimental Edition

MATERIALS FOR SESSION VI

Packet A

bag 1 - 24 chips
bag 2 - 18 chips

Second Experimental Edition

SESSION VII

RULE OF COMPENSATION - ADDING WHOLE NUMBERS

OBJECTIVES:

At the end of this session the learner should be able to:

(1) demonstrate each step in the algorithm of compensation for constructing the sum of whole numbers as they would be carried out by a machine.

(2) construct an explanation based on a physical situation for the algorithm of compensation for constructing the sum of whole numbers.

(3) construct an explanation based on the game rules for the algorithm of compensation for constructing the sum of whole numbers.

Recall that there are many names for the same number, such as 5 = 3 + 2, 3 + 1 + 1, 4 + 1, 1 + 4, 2 + 2 + 1, and 2 + 3. This is quite a few names, so, for the present, let us limit the number of names by restricting ourselves to names with no more than two addends greater than zero. What other name can you think of for the number 4?

(1) ________________. Are there any other possibilities?

(2) ________________. If so, list them. (3) ____________________.

Did you list 3 + 1, 2 + 2, and 1 + 3? These are common acceptable responses.

ACTIVITY ONE

Place the materials from packet B on the table. Now before you is a group of physical objects. How many are in the total group?

(4) ________________. You should have 8 objects. Regroup these objects so they express one of the other names for 8. Write how you regroup them.

(5) ____________________________. What other possible regroupings are there? Regroup the objects and write your answers. (6) ____________

____.

All the possible regroupings are: 7 + 1, 6 + 2, 5 + 3, 4 + 4, 3 + 5, 2 + 6, and 1 + 7. Return the materials to packet B after you have finished.

ACTIVITY TWO

Since we are going to base the physical representation of our algorithm on regrouping of objects, let's try a second example of regrouping. Place the contents of packet C on the table. There are 16 objects this time. Regroup these 16 objects so they express another name for 16. Now write down all the possibilities as you identify them.

(7)____

____

Check to be sure your answers are complete. The correct regroupings are: 1 + 15, 2 + 14, 3 + 13, 4 + 12, 5 + 11, 6 + 10, 7 + 9, 8 + 8, 9 + 7, 10 + 6, 11 + 5, 12 + 4, 13 + 3, 14 + 2, and 15 + 1. Please replace the objects in packet C.

ACTIVITY THREE

Place the materials from packets D and E in separate groups on the table. How many objects are in each group? (8)_________________________. We want to construct the sum for these groups of 13 and 18 objects. The procedures we might ordinarily use to combine these groups are not necessarily the easiest ones. What way of regrouping these objects can you think of so as to make the combining easier? (9)____________________ It is usually quite easy to work with multiples of ten, isn't it? How could you regroup one of your groups so as to make the other group a multiple of ten? (10)_______________

______________________________. There seem to be two possibilities, (a) regroup the 13 group as 11 + 2 and then combine the 2 with the 18 group, or (b) regroup 18 as 11 + 7, and then combine the 7 with the 13. Since (a) makes it necessary to move only two objects, let's use it.

We now have a group of 20 objects. But what is the advantage of having 20?

(11)____

____

Don't you think that having that zero to work with is easier than working with the 8 you had before in 18. What is the total number of objects? (12)____________________ The answer is obviously 31. Please return the objects to the packets before going on!

ACTIVITY FOUR

Let's identify how this regrouping facilitates finding the sum with more than two addends. Place the contents of packet F, G, H, and I in separate groups on the table. One packet has 28 objects in it, one 172, one 94, and the last 79. You will notice that we have three types of objects, hundred bundles, ten bundles, and units. Using these objects in groups, let's work a problem step by step and note the procedures we go through.

The problem is 28 + 172 + 94 + 79. (We will use the horizontal form to make identifying the game rules easier.) How could you regroup to make some of the addends a multiple of ten? Don't hesitate to try it. It isn't really very hard. (13)________ ____. One possible solution would be 28 + (2 + 170) + 94 + (6 + 73) = (28 + 2) + 170 + (94 + 6) + 73. From where did the 2 come? (14)____________________. The 6? (15)____________________ You can see that the 2 came from the 172 and the 6 from the 79. After combining the 6 and 94, what would the problem be? (16)____ Did you get 30 + 170 + 100 + 73? Good!

Can we regroup this new set of addends to come up with some addends that are multiples of a hundred? Try it! What was you conclusion? Yes or no? (17)________ When you regrouped did you get the result (30 + 170) + 100 + 73? Fine! Why was the 30 grouped with the 170? (18)____ Naturally, the 30 was regrouped with the 170 to get an addend of a multiple of a hundred. What do you get when you regroup 30 and 170? (19)____________________ Naturally, you get 200. Now then, what will the final addends be? (20)__________ ______________________. Sure, they're 200 + 100 + 73. What result do you get when you carry through this addition? (21)____________________. Right! 373.

If you were skeptical as to the advantage of our regrouping when we used only two addends, the advantage should be clearer now. Compare the ease of seeing the final answer in the original problem: 28 + 172 + 94 + 79 with 200 + 100 + 73. You should agree that the answer is seen much easier with the zeroes than without. Please return the materials to the packets.

ACTIVITY FIVE

Let's now discuss the game rules. Did we use the game rule of closure? If so, how? (22)____

____

Of course, when we combine two groups of objects each of which has a whole number of objects in it, we will always get the result of a third group that also has a whole number of objects in it. Did we use arrangement any place? If so, where? (23)________

____

____

We used the arrangement game rule to put the 2, which was originally with the 170, with the 28, and also to put the 6, which was first with the 73, with the 94. This is shown in more detail in the example below.

$$
\begin{aligned}
28 + 172 + 94 + 79 &= 28 + (2 + 170) + 94 + (6 + 73) \\
&= (28 + 2) + 170 + (94 + 6) + 73 \\
&= 30 + 170 + 100 + 73 \\
&= (30 + 170) + 100 + 73 \\
&= 200 + 100 + 73 \\
&= 373
\end{aligned}
$$

Renaming is not a game rule, but did we use it? Yes or no? (24)____________

Where did we use renaming? (25)____

____

We renamed where we regrouped, so we renamed 172 as 170 and 2 and also we renamed 79 as 6 and 73. Did we use any other game rules? (26)____________________

Since we are not interested in whether or not any other game rules hold for this method of combining, but whether or not any others were actually used, the answer is no.

PRACTICE EXERCISE

1. Place the contents of packet J on the table. Physically regroup the 18 objects which are in front of you. Write down as many different regroupings as you can.

2. Demonstrate the algorithm of compensation as shown in the example in Activity Five using 276 + 454 + 82 + 69. Show all steps.

3. Identify each step in which you used a game rule in the demonstration of the algorithm in part two above. Name the game rules and list the steps in which they were used.

ANSWERS

1. 17 + 1, 16 + 2, 15 + 3, 14 + 4, 13 + 5, 12 + 6, 11 + 7, 10 + 8, 9 + 9, 8 + 10, 7 + 11, 6 + 12, 5 + 13, 4 + 14, 3 + 15, 2 + 16, and 1 + 17.

2. This is a possible procedure:

   276 + 454 + 82 + 69 =

   Step 1. 276 + (4 + 450) + (81 + 1) + 69 =

   Step 2. (276 + 4) + 450 + 81 + (1 + 69) =

   Step 3. 280 + 450 + 81 + 70 =

   Step 4. 280 + (20 + 430) + 81 + 70 =

   Step 5. (280 + 20) + 430 + (81 + 70) =

   Step 6. (280 + 20) + 430 + (70 + 81) =

   Step 7. (280 + 20) + (430 + 70) + 81 =

   Step 8. 300 + 500 + 81 =

   Step 9. 881

3. Arranging was used in order to get steps 2, 5, and 7.

   Reversibility was used in order to get step 6.

   Closure was actually used in order to write all the steps since closure is needed to be sure the sum of whole numbers is a whole number.

Second Experimental Edition

MATERIALS FOR SESSION VII

Packet B

8 chips

Packet C

16 chips

Packet D

13 chips

Packet E

18 chips

Packet F

2 bundles of 10 toothpicks + 8 single toothpicks

Packet G

1 bundle of 100 toothpicks, 7 bundles of 10 toothpicks, + 2 single toothpicks

Packet H

9 bundles of 10 toothpicks + 4 single toothpicks

Packet I

7 bundles of 10 toothpicks + 9 single toothpicks

Packet J

18 chips

Packet K

4 bundles of 100 toothpicks, 13 bundles of 10 toothpicks, + 13 single toothpicks

Appraisal - Adding Whole Numbers

Have you ever seen people solving arithmetic problems in ways that looked quite unusual to you? There are many unique procedures for the arithmetic we usually take for granted. The interesting and valuable fact, though, is that these procedures can all be explained in terms of the game rules. This is one reason these rules are so useful and important.

Consider the following algorithm:

~~1~~~~6~~~~8~~
~~9~~~~1~~
+ ~~4~~~~7~~
~~1~~~~9~~6
~~1~~~~1~~
~~2~~0
~~1~~
+ 3
306

The steps taken in performing the above algorithm might have looked like this:

1\.
168
91
+ 47

2\.
~~1~~68
91
+ 47
1

3\.
~~1~~~~6~~8
~~9~~1
+ ~~4~~7
19
1

4\.
~~1~~~~6~~8
~~9~~1
+ ~~4~~7
~~1~~~~9~~
~~1~~
2

5\.
~~1~~~~6~~~~8~~
~~9~~~~1~~
+ ~~4~~~~7~~
~~1~~~~9~~6
~~1~~1
2

6\.
~~1~~~~6~~~~8~~
~~9~~~~1~~
+ ~~4~~~~7~~
~~1~~~~9~~6
~~1~~~~1~~
20
1

7\.
~~1~~~~6~~~~8~~
~~9~~~~1~~
+ ~~4~~~~7~~
~~1~~~~9~~6
~~1~~~~1~~
~~2~~0
~~1~~
3

8\.
~~1~~~~6~~~~8~~
~~9~~~~1~~
+ ~~4~~~~7~~
~~1~~~~9~~6
~~1~~~~1~~
~~2~~0
~~1~~
3
306

This is called the scratch method for adding whole numbers.

Now look at this example:

1.	2.	3.	4.	5.	6.
49	49	49	49	49	49
+384	+384	+384	+384	+384	+384
3	32	32	323	323	323
	1	1	11	11	11
		4	4	43	43
					433

(Objective 1) (2) Demonstrate this example in horizontal form:

____

(Objective 1) (3) Now name the game rules that were used in going from one step to another in your horizontal form. This response can be recorded below or beside the steps in your horizontal example above.

____

(Objective 2) (1) Open packet K. Use the objects that you find in this packet to construct an explanation for the algorithm you have just seen. Draw pictures and/or write an explanation below to tell what your explanation is.

(Objective 1) (4) Solve the following problem, demonstrating the same algorithm (scratch method) as we used at the beginning of the appraisal:

$$\begin{array}{r} 328 \\ 169 \\ +\ 43 \\ \hline \end{array}$$

Second Experimental Edition

SESSION VIII

WHERE?

OBJECTIVES:

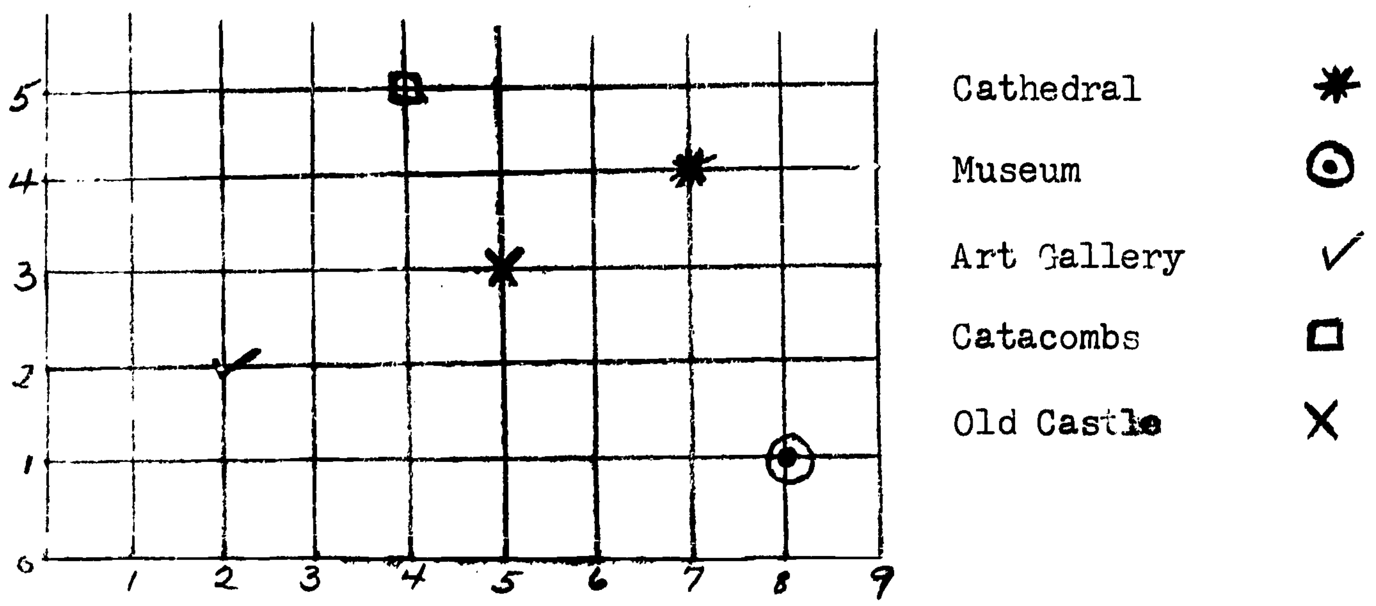
At the end of this session the learner should be able to:

(1) demonstrate the addition of ordered pairs of whole numbers.

(2) construct a physical explanation for the algorithm for adding ordered pairs of whole numbers.

(3) demonstrate some of the game rules for adding ordered pairs of whole numbers.

Last week I took a trip to a fascinating town with some rather unusual features. It is called "Where", and there are several outstanding sights to see there; but before we were able to see any of them, we had some problems to solve. The bus had let us out at the corner formed by the intersection of two streets, both of which seemed to be named "0". We hailed a cab and then took a look at the map we had been given. The map indicated the locations of some of the places to see, and we decided we would like to go first to the Cathedral; that's when our troubles began. The streets were laid out and labeled as shown below. What should we have told our driver?



How would you have directed someone to get to the Cathedral? (1)________________

Sure, I know; and we did try by-passing directions and said, "Take us to the Cathedral." The cabdriver answered, "Which one?" Of course, at that point we almost resorted to pointing to the map, but somehow that didn't seem fair. Anyway, it wouldn't help for the next time. The result was that everyone tried a different set of directions and no one understood anyone else. By this time we weren't even sure of East-West, so that suggestion went down the drain. Then came the gripping. Why didn't they letter or name one set of streets??? We couldn't even use going "ahead" or "left" since we didn't know how to say which was our position. However, in our discussion, the reason for having two sets of names for two sets of streets did become apparent.

After all, everyone would know how to go to 2nd and B Sts. Our driver, who had been taking all this in as he smoked and lounged, suddenly stuck his oar in and asked "What is clearer about telling a place by a letter and number rather than using 2 numbers? It's perfectly clear to us." What kind of rule or clue do you think they had that made it clear to them? (2)____

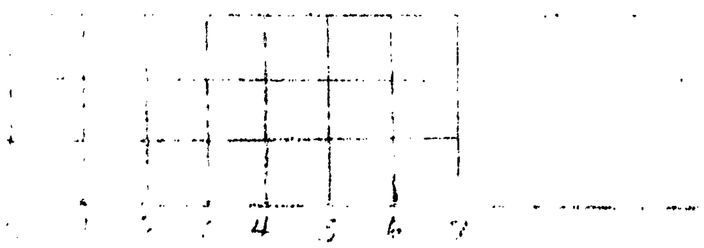
____

Come on, guess. And what would you have said to the driver? Everyone turned on him and demanded "How do you know which of the two streets with the same name to go to?" To our chagrin, he laughed till he wept and said, "Don't tell me no one remembered to tell you?" "Tell us? Tell us what?", we cried in fury. I guess he decided it would be safer to tell us the secret. What do you think he said? Any of a number of methods might work, so list a few and see if one agrees with the people of "Where". (3)________________

____

____

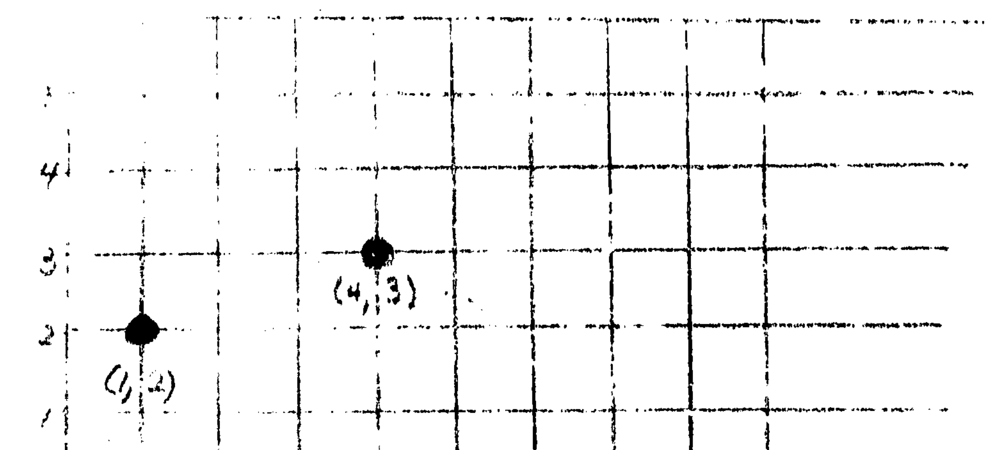
When they spoke of an intersection, the natives used the numbers of the two streets, but they always gave first the number designating the labels on the horizontal of the grid shown below. Sometimes this direction is called "Over", and the second direction "Up". So, for the remainder of our stay in "Where", when we saw something like "five, three" or (5,3), we knew we first went five streets in the direction of the arrow.



→ First number ——→ (5,)

Despite the wasted time, we had a lovely visit, and what's more we learned to receive and give directions for getting around "Where".

Just in case you decide to visit this delightful place, let's practice getting around there. First, this notation has a name, and since it involves a pair of numbers written in a specific order, what is more natural than "ordered pair". The ordered pairs (1,2) and (4,3) have been graphed below.



From the packet of graph papers, take Grid I and on it mark (3,1). Come on, don't hold back. Give it a try and, if necessary, guess. When you have marked it, look at Grid II to see if you agree. If your dot isn't in the same place, go to Grid III and try some more. If you were right, go on.

Using the ordered pair notation, let's take a few trips. This notation allows moving on streets only (no crossing empty lots!). Before we start, take our Grid IV. On the first trip, we'll start at (0,0) with destination (8,7)--but don't move yet, because we have to make a stop on the way, so let's go to (2,5), which can also be read as 2 "Over" and 5 "Up". You'll see this part of the trip marked on Grid IV. The second part of the trip to (8,7) will be written in the same way. What trip would

you take from (2,5) to get to (8,7)? (4)________________ "Over" and (5)________________ "Up". Do we agree? Did you get (6,2)? Good, you've earned the trip; starting where the marking ends at (2,5) on Grid IV, you put in the second part of the trip. We made this trip to (8,7) in two parts, but what if we didn't need to make a stop on the way? What single "Over and Up" trip would you take?

(6)________________. Was it (8,7)? Of course. Mark it on Grid IV.

You're doing so well, it's time for a solo. Using Grid IV and starting at (0,0) again, mark this two-part trip on the street plan: (4,3) and then (4,4).

And where did you end this time? (7)________________ Same place (8,7)?

Very good. We've only taken two trips with different stopovers but the same destination. Now look at the Grid and just think of the number of trips you could take in two parts and always end on (8,7). Quite a few, aren't there? And I've been wondering, did we travel any further, or perhaps less, in a two-part trip than a one-part trip to the

same place? (8)________________. If we look at the Grid, it shows that you travel the same number of streets over and the same number up whether we go there in the most direct manner or make a stopover on the way. There ought to be some big discovery we can make from this! Let's look again at the ordered pairs.

First trip: (2,5), then (6,2), to (8,7)

Second trip: (4,3), then (4,4), to (8,7)

Our markings on the grid and the ordered pairs seem to be saying the same thing. Take a crack at wording what you see. (9)________________________________

____

How does the following compare with what you said? (The sum of the "Overs" in the two-part trip equals the "Over" on the one-part trip, and the same is true of the "Ups"). No? You don't agree? Please, say it isn't so, but if you really think the statement is false, please go over the last couple of paragraphs. When this is clear, see if your agreement will include replacement of the word then with plus and the word to with equals?

This would give us (2,5) + (6,2) = (8,7) and (4,3) + (4,4) = (8,7). Has this been true for every example you have tried? Yes or no? (10)________________________

Then, for the time being, can we agree that for two ordered pairs of whole numbers:

(a,b) + (c,d) = (a + c, b + d)? Yes or no? (11)________________. If you would like to test more or just practice use Grid V to graph some of the trips listed there. Be sure to fill in the blanks for the trips and compare with the trip checks at the bottom of the grid AFTER you graph and write each trip.

In all of this exploration today what have we been working with? What elements?

(12)____

Sure, ordered pairs. What operation have we been performing? (13)________________

________________________. What, all this time and we're still on addition! So our objects or elements are ordered pairs and our operation is addition. When we add two or more of our elements will we always get another element? Or put another way - when we add two ordered pairs will we always get another pair? Yes or no? (14)____________.

It is true that we have not tried all pairs or carried out a formal proof, so we must hedge our answer and say that as far as our experience goes, we always get another ordered pair when we add two or more ordered pairs. Hence, we can say that the game rule of closure appears to hold for adding ordered pairs.

Now add (3,2) and (5,8). (15)____________________. Try it in reverse by adding (5,8) and (3,2). Does it matter in which order you do the addition? Yes or no? (16)________________. Try at least one more example with other pairs and check. (17)____

____

Which game rule are you testing? (18)____

The reversible game rule is being tested here. Can you find an example of addition of ordered pairs which is not reversible? Yes or no? (19)________________________

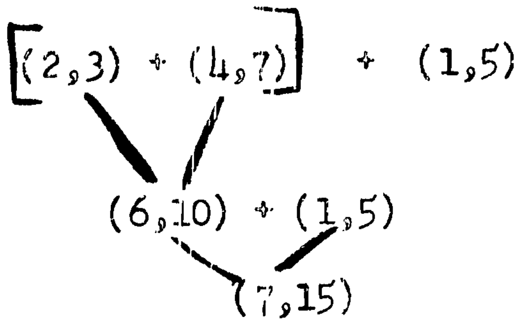
Since your answer is no, you probably feel that the reversibility game rule holds for adding ordered pairs. We do have a fairly good argument, but we cannot be sure that an example will not turn up sometime later which will force us to a different conclusion.

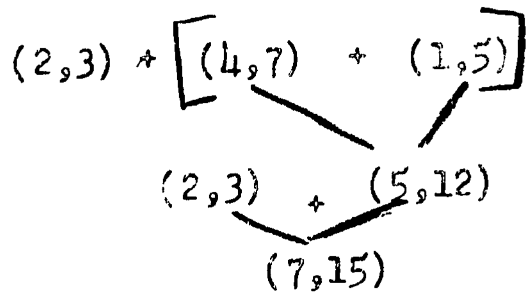
Try adding any three ordered pairs. Did the answer depend upon which two pairs you arranged together and added first? Yes or no? (20)________________. What support do you have for your answer? (21)____

____

____

Your examples probably look something like this:





What game rule are you testing? (22)____

Does this game rule hold for addition of ordered pairs? (23)____________________________

____

Did you place any restrictions on your conclusions. (24)______________________________________

____________________________________. If yes, which restrictions? (25)____________________________

____

You can always support your answers by trying some examples. However, our use of a few examples does not prove that it holds for all cases, so again it is best to add restrictions. Then let's say, until we find evidence to the contrary, the reversibility and arrangement game rules hold. In that case, we have constructed the sum of ordered pairs; we have constructed a convincing explanation of the addition using the physical representation of the grid; we have demonstrated some of the game rules for the operation of addition.

Congratulations!

Performance Tasks

(Using the algorithm for addition of ordered pairs of whole numbers.)

1) (20,12) + (72,91) =

2) (13,16) + (7,8) + (22,41)

3) (4132,271) + (2,3) + (7,30) + (21,420) =

4) Which "game rules" can we demonstrate in this addition?

5) Can you think of another way to physically represent this algorithm? If you can, construct an outline of the procedure.

Answers to Performance Tasks

1) (92,103)

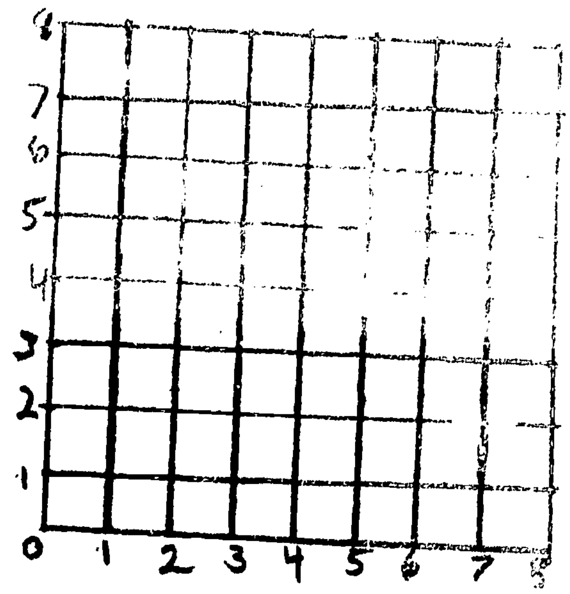
2) (42,65)

3) (4162,724)

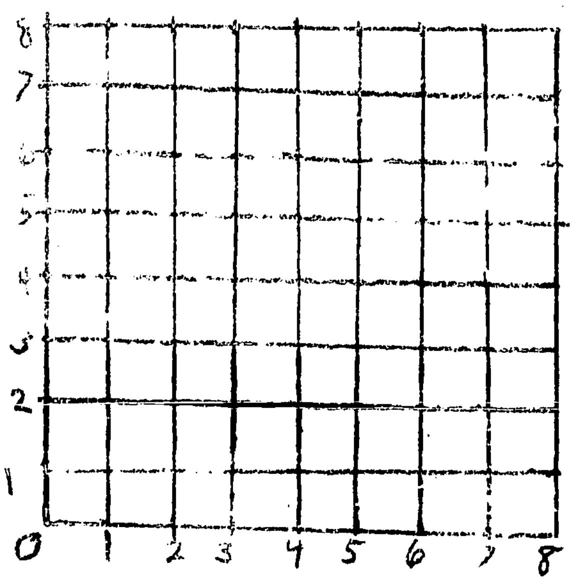
4) An additive identity exists; every element has an additive inverse; the "closure", "arrangement" and "reversibility" game rules apply.

5) We cannot state a "correct" answer for this item since there may be many answers. If you could not think of an answer, just keep in mind the grid idea which we used.

GRID I

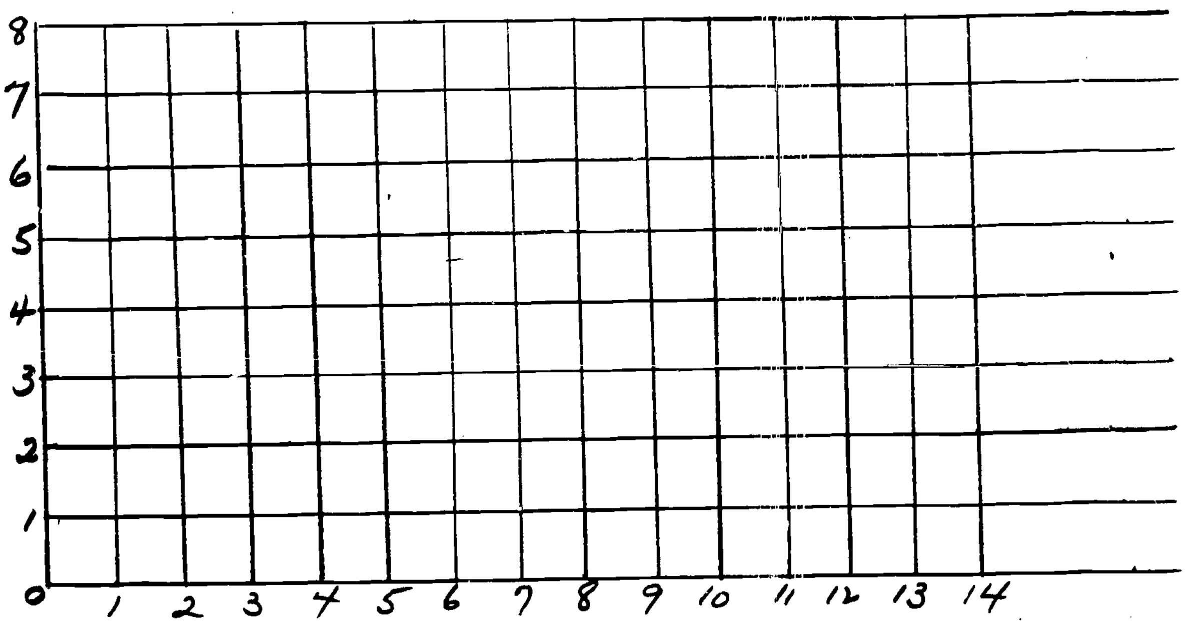


GRID II

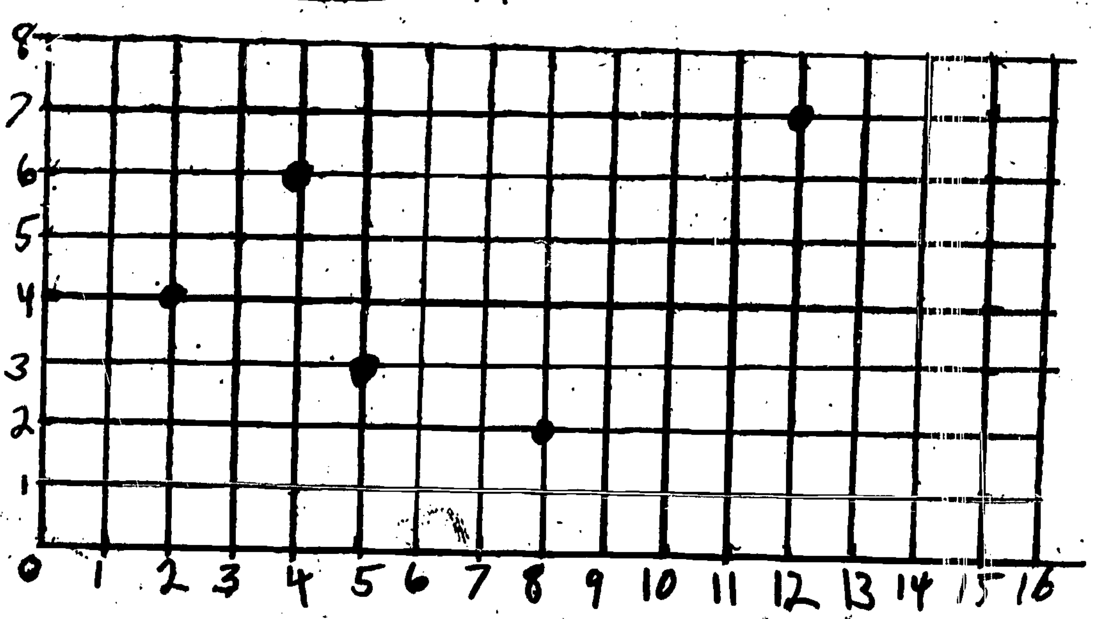


GRID III

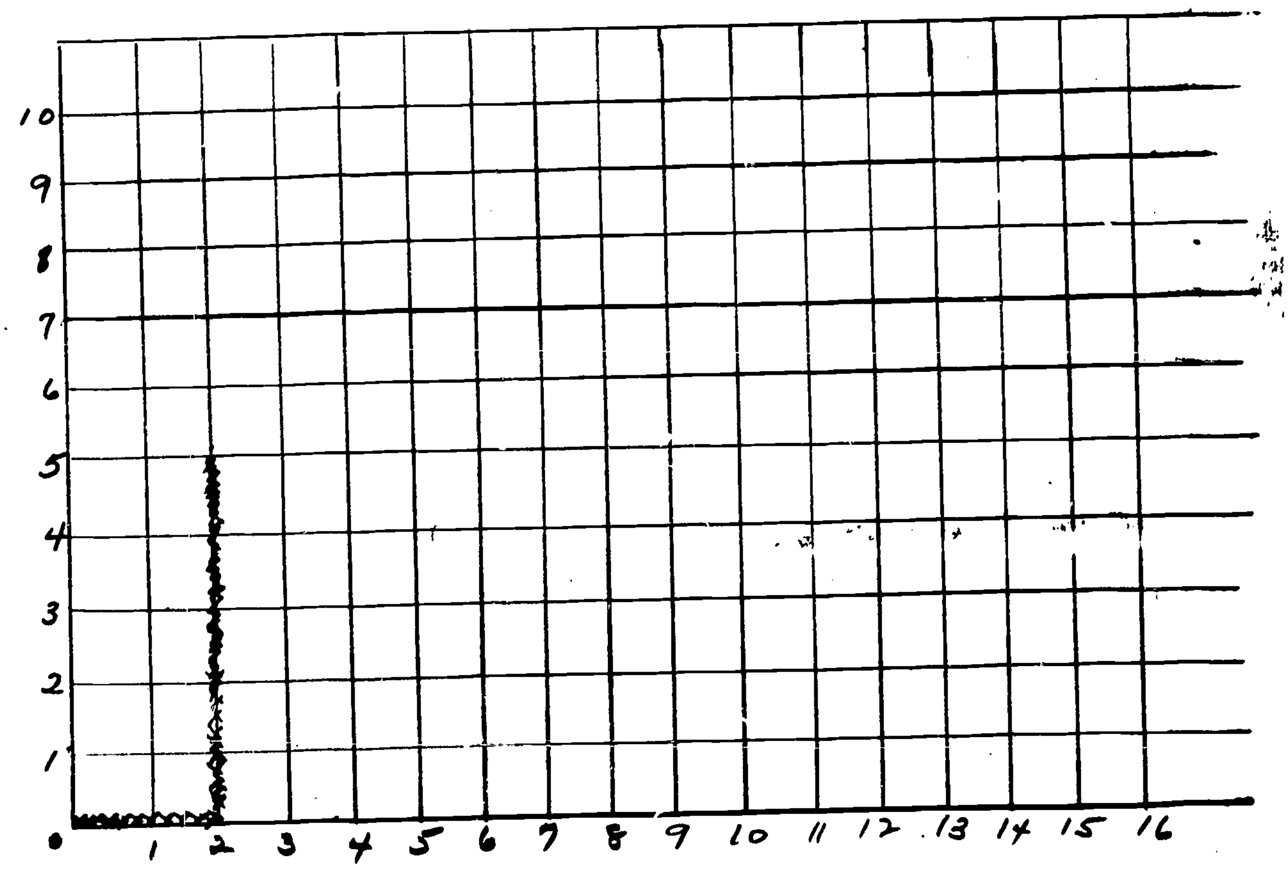
Remember: The first component of an ordered pair means "go to the right along the horizontal line". Go to 5 along the horizontal 0-line on the grid below; now go "up" from there 3 lines. You just plotted (5,3). Try another, (2,4). Want more? Okay: (12,7), (4,6), (8,2). Answers are on Grid III-A.



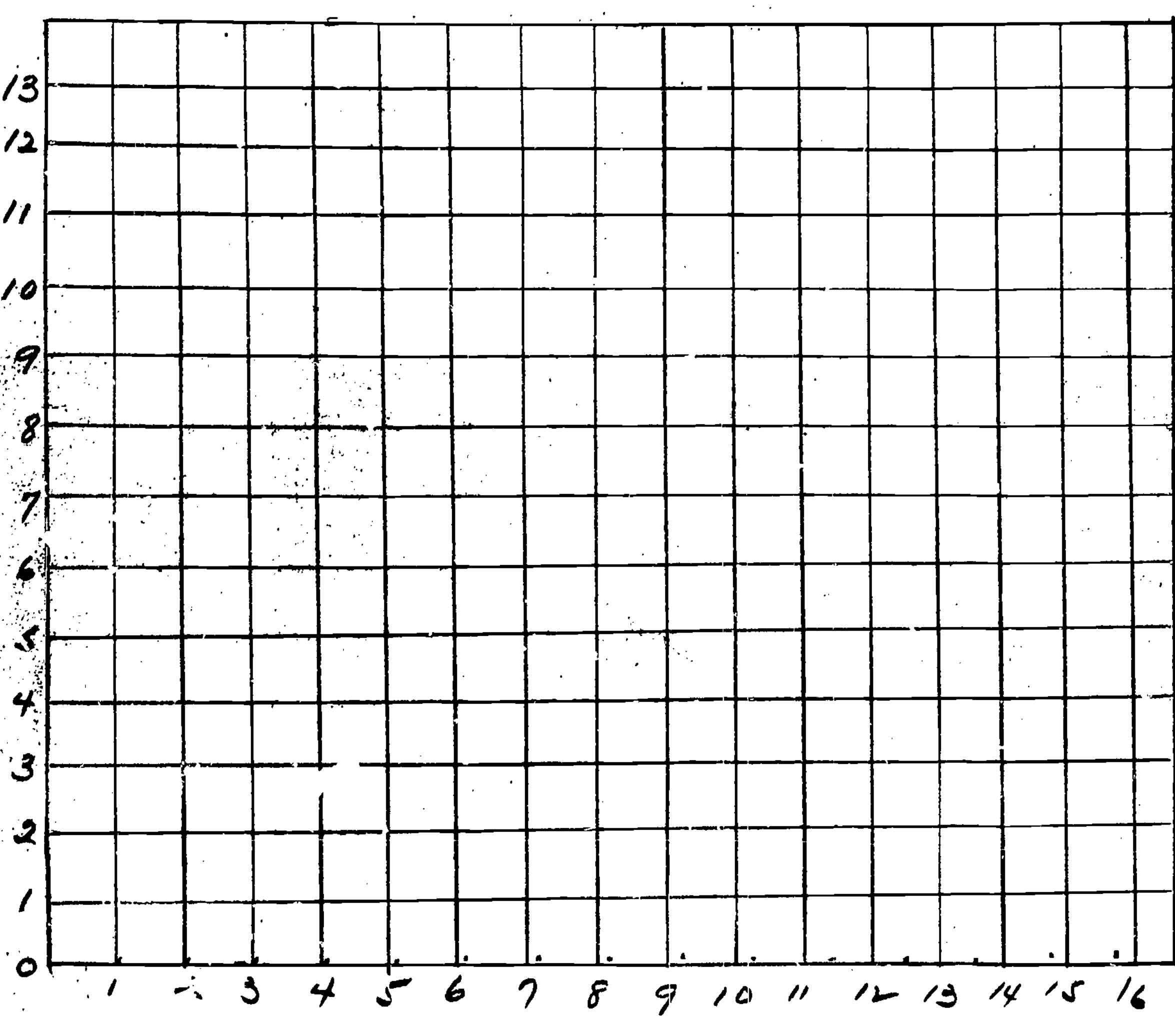
GRID III-A



GRID IV



GRID V Fill in blanks for the 6 trips to the right of the grid; graph each trip; check trip notation at bottom.



Trip notations

1. (2,1), (   ), (10,3)
2. (   ), (10,1), (10,3)
3. (8,4), (5,8), (   )
4. (6,6), (   ), (13,12)
5. (1,8), (11,1), (   )
6. (   ), (5,6), (12,9)

Check: 1. (2,1), (8,2), (10,3)
2. (0,2),(10,1), (10,3)
3. (8,4), (5,8), (13,12)
4. (6,6), (7,6), (13,12)
5. (1,8),(11,1), (12,9)
6. (7,3), (5,6), (12,9)

Second Experimental Edition

MATERIALS FOR SESSION VIII

6 grids attached to last page of Session.

Second Experimental Edition

SESSION IX

MORE WHERE?

OBJECTIVES:

At the end of this session the learner should be able to:

(1) demonstrate the addition of integers named by ordered pairs of whole numbers.

(2) construct physical explanation for the algorithm for adding integers named as ordered pairs of whole numbers.

(3) demonstrate the game rules for adding integers named as ordered pairs of whole numbers.

Last session we visited the town of "Where" and made some interesting discoveries about the streets and trips that could be made on them. Look at Grid VI and let's review briefly. If we wanted to make a trip to the X on

this grid, we could do it by naming an (1) ______________________

which would tell us how to get there if we started at the point of intersec-

tion (2) ____________________. You are remembering accurately if you said we would name an "ordered pair" and that our starting point would be (0,0).

What ordered pair would name the location of the Δ on Grid VI? (3) ________

________________. We should keep in mind that an ordered pair can be used

in two ways; it can name the parts of a trip or it can name a particular location on the grid or map. If you said the Δ location could be named by (10,5) you were correct.

Another thing we did with the trips last session was to combine two or more trips to get to some specific location and to study the various ways this could be done. For example, on Grid VI again a trip of (4,1) followed by a

trip of (1,3) would get you to which point on the grid? (4) ______________

______________. The correct point is the circled point on the grid--the ordered pair (5,4). We represented the total trip as (4,1) + (1,3) = (5,4)

and notice that (a,b) + (c,d) = (5) ______________________ for any case of adding ordered pairs or making two consecutive trips with the first one starting at (0,0). The last answer should have been (a + c, b + d). If you did not get this it might be well to get out Session VIII again and review the first few pages.

There are some other interesting characteristics of travelling in "Where" that can be brought out more clearly if we study the map or grid which we have been using to indicate our trip. Look at Grid VII. This is a grid of the town on which some diagonal lines have been drawn. We shall call these diagonal lines, "avenues", but they shall differ from ordinary avenues in that we shall not travel on them. We shall use them only to call our attention to the particular street intersections which they connect. Choose any one of the avenues; say perhaps the one through the point (2,0). Have you located it?

(6)________________. If not, you may notice that there is an arrow pointing to it along the lower edge of the grid. Is that the one you had already picked? Good! Write ordered pairs which name two of the points on the avenue.

(7)________________. You may have written (2,0), or (3,1), or (4,2), or (5,3), or (10,8), or any of many other choices. Now, use the two points (4,2) and (5,3) to take two trips; first, use one of the ordered pairs to take your first trip, then the other for your second trip as we have done previously by starting at (0,0). What is the ordered pair? (8)________________. What avenue did you land on? (9)________________. I'll bet I can tell you! I'll bet you landed on the avenue which goes through the point named by the ordered pair (9,5). Right? Now pick any other two points on this avenue through (2,0) and use them to make two more trips in the same fashion. (10)________________

________________. What avenue did you land on this time?

(11)________________. You may not believe it, but I'll bet I can still tell you. You landed on the avenue through the point (4,0) didn't you?

(12)________________. Remember I am just naming the avenue, not the specific point! Now, am I right? (13) ________________. Good! It's the same avenue as we previously landed on, isn't it? (14)________________.
Do you wonder how I guessed that since I didn't know the <u>particular</u> points <u>you</u> had named? No, it's not magic, and I didn't peek over your shoulder. Let's try some other cases. Consider the avenue through the point (3,2). Name any point on that avenue. Have you made your choice? Now name any point on the avenue through the point (7,4). (15)________________. Using the ordered pairs that name the points you have chosen, take the indicated trips and decide which avenue you land on. Remember that you may actually count off the trips on the grid or you may add the ordered pairs as we did last session. Which avenue did you land on? (16) ________________. I'll bet it's the same avenue that the location (10,6) is on. Is it? How could I predict your answer not knowing the specific points you started with? (17)________________

____

Do you still have some uncertainty as to my method? If so, let's look at another example. Consider the avenue through (6,1) and the avenue through (2,1). Name a point from each one. We could use (6,1) and (2,1) themselves. Take the trips or add the ordered pairs. (18)________________. Did you write (6,1) + (2,1) = (8,2)? Good! Try two other locations on these same avenues. Say (8,3) and (3,2). Add the trips or add the ordered pairs.

(19) ______________________. Is your answer the same as mine?

I got (11,5). Good! Now try two other locations on the same two avenues. Let's use (9,4) and (1,0). This time the sum or result of the trips is

(20) ______________________. I got (10,4). Did you? Good!

Now, what were the three answers? (21) ______________________

Right! We have (8,2), (11,5), and (10,4). Locate these three points on the grid if you haven't already done so. What do you notice about them?

(22) ______________________

____

Did you notice that all three answers lie on the same avenue? Good! Now do you see how I knew what your answers were in the previous examples? Can you make a statement about these results? (23) ______________________

____

____

You should have said something like the following: Any time you are allowed to name points from a given pair of avenues or a single avenue and add them the resulting locations or points will be on the same avenue, whatever the choice of ordered pairs. Let's see then; if we add ordered pairs on the avenue through (4,7) to ordered pairs on the avenue through (4,3), the result will be on the avenue through (24) ______________________

Did you give as an answer (8,10) or some other ordered pair on that same avenue? It is as though we were actually adding the avenues when we added the representative ordered pairs. We actually can think of it in that way. Do you see why the avenues seemed rather interesting to me?

Last session we added ordered pairs and discovered what seemed to be some appropriate game rules for the addition. What were these game rules?

(25) ______________________

____

Did you write down just three game rules? They were closure, reversibility and arranging. Consider the following examples. Write the name of the game rule which each example illustrates.

1. (8,6) + (3,2) = (11,8) = (3,2) + (8,6) (26) ______________________

2. [(3,1) + (4,2)] + (5,3) = (7,3) + (5,3) = (12,6)

   and (3,1) + [(4,2) + (5,3)] = (3,1) + (9,5) = (12,6)

   (27) ______________________

3. (3,5) + (6,1) = (9,6)
   and (a,b) + (c,d) = (a + c, b + d)
   for all ordered pairs (a,b), (c,d)
   and (a + c, b + d)

(28) ____________________

Did you write in your answers? If not be sure to do so before looking at my answers. The first one is an example of reversibility and the rule of closure. The second one illustrates arrangement and the rule of closure. The last one illustrates the rule of closure.

Do you think that these same game rules would hold if we were thinking of avenues instead of just the ordered pairs? (29) ____________________

How could we distinguish between adding ordered pairs which represent avenues and just adding ordered pairs? Well, we might make a special mark on the ordered pairs representing avenues. For example, we might say the ordered pair (3,2) represents the avenue when we write it this way: $(\overline{3,2})$. Now, if we put these marks over the ordered pairs which represent avenues, will the previous three examples hold for avenues as they did for ordered pairs?

1. $(\overline{8,6}) + (\overline{3,2}) = (\overline{11,8}) = (\overline{3,2}) + (\overline{8,6})$
2. $[(\overline{3,1}) + (\overline{4,2})] + (\overline{5,3}) = (\overline{3,1}) + [(\overline{4,2}) + (\overline{5,3})]$
3. $(\overline{a,b}) + (\overline{c,d}) = (\overline{a + c, b + d})$

Has the reasonableness of the answers changed? Yes or no? (30) ____________

No! It appears that everything still holds. Does this prove that closure, reversibility and arranging hold for the addition of avenues? (31) ____________

____________________. Well, we would have to say it does not really prove the fact, but it does give us evidence to make us confident that they probably hold. As a matter of fact, until someone proves they do not hold we shall accept them and use them as if they do hold.

Now, do we have any other interesting properties when we add these avenues? Get out Grid VII again. Consider the following examples and find the answers by taking the trips or by adding the ordered pairs and then locating the resulting avenue.

$(\overline{3,1}) + (\overline{2,2}) =$ (32) ____________________

$(\overline{5,2}) + (\overline{1,1}) =$ (33) ____________________

$(\overline{7,2}) + (\overline{5,5}) =$ (34) ____________________

Did you get the answers $(\overline{5,3})$, $(\overline{6,3})$, and $(\overline{12,7})$ respectively? Compare the avenue you started on in each case to the avenue you landed on. What seems to be true? (35) ____

____

Did you find yourself ending up on the same avenue as was named by your first ordered pair? In other words $(\overline{5,3})$ is the same avenue as $(\overline{3,1})$ and $(\overline{6,3})$ is the same avenue as (36) ____.

Did you write $(\overline{5,2})$? Good! Finally $(\overline{12,7})$ is the same avenue as (37) ________________________________. Yes, it is the same as the one we started with in that example, $(\overline{7,2})$. Now what about the avenues named by $(\overline{2,2})$, $(\overline{1,1})$ and $(\overline{5,5})$? (38) ____

Did you notice that these names were all for points on the same avenue? Good!

Let's summarize what we have seen in the last paragraph. We started with an avenue and we landed on (39) ____.

Yes, we landed on the same avenue. In each case we were adding the same avenue to the avenue we started with. This avenue could be identified by the fact that the first and second numbers in the ordered pair were (40) ________________________________. Yes, they were the same. We could put these two facts together and say: (41) ____

____

____

____.

In the above statement you might have said: "Whenever we add the avenue which is represented by ordered pairs with the same first and second elements to any other avenue the result is the same avenue." What game rule does this resemble?

(42) ____.

When we added a number to another number, and the result was the same number we started with, we called this the identity game rule. We might call the avenue with this same characteristic, "identity avenue". Try some more:

$(\overline{2,5}) + (\overline{8,8}) =$ (43) ____________________. Is the result the same avenue named by $(\overline{2,5})$? Yes, since the result is $(\overline{10,13})$.

$(\overline{4,4}) + (\overline{3,3}) =$ (44) ____________________. Is the result the same avenue named by $(\overline{4,4})$? Yes, the result is $(\overline{7,7})$ which represents the same avenue.

Now, with "identity avenue" which game rules hold? (45) ____________________

____

____

Did you write closure, reversibility, arranging, and identity? Good!

What other game rule did we know which involved a single operation? (46) ______________________. Remember it was closely tied in with the identity game rule. Does that hint help? Yes, we had an inverse game rule for some of our games. Did the inverse game rule hold for the addition of whole numbers? Yes or no? (47) ____________________________. Write an example to illustrate what we would look for if we were trying to find an inverse for a whole number. (48) ____

____.

Did you write something like 3 + ☐ = 0? What can go in the box? (49)______

______________________. That's right. There is no whole number which will go in the box and give us a true sentence. Therefore, we say that the inverse game does not hold for addition of whole numbers. To have an inverse we must get the (50) ______________________ for our sum. Did you write "identity" in the blank? Good!

Do we have inverses in this new system of avenues? What is the "identity avenue"? (51) ___________________________________. Yes, we recognize that "identity avenue" is $(\overline{8,8})$ or we could name it by any other ordered pair whose first and second elements are the same. Could we get "identity avenue" as a result when we add two other avenues? Consider $(\overline{2,3})$ in the following example:

$(\overline{2,3})$ + (52) ____________________ = $(\overline{5,5})$. We used one of the names for "identity avenue", but we know we could have chosen from many other names. Did you fill in the blank? Go back and try if you haven't done so. The blank could be filled in with $(\overline{3,2})$ couldn't it? Try another example:

$(\overline{5,1})$ + (53) ____________________ = $(\overline{7,7})$. This time the blank can be filled in with $(\overline{2,6})$.

Do you suppose we can always get "identity avenue" as a result, no matter what avenue we start on? If you are uncertain, try several examples like the ones just given.

Since we can start with any avenue and find an avenue to add to it and get "identity avenue" as a result which game rule appears to hold for these avenues. (54) ______________________. Yes, the inverse game rule, which we did not have for the whole numbers, appears to hold for these avenues.

We can now summarize again by listing all the game rules which appear to hold for addition of these avenues: (55) ______________________________________

____

____

Did you list closure, arranging, reversibility, identity, and inverse? Good!

PRACTICE EXERCISE

1. Show the following examples as trips on Grid VIII.

   a) $(\overline{3,6}) + (\overline{4,7}) = (\overline{4,7}) + (\overline{3,6})$

   b) $[(\overline{1,2}) + (\overline{3,1})] + (\overline{2,4}) = (\overline{1,2}) + [(\overline{3,1}) + (2,4)]$

   c) $(\overline{8,6}) + (\overline{2,2}) = (\overline{10,8})$

   d) $(\overline{5,1}) + (\overline{1,5}) = (\overline{6,6})$

2. Find the sum of the two avenues by using the algorithm for adding avenues named by ordered pairs.

   a) $(\overline{17,9}) + (\overline{72,91}) =$

   b) $(\overline{11,14}) + (\overline{7,8}) + (\overline{21,40}) =$

   c) $(\overline{4132,271}) + (\overline{1,2}) + (\overline{7,30}) + (\overline{17,416}) =$

   d) Which game rules can we demonstrate hold for this addition of avenues?

   3) Can you think of another way to physically represent this algorithm? If you can, construct an outline of the procedure.

ANSWERS

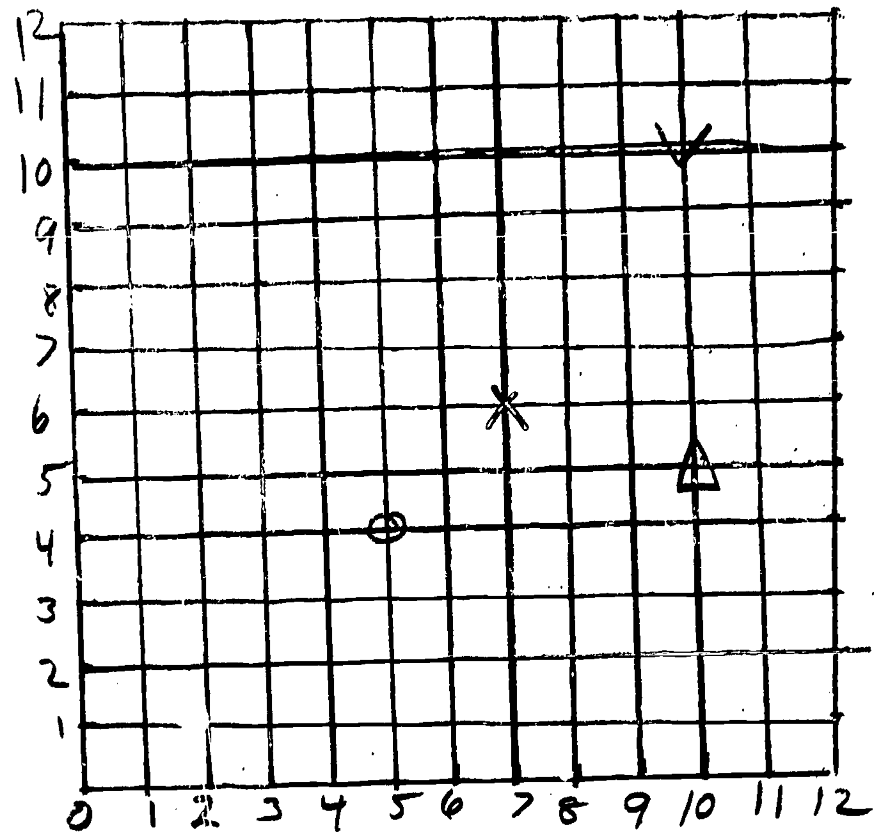
2. a) $(\overline{89,100})$

   b) $(\overline{39,62})$

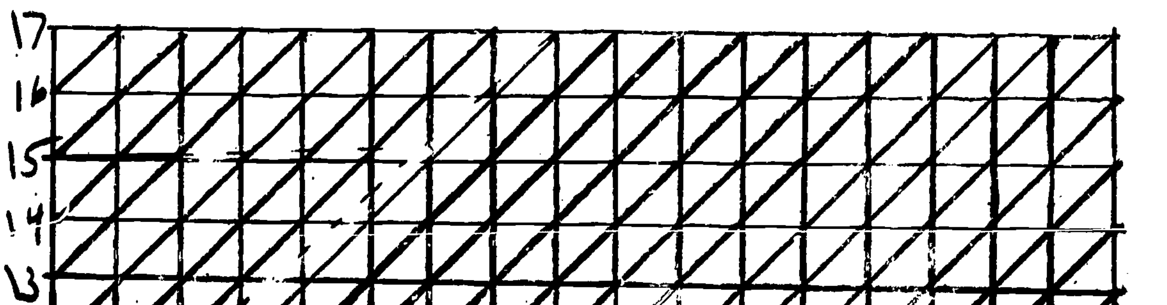
   c) $(\overline{4157,719})$

   d) An identity exists for addition; every element has an inverse for addition; the closure, arranging, and reversibility game rules apply for addition.

GRID VI

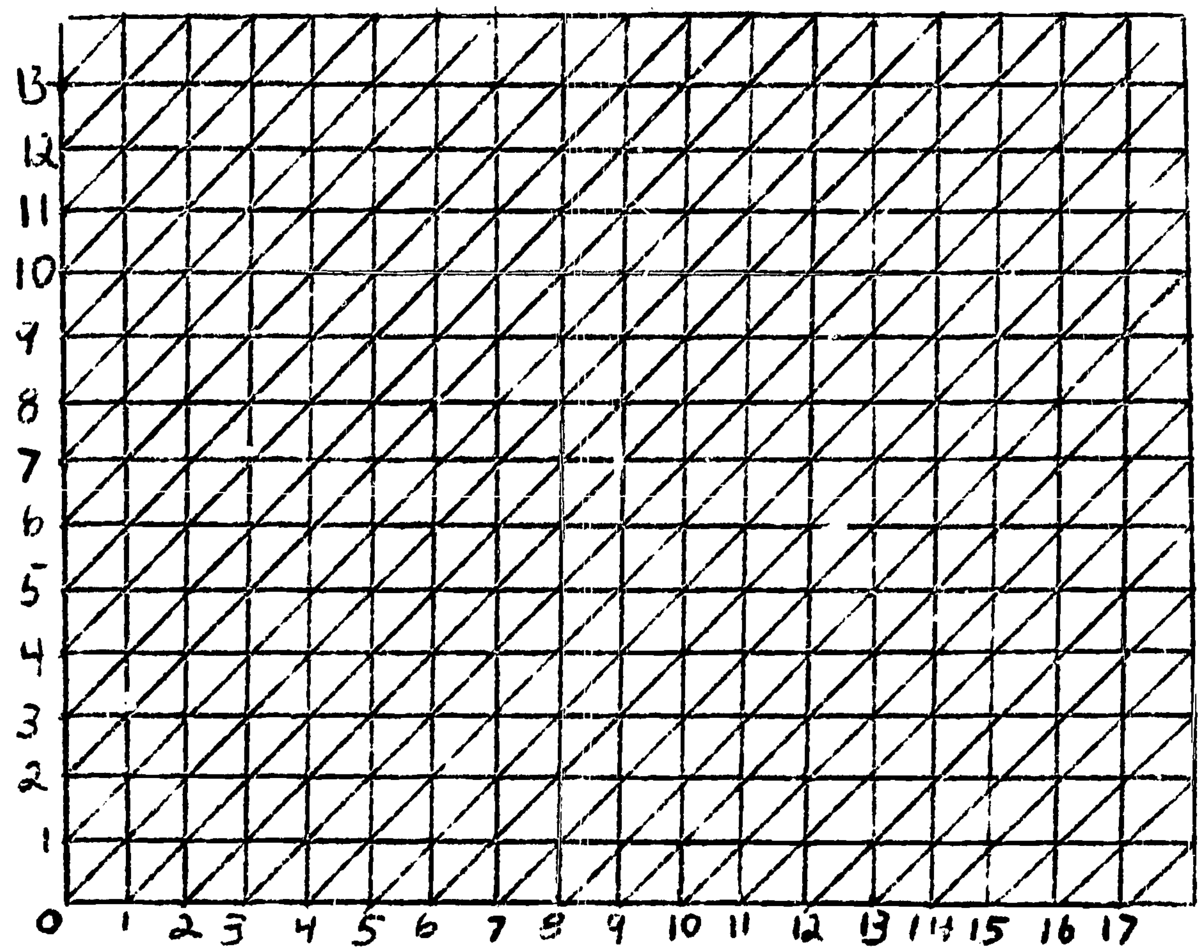


GRID VII



GRID

VIII



Second Experimental Edition

MATERIALS FOR SESSION IX

3 grids attached to last page of Session

SESSION X

ADDING ARROWS

OBJECTIVES:

At the end of this session the learner should be able to:

(1) demonstrate the procedures of an algorithm for finding the sum of two positive integers, two negative integers, a positive and a negative integer when they have the same magnitude, and a positive and a negative integer when the positive integer has the greater magnitude.

(2) construct a convincing explanation that appeals to observations based upon the manipulation of arrows for each procedure in the algorithm for constructing the sum of two integers.

(3) construct a convincing explanation that appeals to the game rules for each procedure in the algorithm for constructing the sum of two integers.

When we were examining some of the algorithms for adding whole numbers, we made use of three of the game rules. The game rules of closure, arranging, and reversibility were all used to explain the algorithms with whole numbers. There are three other game rules which we didn't mention in connection with the algorithms for addition with whole numbers: identity, inverse, and distributive. However, we have observed that the identity game rule holds for addition of whole numbers.

What is the identity for addition of whole numbers? (1)____________________

The number is zero of course. In the last session we observed that the inverse game rule did not hold for adding whole numbers. The whole number 5 does not have an inverse for the operation of addition since there is no whole number which we could find which when added to 5 results in zero - the identity for addition.

Well, this takes care of the identity and inverse game rules. The identity game rule holds and the inverse game rule doesn't hold for whole numbers. What about the distributive game rule? Does this game rule hold for addition of whole numbers?

Yes or no? (2)________________. We can't answer yes to this because the distributive game rule involves more than addition of whole numbers. It involves two operations.

We can see that our set of whole numbers is deficient for the operation of addition. One of the game rules simply doesn't hold for addition of whole numbers - the inverse game rule. Notice something else. One of the following number sentences cannot be made into a true sentence by writing the name of a whole number in the box:

$2 + \square = 8$ $\quad$ $\square + 5 = 8$ $\quad$ $\square + 5 = 7$ $\quad$ $3 + \square = 7$ $\quad$ $3 + \square = 2$

Which ones cannot be made into true sentences? (3)____________________

$3 + \square = 7$ can be made into a true sentence by writing 4 in the box. However, you are right if you said that there was no whole number which you could add to three so that the result would be two.

What we need is a better number system than the whole numbers. The addition of avenues named by ordered pairs which we looked at in the last session doesn't give us this trouble. For instance, examine $(\overline{2,1}) + \square = (\overline{4,5})$. What is the ordered pair which we would write in the box in order to name an avenue that makes the sentence a true sentence? (4)______________. $(\overline{4,2})$ wouldn't be correct, but $(\overline{2,4})$ would be. There are other names for the avenue named by $(\overline{2,4})$. $(\overline{4,6})$ would be one of those names since $(\overline{2,1}) + (\overline{4,6}) = (\overline{6,7})$ and $(\overline{6,7})$ is another name for the avenue named by $(\overline{4,5})$. In fact, we can always find an ordered pair that will make any sentence involving addition true.

The system of numbers involving avenues named by ordered pairs with addition seems to be superior to whole numbers with addition. Can you recall what game rules appeared to hold for adding avenues named by ordered pairs? Examples for these game rules are given below. See if you can supply the names.

There is an avenue named by $(\overline{2,2})$ such that $(\overline{3,4}) + (\overline{2,2}) = (\overline{5,6})$ where $(\overline{3,4})$ and $(\overline{5,6})$ name the same avenue. (5)____________________________

The sum of two avenues, such as $(\overline{2,1})$ and $(\overline{3,8})$, is an avenue, $(\overline{5,9})$. (6)____________________. $(\overline{3,6}) + (\overline{2,8}) = (\overline{2,8}) + (\overline{3,6})$ (7)________________________

For each avenue such as $(\overline{3,7})$, there is an avenue $\square$ such that $(\overline{3,7}) + \square = (\overline{9,9})$. (8)______________________________

$(\overline{2,1}) + (\overline{3,9}) + (\overline{1,4}) = (\overline{2,1}) + (\overline{3,9}) + (\overline{1,4})$. (9)______________________________.

The correct responses in reverse order are: arranging, inverse, reversibility, closure, and identity.

All of the game rules which held for the whole numbers - remember we listed them earlier: closure, arranging, reversibility, and identity - also hold for addition of avenues named by ordered pairs. Since the inverse game rule also holds for addition of avenues named by ordered pairs, it looks like these avenues are really superior to whole numbers.

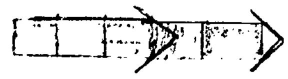
We call this new set of numbers which we created in the last session integers. All the game rules hold for addition of integers.

Last session we talked about integers as avenues named by ordered pairs. Another way we can represent integers is to use arrows. There are a number of arrows in packet A. Take them out and we'll see how we can use them. First, arrange them by size. We'll call the shortest arrow a one arrow. What is the relationship between the shortest arrow and the next longer arrow? (10)______________________________

Since the next longer arrow is twice as long as the shortest arrow, we will call it a two arrow. The next longer we will call a three arrow, the next a four arrow, and the longest a five arrow.

In the packet you should have found two each of the following: one arrow, two arrow, three arrow, four arrow, and five arrow. There is another kind of arrow in the packet. Since it didn't take up any room, we slipped in a zero arrow. It might be handy to have an arrow which has no length at all.

Now let's try to add two of these arrows. Take a three arrow and point it to the right. Now place a two arrow with its tail at the head of the three arrow and point it to the right as in the diagram below.



What arrow could be used in place of both of these arrows so that it begins at the tail of the three arrow and ends at the head of the two arrow? (11)________________

You are right that the five arrow could be used to replace the three arrow and the two arrow laid end to end. But since we have introduced the idea of direction for the three arrow and the two arrow, we must consider the direction of the five arrow. Naturally, we would want it to point to the right. Hence, when we add a three arrow pointing to the right, and a two arrow pointing to the right, the resultant arrow is a five arrow pointing to the right.

What would be the resultant arrow if we added a two arrow pointing to the right and a one arrow pointing to the right? (12)________________. You are right. This is a three arrow pointing to the right. What would be the resultant arrow if we added a three arrow pointing to the right and a zero arrow? (13)________________

Note that we did not have to supply the direction of the zero arrow. You just can't tell which way a zero arrow is pointing. Hence, the sum of a three arrow pointing to the right and a zero arrow is a three arrow pointing to the right. Of course the question can be asked the other way around. What two arrows could be added to give a resultant three arrow pointing to the right? (14)________________

____

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Naturally, one of the possible pairs of arrows would be a three arrow pointing to the right and a zero arrow. Other possible pairs would be a two arrow pointing to the right and a one arrow pointing to the right, a one arrow pointing to the right and a two arrow pointing to the right, and a zero arrow and a three arrow pointing to the right.

The two arrow pointing to the right represents an integer. We could write the integer in a shorthand notation. We need to talk about two things in connection with the integer. What are they? (15)________________

____

We are concerned with both the magnitude of the arrow and the direction. There are many ways in which we could express this. One way would be to write $\vec{2}$. How could we write the number sentence which expresses the sum of a three arrow pointing to the right and a two arrow pointing to the right as a five arrow pointing to the right?

(16)______________________________. The correct response is $\vec{3} + \vec{2} = \vec{5}$.

Write in the integer in shorthand notation which will make each of the following sentences true:

$$\vec{4} + \vec{1} = \square$$
$$\vec{2} + \vec{2} = \square$$
$$\square = \vec{1} + \vec{4}$$
$$\vec{7} + \vec{4} = \square$$
$$\square = \vec{2} + 0$$
$$0 + \vec{2} = \square$$

The correct responses are $\vec{5}$, $\vec{4}$, $\vec{5}$, $\vec{11}$, $\vec{2}$, and $\vec{2}$.

Write in the names of a pair of integers in shorthand notation which will make each of the following sentences true:

$$\vec{3} = \square + \triangle$$
$$\bigtriangledown + \triangle = \vec{3}$$
$$\vec{4} = \square + \bigtriangledown$$
$$\vec{2} = \square + \triangle$$
$$\vec{15} = \triangle + \bigtriangledown$$

There are many pairs of integers for each sentence. We could say $\vec{3} = \vec{3} + 0$ or $\vec{3} = \vec{2} + \vec{1}$ for the first one. We could write $\vec{4} = \vec{3} + \vec{1}$, $\vec{4} = \vec{1} + \vec{3}$, or even $\vec{4} = \vec{2} + \vec{2}$ for the third one. We could call all of these number sentences examples of adding right numbers.

Did we need any game rules in order to add a couple of right numbers?

List them. (17)____

____

If you think about it carefully, you will realize that the only game rule that was used was closure. When you add two integers, you always end up with an integer. In fact, in this case when you add two right numbers, you get a right number as a resultant.

However, we can observe that other game rules hold for addition of right numbers. Give an example of a number sentence involving addition of right numbers using the reversing game rule in shorthand notation.

(18)____

Your example might look like this: $\vec{2}+\vec{3}=\vec{3}+\vec{2}$ or $(\vec{4}+\vec{2})+\vec{1}=\vec{1}+(\vec{4}+\vec{2})$. Note that in this last example the $(\vec{4}+\vec{2})$ was reversed with the $\vec{1}$. Can you give an example of a number sentence involving addition of right numbers using the arranging game rule? Yes or no? (19)________________. Your answer should once again be yes, and you should be able to write the example using the shorthand notation. (20)____

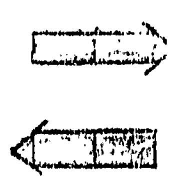
There are many examples you might have chosen here. Perhaps you picked an example like this: $\vec{3}+(\vec{2}+\vec{1})=(\vec{3}+\vec{2})+\vec{1}$ or $(\vec{4}+(\vec{3}+\vec{2}))+\vec{1}=\vec{4}+((\vec{3}+\vec{2})+\vec{1})$. Here in the last example $(\vec{3}+\vec{2})$ was considered as a single number in the arranging game rule.

Is there an identity when you add integers? Yes or no? (21)________________ If you said yes, you were probably thinking of zero since zero is the number which when added to any number, say n, gives a result which is n. If you said no, you were probably not thinking of zero as an integer. In any case, we really want to have zero as an integer even though we cannot determine the direction of zero. Thus, zero is the identity when adding integers.

Now **recall the game rule** which was found to hold for integers when written as ordered pairs even though it did not hold for whole numbers. What was the game rule? (22)________________________. The only game rule which holds for addition of integers written as ordered pairs but does not hold for addition of whole numbers is the inverse game rule. How could we show this game rule with arrows? (23)____

____

Did your thinking go something like this? If you are looking for the inverse of an arrow, you want to find an arrow which when added to the given arrow results in the zero arrow. If there was a two arrow pointing to the right, you could place a two arrow pointing to the left in order to have a result that is the zero arrow as indicated in the diagram below.



Now, we see that there are not only right arrows and a zero arrow, but also left arrows. If there were a four arrow pointing to the right, what arrow could be added to it in order to get the zero arrow for a result?

(24)________________. If there were a three arrow pointing to the left, what arrow could be added to it in order to get the zero arrow for a result?

(25)________________.

If there were a zero arrow, what arrow could be added to it in order to get the zero arrow for a result? (26)__________________. The correct responses to these last three questions are "four arrow pointing to the left," "three arrow pointing to the right", and "zero arrow".

How could you write these expressions of the inverse game rule in shorthand notation? Let's try with the first example when you searched for the inverse of a two arrow pointing to the right. (27)____________________________
The number sentence which you start with looks like this: $\overrightarrow{2} + \square = 0$. The number which makes this number sentence true is the inverse of a right two. The inverse of a right two is left two. This could be written $\overleftarrow{2}$. $\overrightarrow{2} + \overleftarrow{2} = 0$ expresses an example of the inverse game rule.

Write in the integer in shorthand notation which will make each of the following sentences true:

$$\overrightarrow{4} + \square = 0$$

$$\overleftarrow{3} + \square = 0$$

$$0 + \square = 0$$

$$\overleftarrow{12} + \square = 0$$

The correct responses are $\overleftarrow{4}$, $\overrightarrow{3}$, 0, $\overrightarrow{12}$.

Write in the names of a pair of integers in shorthand notation which will make each of the following sentences true:

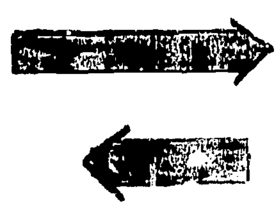
$$0 = \triangle + \square$$

$$\square + \triangledown = 0$$

There are many pairs which could be chosen. $0 = \overrightarrow{5} + \overleftarrow{5}$ is one true sentence. $0 = \overleftarrow{7} + \overrightarrow{7}$ is another.

We can now construct the sum of any two right numbers. We also can add a right number and a left number as long as they can be represented by arrows that are the same length. Let's try some of the examples for adding arrows when we have one arrow pointing to the right and one arrow pointing to the left.

Take a five arrow and point it to the right. Now add a three arrow pointing to the left in the following way. Place the tail of the three arrow at the head of the five arrow. The resultant arrow is an arrow which could be placed such that its tail would be at the tail of the five arrow and its head at the head of the three arrow as in the diagram below.



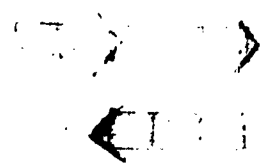
What is the resultant arrow? (28)____________________. What is the magnitude? (29)____________________. What is the direction? (30)________________

Let's systematically write down how you would use arrows to explain this result. First, since the three arrow is pointing to the left, is there an arrow I can put with it which will result in only the zero arrow? (31)________________

Naturally, such an arrow is a three arrow pointing to the right. Could I then replace the five arrow pointing to the right with a [illegible] f arrows - one of which is the three arrow pointing to the right? Try draw[illegible] gram for the replacement for the five arrow pointed to the [illegible]. ________________

____

The five arrow could be replaced by a two arrow and a three ar[illegible] as in the diagram below.

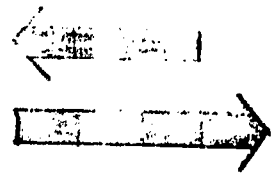


Now arrange the arrows so that the pair of three arrows are grou[illegible] [illegible]ether.

What is the result of adding the pair of three ar[illegible]ows? (33)________________

Since the pair are pointing in opposite directio[illegible], the result [illegible] zero arrow. Now the only remaining arrow is the tw[illegible] [illegible]row pointing [illegible] [illegible] ight. This is the resulting arrow when a five arrow p[illegible]ing to the ri[illegible] i[illegible] added to a three arrow pointing to the left.

Now go through this algorithm again u[illegible]in[illegible] the example ad[illegible] [illegible] t[illegible]ee arrow pointing to the left and a four arrow pointing to the right as [illegible] the diagram below.



Which arrow would be replaced with two separate arrows this time? (34)________________. The four arrow should [illegible] replaced. The longe[illegible]t [illegible] [illegible] the obvious choice for replacing. What wo[illegible]ld you replace i[illegible] w[illegible]t ________________. Since you want one of the replacement a[illegible] be a three arrow, you should replace the four arrow with a three arrow and a one arrow both pointing to the right. Now arrange the two three arrows in a grouping. What is the result of adding a three arrow pointing to the left and a three arrow pointing to the right? (36)____________________. Your result is the zero arrow just as planned.

And now what is the remaining arrow? (37)______________. You have a resultant arrow **which** is a one arrow pointing to the right.

We can also expect to write examples of adding arrows using our shorthand notation. Let's look at the last example. How would you write the **example in** shorthand notation? (38)______________________________

Your response should be similar to this: $\overleftarrow{3} + \overrightarrow{4} = \square$. The first thing that you did when you added the arrows was to replace the four arrow with a three arrow and a one arrow both pointing to the right. We could now write $\overleftarrow{3} + \overrightarrow{4} = \overleftarrow{3} + (\overrightarrow{3} + \overrightarrow{1})$. The next move you made with the arrows was to arrange the $\overleftarrow{3}$ and the $\overrightarrow{3}$ together. How would you write this? (39)______________________
The best way to show the $\overleftarrow{3}$ and the $\overrightarrow{3}$ together is to use parentheses. We could write $\overleftarrow{3} + (\overrightarrow{3} + \overrightarrow{1}) = (\overleftarrow{3} + \overrightarrow{3}) + \overrightarrow{1}$. But we know that the two three arrows added together result in the zero arrow. Therefore, we could write $(\overleftarrow{3} + \overrightarrow{3}) + \overrightarrow{1} = 0 + \overrightarrow{1}$. And finally the remaining arrow is the one arrow pointing to the right. We could write $0 + \overrightarrow{1} = \overrightarrow{1}$.

If we summarized the steps aboce they might look like this:

Step 1. $\overleftarrow{3} + \overrightarrow{4} = \overleftarrow{3} + (\overrightarrow{3} + \overrightarrow{1})$

Step 2. $\overleftarrow{3} + (\overrightarrow{3} + \overrightarrow{1}) = (\overleftarrow{3} + \overrightarrow{3}) + \overrightarrow{1}$

Step 3. $(\overleftarrow{3} + \overrightarrow{3}) + \overrightarrow{1} = 0 + \overrightarrow{1}$

Step 4. $0 + \overrightarrow{1} = \overrightarrow{1}$

What are the game rules for integers which we are using in finding the sum of two integers? Let's look at each step. In the first step $\overrightarrow{4}$ was renamed as $\overrightarrow{3} + \overrightarrow{1}$. This is not a game rule, but notice that this depends on our being able to add two right numbers. In the second step what is the game rule involved?

(40)______________________________________

Here we used the arranging game rule to group the two threes together. In the third step, the two threes were added. Since one is a left number and the other is a right number, they are inverses of each other and their sum is zero. Hence, we are using the inverse game rule in the third step. What is the game rule in the fourth step? (41)______________________________

Here we used the identity game rule. Remember that 0 is the identity for integers.

Now use the algorithm we have developed in shorthand notation to add $\overrightarrow{3}$ and $\overleftarrow{1}$.

(42)______________________________________

____

____

____

____

____

Now go back and supply the game rules for each step.

Your steps should look something like this:

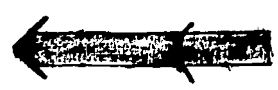
1) $\overleftarrow{3} + \overrightarrow{1} = (\overrightarrow{2} + \overrightarrow{1}) + \overleftarrow{1}$ Renaming $\overrightarrow{3}$

2) $(\overrightarrow{2} + \overrightarrow{1}) + \overleftarrow{1} = \overrightarrow{2} + (\overrightarrow{1} + \overleftarrow{1})$ Arranging

3) $\overrightarrow{2} + (\overrightarrow{1} + \overleftarrow{1}) = \overrightarrow{2} + 0$ Inverse

4) $\overrightarrow{2} + 0 = \overrightarrow{2}$ Identity

Let's see how we would add to arrows when they both point to the left. Take a two arrow and point it to the left. Now place a three arrow pointing to the left with its tail at the head of the two arrow as in the diagram below.



What arrow could be used in place of both of these arrows so that it begins at the tail of the two arrows and ends at the head of the three arrow? (43)__________

__________. You are right if you said a five arrow pointing to thee left would do the job.

What would be the resultant arrow if we added a three arrow pointing to the left and a one arrow pointing to the left? (44)__________

What would be the resultant arrow if we added a two arrow pointing to the left and a zero arrow? (45)__________

These are easy. The first answer was a four arrow pointing to the left and the second answer was a two arrow pointing to the left.

Naturally, these addition examples can be written in our shorthand notation. Try writing the addition example in the last diagram in this form. (46)__________

__________. You might have writt[en] $\overline{\ } + \overleftarrow{2} = \overleftarrow{5}$. However, this is not correct. We actually started with the two arrow. $\overleftarrow{2} + \overleftarrow{3} = \overleftarrow{5}$ is the correct response.

What game rules did we use in adding left numbers? List them. (47) ______

____.

Just as when we added right numbers, the only game rule used was closure for integers. When you add two integers, you always end up with an integer. In this case we added two left numbers, and the result was a left number.

We have now added several different combinations of integers. We have added two left numbers, two right numbers, a right number and a left number which can be represented by arrows of the same length, and a right number and a left number with the arrow for the right number longer than the arrow for the left number. We have also presented an argument for these algorithms based upon arrows as well as the game rules.

Try a few of these:

$\overrightarrow{1} + \overrightarrow{2} = \square$	$\overrightarrow{2} + \overrightarrow{2} = \square$	$\overleftarrow{3} + \overleftarrow{3} = \square$
$\overleftarrow{6} + \overleftarrow{2} = \square$	$\overrightarrow{12} + \overleftarrow{7} = \square$	$\overrightarrow{5} + 0 = \square$
$\overleftarrow{3} + 0 = \square$	$0 + \overleftarrow{7} = \square$	$\overleftarrow{4} + \overrightarrow{4} = \square$

In this session we mentioned that the avenues named by ordered pairs that you learned to add are called integers. Certainly if both ordered pairs and arrows represent integers, we should be able to show the relationship between them. So let's look. We demonstrated that the same game rules hold for addition as for integers expressed as ordered pairs and integers expressed as arrows. Now let's play around a little and see if we can figure out which arrow belongs to which avenue named by an ordered pair.

$\overrightarrow{6} + \overrightarrow{2} = \overrightarrow{8}$

$(\overline{2,8}) + (\overline{1,3}) = (\overline{3,11})$

$\overrightarrow{6} + \overleftarrow{2} = \overrightarrow{4}$

$(\overline{2,8}) + (\overline{3,1}) = (\overline{5,9})$

$\overrightarrow{4} + \overleftarrow{4} = 0$

$(\overline{3,7}) + (\overline{7,3}) = (\overline{10,10})$

$\overrightarrow{4} + 0 = \overrightarrow{4}$

$(\overline{3,7}) + (\overline{0,0}) = (\overline{3,7})$

$(\overline{3,7}) + (\overline{5,5}) = (\overline{8,12})$

As you look at these puzzles, can you see that the pair of sentences in each bracket says the same thing? This is also true of the three sentences in the last beacket, because an integer can be named by more than one ordered pair. $\overrightarrow{6}$ and $(\overline{2,8})$ are simply two different names for the same integer which is often called $^{+}6$, positive six. Similarly, $\overleftarrow{2}$ and $(\overline{3,1})$ are just different names for the integer $^{-}2$, negative two. Of course $(\overline{3,7})$, $(\overline{8,12})$ and $\overrightarrow{4}$ are different names for the integer $^{+}4$, positive four.

$(\overline{3,4}) + (\overline{8,7}) = (\overline{11,11})$. How would you write this sentence using arrows?

(48) ____

$\overrightarrow{1} + \overleftarrow{1} = 0$ is correct. How would you write this sentence using positive and negative integers? (49)____________________________

$^{+}1 + {}^{-}1 = 0$ is correct. Try another, $(\overline{3,7}) + (\overline{4,1}) = (\overline{7,8})$ using arrows.

(50)____

This would be $\overrightarrow{4} + \overleftarrow{3} = \overrightarrow{1}$.

Now, reverse the process and go from arrows to ordered pairs. $\overrightarrow{3} + \overrightarrow{2} = \overrightarrow{5}$. Express this sentence about integers using ordered pairs. (51)______________

____

Can't decide which pairs to use? Just pick any that name arrows. Although it is true that many ordered pairs can be used to name each integer, in this case any one will do, so long as your result agrees with the integer named by the resultant arrow. Also express this sentence using positive and negative integers. (52)____________________________________

How about $\overleftarrow{6} + \overrightarrow{7} = \overrightarrow{1}$ expressed as a sentence using ordered pairs? (53)______

____

What you just did indicates that anything you can write with arrows that name integers, you can also write with ordered pairs and vice versa. Every arrow can be named by an ordered pair and vice versa.

First Experimental Edition

MATERIALS FOR SESSION X

Packet A

10 arrows:

2	orange	5 inch arrows
2	yellow	4 inch arrows
2	green	3 inch arrows
2	red	2 inch arrows
2	blue	1 inch arrows

SESSION XI

THE CASE OF THE MISSING ARROW

OBJECTIVES:

At the end of this session the learner should be able to:

(1) demonstrate the procedures of an algorithm for finding the difference of two integers.

(2) construct a convincing explanation that appeals to observations based upon the manipulation of arrows for each procedure in the algorithm for constructing the difference of two integers.

(3) construct a convincing explanation that appeals to the game rules for each procedure in the algorithm for constructing the difference of two integers.

Last session we represented integers by arrows and constructed the sums of different combinations of integers. Let us review briefly. What would be the resultant arrow if we added a three arrow pointing to the right and a two arrow pointing to the right? (1)____________ You are remembering correctly if you said a five arrow pointing to the right. Don't forget. We were concerned with not only the magnitude of the arrow but also its direction. A three arrow pointing to the right was represented by $\vec{3}$. Were there any other arrows? (2)____________ Sure. Remember the arrow in the packet that you couldn't find. This was the zero arrow. If there was a three arrow pointing to the right, what arrow could be added to it to get the zero arrow for a result? (3)____________ The correct response is a three arrow pointing to the left. So we see that we also have left arrows. A four arrow pointing to the left was represented by $\overleftarrow{4}$. The resultant of a one arrow pointing to the right and a five arrow pointing to the left is (4)________________? Good. The resultant is a four arrow pointing to the left.

In shorthand notation, our three above problems would look like the following:

(a) $\overrightarrow{3} + \overrightarrow{2} = \overrightarrow{5}$

(b) $\overrightarrow{3} + \overleftarrow{3} = 0$

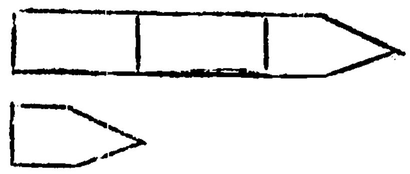
(c) $\overrightarrow{1} + \overleftarrow{5} = \overleftarrow{4}$

We should note that in sentence (b) above, $\overleftarrow{3}$ is the inverse of $\overrightarrow{3}$.

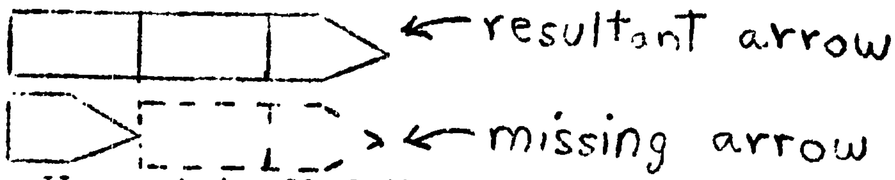
We are now going to introduce a new operation into our set of integers. We are already familiar with the operation of addition which assigns to a pair of integers, a third integer called the sum: Subtraction is the operation of finding the missing addend when the sum and other addend are known. We agree to call the missing addend the difference. So we can see that solving a subtraction problem is really undoing an addition problem. It is said that subtraction and addition are inverse operations. We already said that we can represent integers with arrows. Now let's try to find the difference of two arrows. Remember, the difference was our name for the missing addend. In working with arrows, let us agree to name the difference, the missing arrow. Take out our familiar packet A. In it, you will find arrows of different sizes. Don't forget about the zero arrow. They get lost among the larger arrows. In finding the difference of two arrows, it will be helpful to keep in mind what we learned from out experiences of adding arrows. When we add two arrows, the second arrow is placed at the head of the first arrow, and the resultant arrow begins at the tail of the first arrow, and ends at the head of the second arrow.

Consider the problem of finding the difference of a three arrow pointing to the right and a one arrow pointing to the right. Let us agree to name the first arrow in a given subtraction problem the resultant arrow and the second arrow the known addend.

We already agreed that the difference is the missing arrow. Our definition for subtraction tells us that our problem is to find the arrow, such that when added to the one arrow pointing to the right we obtain the resultant three arrow pointing to the right. Take the resultant three arrow and point it to the right. Now take the one arrow pointed to the right, and place it at the tail of the resultant three arrow.

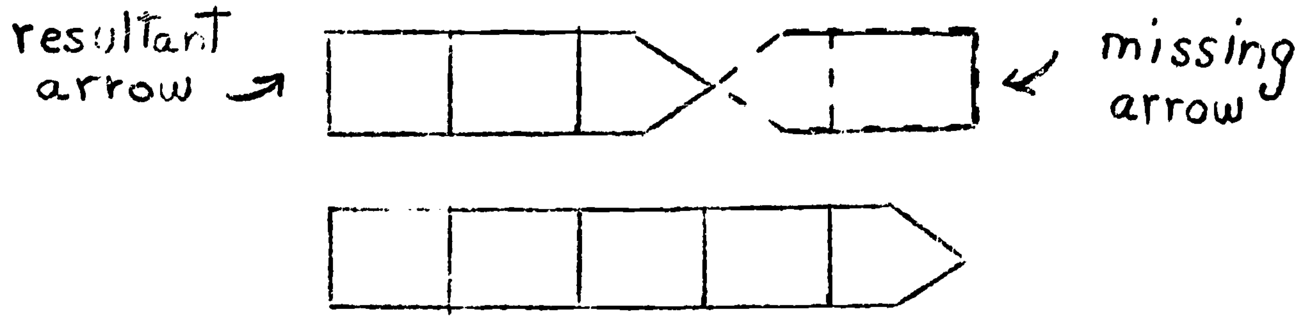


Now what arrow can we place at the head of the one arrow so that its head is at the head of the resultant arrow? (5)_______________ You are doing great if you said that the missing arrow is a two arrow pointed to the right.



Let's try another problem. We want to find the difference of a three arrow pointed to the right and a five arrow pointed to the right. Again, our problem is finding the arrow such that when added to the five arrow pointing to the right, we obtain the resultant three arrow pointing to the right. Take the resultant three arrow and point it to the right. Now take the five arrow pointed to the right and place it at the tail of the resultant three arrow. Now what arrow can we place at the head of the five arrow so that its head is at the head of the resultant arrow? (6)___________________________

Excellent! The missing arrow is a two arrow pointed to the left.



In shorthand notation, our problems look like this:

(1) $\overrightarrow{3} - \overrightarrow{1} = \square$

(2) $\overrightarrow{3} - \overrightarrow{5} = \square$

Note: We represent the operation of finding the difference of two arrows by the symbol ' - '. Did we need any game rules in order to find the difference of two arrows? Let us consider what we have done in problem (1). Our first step was to change the form of our subtraction problem to a form which uses the familiar operation, addition. How would you rewrite our subtraction problem using the operation of addition? (7)____________________________ Your response should be similar to this: $\overrightarrow{1} + \square = \overrightarrow{3}$. Our problem is now one of finding the missing arrow, represented by $\square$, which will make the statement $\overrightarrow{1} + \square = \overrightarrow{3}$ true. We might note the similarity between such a statement as $\overrightarrow{1} + \square = \overrightarrow{3}$ and a balance scale. The sum of $\overrightarrow{1} + \square$ balances $\overrightarrow{3}$ as do the weights on a balance scale. If we add some quantity to the left side of our statement we must add the same quantity to the right side in order to "keep things balanced." Since we are looking for the missing arrow represented by $\square$, it might be helpful to do something to the left side of our statement to get $\square$ to stand alone. What arrow can we add to the left side of our statement so that $\square$ is all that remains on the left side? (8)____________________________ Oh, so you say we should get rid of $\overrightarrow{1}$. We can't just make it disappear! Now you are thinking. If

If we add the inverse of $\overrightarrow{1}$, namely $\overleftarrow{1}$ to $\overrightarrow{1}$ we get the zero arrow for a result. Don't think you're finished. What haven't we done? (9)________

____

Right. We must also add $\overleftarrow{1}$ to the right side of our statement to balance the left side. Our move would look something like this:

$$\overleftarrow{1} + (\overrightarrow{1} + \square) = \overleftarrow{1} + \overrightarrow{3}$$

The next move you made was to arrange the $\overleftarrow{1}$ and the $\overrightarrow{1}$ together. We could write $\overleftarrow{1} + (\overrightarrow{1} + \square) = (\overleftarrow{1} + \overrightarrow{1}) + \square$. But we know that the two one arrows added together result in the zero arrow. Therefore, we could write $(\overleftarrow{1} + \overrightarrow{1}) + \square = 0 + \square$. We could write $0 + \square = \square$. Our final statement is $\square = \overleftarrow{1} + \overrightarrow{3}$. We're almost done now. You can easily name the missing arrow. The missing arrow would be a (10)____________ The correct response is $\overrightarrow{2}$ since the sum of $\overleftarrow{1}$ and $\overrightarrow{3}$ is $\overrightarrow{2}$.

If we summarized the above steps they might look like this:

Problem:	$\overrightarrow{3} - \overrightarrow{1} = \square$
Step 1	$\overrightarrow{1} + \square = \overrightarrow{3}$
Step 2	$\overleftarrow{1} + (\overrightarrow{1} + \square) = \overleftarrow{1} + \overrightarrow{3}$
Step 33	$(\overleftarrow{1} + \overrightarrow{1}) + \square = \overleftarrow{1} + \overrightarrow{3}$
Step 4	$0 + \square = \overleftarrow{1} + \overrightarrow{3}$
Step 5	$\square = \overleftarrow{1} + \overrightarrow{3}$
Step 6	$\square = \overrightarrow{2}$

What are the game rules for integers which we are using in finding the difference of two integers? Let's look at each step. In the first step we rewrote our problem using the definition of subtraction. What game rules are involved in the second step? (11)____________________________ Very Good. We used the game rule of closure. In the third step what game rule is involved? (12)_________________ Here we used the arranging game rule to group the two ones together. In the fourth step, the two ones are added and their sum is zero. Which game rule is involved here? (13)______________________ Right. We are using the inverse game rule in the fourth step since the two ones are inverses of each other. In the fifth step, we used the identity game rule. Finally, what did we use in the sixth step? (14)______________________ We used the algorithm we developed for the addition of arrows.

Now use the algorithm we have just developed in shorthand notation to find the difference of $\overrightarrow{3}$ and $\overrightarrow{5}$. Also, supply the game rules for each step.

(15)____

____

____

____

____

____

Your steps should look something like this:

Problem: $\overrightarrow{3} - \overrightarrow{5} = \square$

1) $\overrightarrow{5} + \square = \overrightarrow{3}$ Definition of subtraction

2) $\overleftarrow{5} + (\overrightarrow{5} + \square) = \overleftarrow{5} + \overrightarrow{3}$ Closure

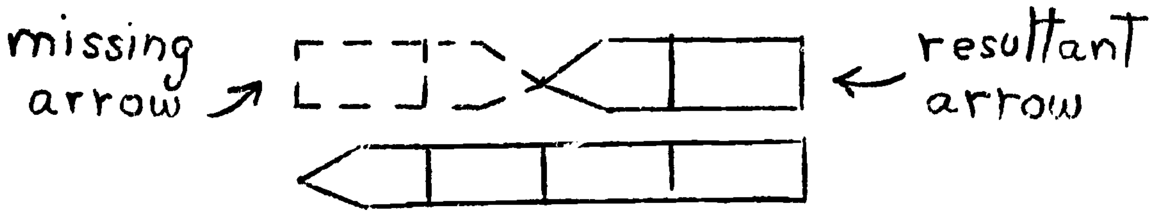
3) $(\overleftarrow{5} + \overrightarrow{5}) + \square = \overleftarrow{5} + \overrightarrow{3}$ Arranging

4) $0 + \square = \overleftarrow{5} + \overrightarrow{3}$ Inverse

5) $\square = \overleftarrow{5} + \overrightarrow{3}$ Identity

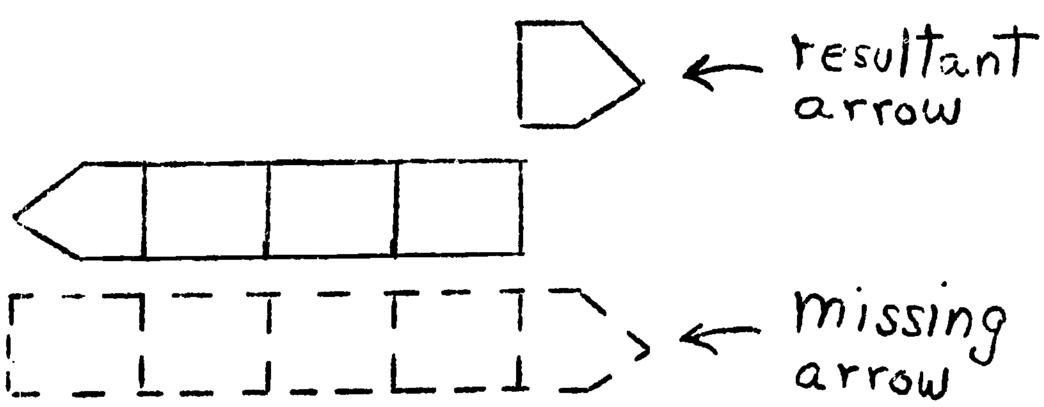
6) $\square = \overleftarrow{2}$ Addition of arrows

Let's consider the problem of finding the difference of two arrows when they both point to the left. Suppose we are given the problem of finding the difference of a two arrow pointing to the left and a four arrow pointing to the left. Can you restate the problem using the definition of subtraction? (16)________________________ Exactly. Our problem is finding the arrow such that when added to the four arrow pointing to the left, we obtain the resultant two arrow pointing to the left. Take the resultant two arrow and point it to the left. Now take the four arrow pointed to the left and place it at the tail of the resultant two arrow. Now what arrow can we place at the head of the four arrow so that its head is at the head of the resultant arrow? (17)________________________ You say you're not sure. Then take a peek at the picture below.



The missing arrow is a two arrow pointed to the right.

What would be the missing arrow if we found the difference of a one arrow pointing to the right and a four arrow pointing to the left? (18)________________________ The correct response is a five arrow pointing to the right. For those of you who had some difficulty, refer to the diagram below.



Try writing the subtraction example in the last diagram in shorthand notation (19)____

Your response should look like this: $\overrightarrow{1} - \overleftarrow{4} = \overrightarrow{5}$.

We might summarize by looking at the game rules which hold for the subtraction of integers. We will show that although certain of the game rules hold for the addition of integers, it is not true that these game rules will also hold for the subtraction of integers. Since the difference of two integers is always an integer, we know that closure holds for subtraction. In one of our examples we found that $\overrightarrow{3} - \overrightarrow{5} = \overleftarrow{2}$. What is $\overrightarrow{5} - \overrightarrow{3}$? (20)__________________________ The answer is not $\overleftarrow{2}$! The correct answer is $\overrightarrow{2}$. Therefore, we see that $\overrightarrow{3} - \overrightarrow{5} \neq \overrightarrow{5} - \overrightarrow{3}$, and subtraction is not reversibility in the integers. Look at the problem: $\overrightarrow{3} - (\overrightarrow{4} - \overrightarrow{2})$. The resultant arrow is $\overrightarrow{1}$. Do we get the same result when we consider $(\overrightarrow{3} - \overrightarrow{4}) - \overrightarrow{2}$? (21)__________________________ Of course not! The resultant arrow is $\overleftarrow{3}$. Since $\overrightarrow{3} - (\overrightarrow{4} - \overrightarrow{2}) \neq (\overrightarrow{3} - \overrightarrow{4}) - \overrightarrow{2}$, we can conclude that the game rule of arranging does not hold for the subtraction of integers.

What is the identity for subtraction of integers? (22)________________

The zero arrow of course.

Before we conclude this session, let's take a look at the statement $\overrightarrow{a} - \overrightarrow{b} = \square$ where a and b are positive integers. Restating this problem, using the definition of subtraction we get $\overrightarrow{b} + \square = \overrightarrow{a}$. Recall, to find $\square$, we must add $\overleftarrow{b}$ to both the left side and the right side of our statement.

$$\overleftarrow{b} + (\overrightarrow{b} + \square) = \overleftarrow{b} + \overrightarrow{a}$$

By the associativity of integers $(\overleftarrow{b} + \overrightarrow{b}) + \square = \overleftarrow{b} + \overrightarrow{a}$. Since $\overleftarrow{b} + \overrightarrow{b} = 0$, we are left with $\square$ on the left side and that $\square = \overleftarrow{b} + \overrightarrow{a}$.

By the reversibility of addition of integers we get $\square = \overrightarrow{a} + \overleftarrow{b}.$ The original statement and our final step show us that $\overrightarrow{a} - \overrightarrow{b} = \square = \overrightarrow{a} + \overleftarrow{b}.$ We can get a similar result by taking any combination of integers which leads us to the general statement: (23)____

Very Good. Subtracting an integer is the same as adding its additive inverse.

Performance Tasks

Use the algorithm we have just developed in shorthand notation to find the difference of $\overrightarrow{1}$ and $\overleftarrow{4}$. Also, supply the game rules for each step.

Answers to Performance Tasks

Problem: $\overrightarrow{1} - \overleftarrow{4} = \square$

1)	$\overleftarrow{4} + \square = \overrightarrow{1}$	Definition of Subtraction
2)	$\overrightarrow{4} + (\overleftarrow{4} + \square) = \overrightarrow{4} + \overrightarrow{1}$	Closure
3)	$(\overrightarrow{4} + \overleftarrow{4}) + \square = \overrightarrow{4} + \overrightarrow{1}$	Arranging
4)	$0 + \square = \overrightarrow{4} + \overrightarrow{1}$	Inverse
5)	$\square = \overrightarrow{4} + \overrightarrow{1}$	Identity
6)	$\square = \overrightarrow{5}$	Addition of arrows

SESSION XII

SO, WHAT'S THE DIFFERENCE?

OBJECTIVES:

At the end of today's session, the learner will be:

(1) demonstrating each step of the equal-additions method of subtraction algorithm.

(2) constructing an explanation for the algorithm using whole numbers and based on a physical situation.

(3) constructing an explanation for the algorithm using whole numbers and based on the rules of the "convincing game".

In our last session we observed that solving a subtraction problem is very similar to solving an addition problem. We defined subtraction as the operation of finding the missing addend when we are given the sum and the other addend. The given sum was named the resultant and the given addend was named the known addend. The number we are interested in finding, the missing addend, was named the difference. In our last session we stated that the set of integers is closed under the operation of subtraction. Since we are working within the set of whole numbers in this session we must note one very important limitation. We are not always able to subtract within the set of whole numbers. Can you say why not? An example would be fine. (1)_______________ Exactly. Consider the example 2-4. The difference is not a whole number. When will the difference be a whole number? (2)____________ You really are sharp today. We are able to perform the operation of subtraction within the set of whole numbers, when the resultant is greater than or equal to the given addend.

When we were working with the rule of compensation, we arranged numbers so that we could construct a sum in an easier manner. Let's look once again at the total arrangement in this problem.

$$28 + 172 + 94 + 89 = 28 + (2 + 170) + 94 + (6 + 83)$$
$$= (28 + 2) + 170 + (94 + 6) + 83$$
$$= 30 + 170 + 100 + 83$$
$$= (30 + 170) + 100 + 83$$
$$= 200 + 100 + 83$$
$$= 383$$

Constructing the sum was made easier by regrouping the numbers to the nearest ten and to the nearest hundred. Where did the six that we added to the ninety-four in order to regroup to the nearest hundred come from? (3)________ Come on, we didn't pull them out of a hat. Good! The six came from eighty-nine. In order to add six to ninety-four we had to take the same number, namely six, from eighty-nine.

Similarly, the rule of equal-additions will enable us to solve a subtraction problem more easily. The result of a subtraction problem remains unchanged if the same number is added to both the resultant and the given addend. Consider the following examples:

$$76 - 18 = (76 + 2) - (18 + 2) = 78 - 20 = 58$$

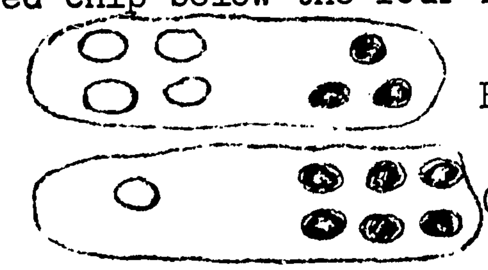
How are we using the rule of equal-additions? (4)________ You are absolutely correct if you said that we added two to both the resultant and the given addend. Let's try one on your own now. Show an easy way to compute 421 - 97. What number did you add to both the resultant and the given addend? (5)__________. If you said three, you are doing fine.

$$421 - 97 = (421 + 3) - (97 + 3) = 424 - 100 = 324.$$

The method of equal-additions is based on the principle that if the same number is added to two numbers, the difference between them remains unchanged.

Activity I

In Packet A you will find red, blue, and yellow chips. Let us agree to name a blue chip "a unit" and a red chip "a ten." One red chip is equivalent to how many blue chips? (6) ________. That's the idea! Since one ten is the same as ten units, the correct response is ten. We are now going to set up a subtraction problem using our chips in which our resultant is forty-three and our given addend is sixteen. Make sure that you have working space in front of you. Set up the resultant by placing three blue chips to the right of four red chips. Let us set up the given addend right below the resultant. Place six blue chips below the three blue chips of the resultant and one red chip below the four red chips of the resultant.



We are ready to begin. Note that the blue chips form a units column and the red chips form a tens column. If we look at the units column, we observe that the upper digit is smaller than the lower digit. In order to subtract six, we are going to have to place additional blue chips with the three blue chips we started with. Place ten additional blue chips with the three blue chips. Suppose we all do this now. According to our principle of increasing the resultant and given addend by the same number, we are going to have to add ten to the given addend. We could join ten blue chips with the six blue chips, but we would again have more blue chips in the given addend than in the resultant. How can we add ten to the given addend? Could we join one red chip to the one red chip that we started with? Would this be adding ten to the given addend? Sure! Add one red chip to the one red chip that we started with in the given addend.

In order to subtract we must compare the number of chips in the resultant with the number of chips in the given addend. Pair each of the blue chips in the resultant with a blue chip in the given addend. What is the remainder? (7) ________. Pair each of the red chips in the resultant with a red chip in the given addend. What is the difference? (8) ________. If you are with us, you should be left with two red chips and seven blue chips or twenty-seven.

Activity II

In addition to what we previously said, let us agree to name a yellow chip "a hundred." One yellow chip is equivalent to how many red chips? (9) ________. Since, one hundred is the same as ten "tens," the correct answer is ten. Now let us take a look at a more complicated example in which our resultant is 304 and our given addend is 167. Set up the resultant above the given addend, as we did in the previous example. Note that the yellow chips form a hundreds column. Once again we observe that the upper digit in the units column is smaller than the lower digit. In order to subtract seven, we are going to have to place ten blue chips with the four that we started with in the resultant. Let us do it together. We know that we must also add ten to the given addend; so let us join one red chip to the six red chips that we started with. But now we observe that there are no red chips in the resultant but there are seven red chips in the given addend. In order to proceed with our subtraction, we are going to have to add ten red chips to what we already have in the resultant. How much did we add to the resultant? (10) ________. Good! Since we joined ten red chips or one hundred to the resultant, we must also add one hundred to the given addend. Go ahead. Place one yellow chip with the yellow chip in the given addend. We are finally able to subtract. Again, we must compare the number of chips in the resultant with the number of chips in the given addend. Pairing the fourteen blue chips of the resultant with the seven blue chips of the given addend leaves us with a remainder of seven blue chips. Similarly, pairing the ten red chips of the resultant with the seven red chips of the given addend leaves us with a remainder of three red chips. Last but not least, pairing the three yellow chips of the resultant with the two yellow chips of the given addend leaves us with a remainder of one yellow chip. Our difference is 1 yellow chip, 3 red chips and 7 blue chips or 137.

Activity III

Look at the steps we have taken in the problem in order to discuss the game rules.

Example 1: $\begin{array}{r} 43 \\ -16 \\ \hline \end{array}$

(1)	$43 - 16 = (40 + 3) - (10 + 6)$	Renaming
(2)	$= (40 + (10 + 3)) - ((10 + 10) + 6)$	Rule of Equal-Additions
(3)	$= (40 + 13) - (20 + 6)$	Addition
(4)	$= (40 - 20) + (13 - 6)$	Rule of Subtraction and Reversibility & Arrangement for Addition
(5)	$= 20 + 7$	Subtraction
(6)	$= 27$	Addition

Now, let us look at our second example:

$\begin{array}{r} 304 \\ -167 \\ \hline \end{array}$

(1) $304 - 167 = (300 + 0 + 4) - (100 + 60 + 7)$

(2) $= (300 + 0 + (10 + 4)) - (100 + (10 + 60) + 7)$

(3) $= (300 + 0 + 14) - (100 + 70 + 7)$

(4) $= (300 + (100 + 0) + 14) - ((100 + 100) + 70 + 7)$

(5) $= (300 + 100 + 14) - (200 + 70 + 7)$

(6) $= (300 - 200) + (100 - 70) + (14 - 7)$

(7) $= 100 + 30 + 7$

(8) $= 137$

In what way did we use renaming? (11) _______. In Step (1) e.g., we renamed 167 as 100 + 60 + 7. In what way did we use the rule of equal-additions? (12) ____. In Step (2) we added ten to the resultant and to the given addend. In Step (4) we added a hundred to the resultant and to the given addend. In Step (6) we used what "game rules" in addition to the rule of subtraction? (13) _______________. Yes, we used reversibility and arrangement for addition. Finally, what process did you use to get an answer? (14) _______________. Of course you constructed the sum!

Performance Task

Demonstrate the solution to this problem using the algorithm of equal-additions. Make use of your chips. Also, demonstrate the solution in horizontal form and identify the game rules that you used.

$$\begin{array}{r} 416 \\ -277 \\ \hline \end{array}$$

Answers for the Performance Task

$416 - 277 = (400 + 10 + 6) - (200 + 70 + 7)$	Renaming
$= (400 + 10 + (10 + 6)) - (200 + (10 + 70) + 7)$	Rule of Equal-Additions
$= (400 + 10 + 16) - (200 + 80 + 7)$	Addition
$= (400 + (100 + 10) + 16) - ((100 + 200) + 80 + 7)$	Rule of Equal-Additions
$= (400 + 110 + 16) - (300 + 80 + 7)$	Addition
$= (400 - 300) + (110 - 80) + (16 - 7)$	Rule of Subtraction and Reversibility & Arrangement for Addition
$= (100 + 30) + 9$	Subtraction
$= 130 + 9$	Addition
$= 139$	Addition

APPRAISAL

The following is an illustration of one method of finding the product of any wo whole numbers. Suppose we wished to find the product of 24 and 3. This is ow we would proceed:

$$
\begin{array}{r} 24 \\ \underline{\times 3} \end{array}
$$

$= (20 + 4) \times 3$

$= (20 \times 3) + (4 \times 3)$

$= 60 + (10 + 2)$

$= (60 + 10) + 2$

$= 70 + 2$

$= 72$

Now show how you would use this <u>same</u> procedure to find the product of 14 and 6.

$$
\begin{array}{r} 14 \\ \underline{\times 6} \end{array}
$$

Suppose you were now called upon to provide an explanation of this method of multiplication using some kind of physical objects such as blocks or chips. Make a series of drawings to illustrate your explanation. The first drawing, showing the objects, might look something like this:

x x x x x x x
x x x x x x x

x x x x x x x
x x x x x x x

x x x x x x x
x x x x x x x

x x x x x x x
x x x x x x x

x x x x x x x
x x x x x x x

x x x x x x x
x x x x x x x

First Drawing

We would be able to agree that the product of 14 and 6 is a whole number because 14 and 6 are each whole numbers and because we have a certain game rule. Which game rule would we be using? ______________________________

____

An explanation of this multiplication procedure was provided by someone else and his explanation in terms of our game rules is given below. However, there is one game rule which has not been identified. Your task is to supply the missing information.

24 x 3	Given
(20 + 4) x 3	Writing an expanded numeral for 24
(20 x 3) + (4 x 3)	Multiplication is distributive over addition
60 + (10 + 2)	Writing another name for products
(60 + 10) + 2	* ______________________
70 + 2	Writing another name for sum of (60 + 10)
72	Writing contracted numeral

*Note: Regrouping is not an acceptable game rule. The game rules we have agreed to use include closure, reversibility or commutativity, arranging or associativity, identity, and inverse.

There are few adults who have not observed the unusual character of the number zero when performing the operation of addition. For example, 6 + 0 = 6 and 0 + 27 = 27. What have we named this particular game rule? ______________

Suppose we were constructing sums with integers, numbers such as 0, $^{+}1$, $^{-}1$, $^{+}2$, $^{-}2$, and so on. Someone was asked to show how to construct the sum of $^{+}4$ and $^{-}3$. This is the procedure he wrote:

$$^{+}4 + {}^{-}3$$
$$(^{+}1 + {}^{+}3) + {}^{-}3$$
$$^{+}1 + (^{+}3 + {}^{-}3)$$
$$^{+}1 + 0$$
$$^{+}1$$

By which of the game rules would we be able to explain $(^{+}3 + {}^{-}3) = 0$?

When asked to provide an explanation of this addition procedure in terms of the game rules, this is what was provided:

$^{+}4 + {}^{-}3$	Given
$(^{+}1 + {}^{+}3) + {}^{-}3$	Writing another name for $^{+}4$
$^{+}1 + (^{+}3 + {}^{-}3)$	______________
$^{+}1 + 0$	______________
$^{+}1$	______________

Your task is to supply the missing game rules.

Read the following descriptions of objectives and identify those which are behavioral objectives by placing a check (√) before those which are behavioral.

____ 1. The student will acquire a better understanding of the addition process.

____ 2. The student will demonstrate a process for finding the quotient of two natural numbers.

____ 3. The student will construct a bar graph given a table of data.

____ 4. The student will gain insight into the solution of equations.

____ 5. The student will inductively discover and use the properties of numbers.

Read the following description of a student performance and then insert the correct action verb in the statement of the objective.

1. The student is asked to illustrate a procedure for finding the product of 12 and 7. The student said nothing but did write:

$$\begin{array}{r} 12 \\ \underline{x7} \\ 70 \\ \underline{14} \\ 84 \end{array}$$

Objective: The student will be able to ______________________ a procedure for finding the product of two whole numbers.

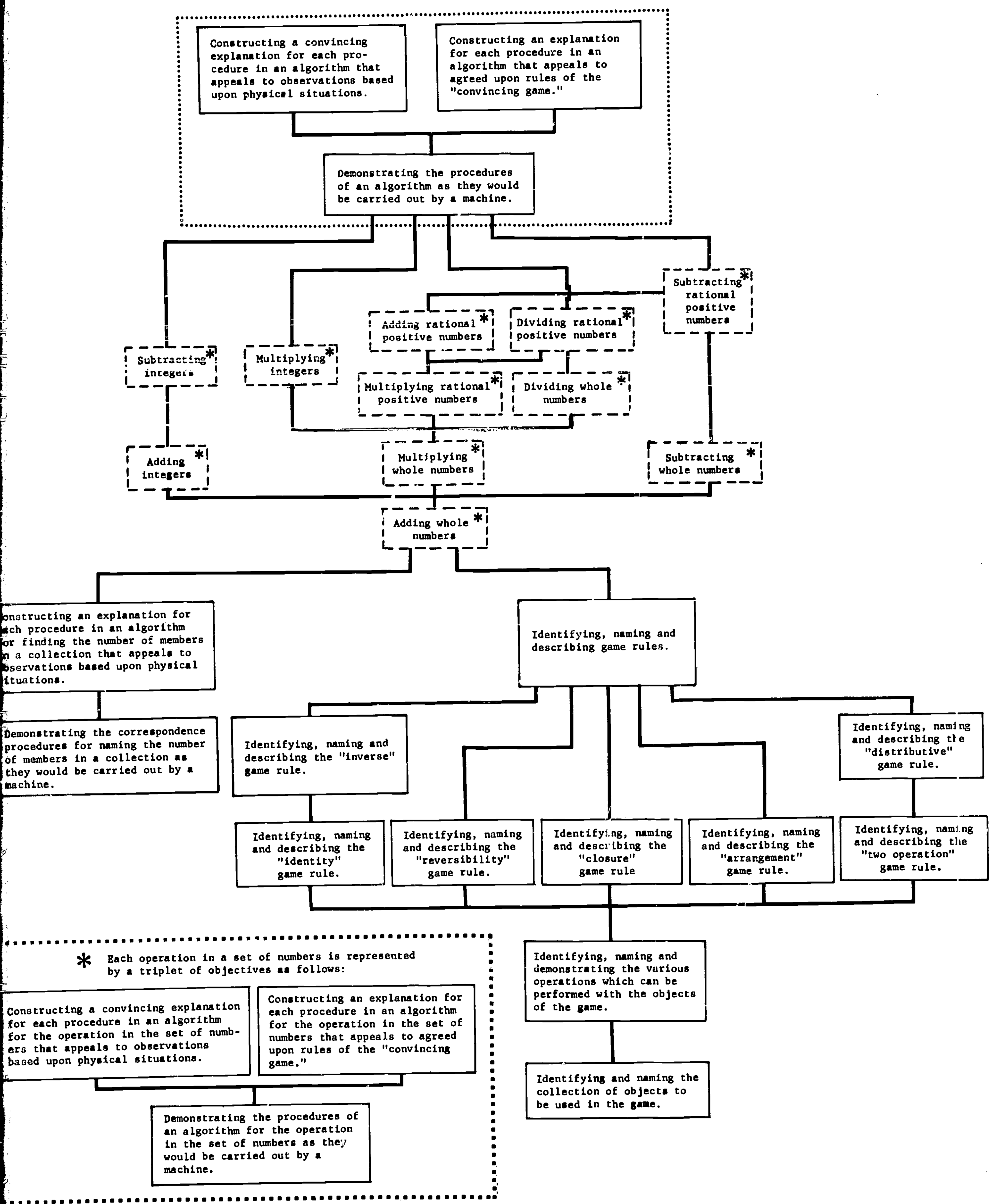
2. The student is asked to select the largest number, given the set $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$. The student says nothing but writes $\frac{1}{2}$.

Objective: The student will be able to ______________________ the largest fraction given several fractional numerals.

The action verbs we have agreed to use include identify, name, construct, demonstrate, order, describe, applying a rule, stating a rule, interpret, and distinguish.

APPENDIX C

ALGORITHMS PROCESS HIERARCHY



APPENDIX D

Partial List of Resource Volumes

1. Banks, J. Huston, Elementary School Mathematics, A Modern Approach for Teachers. Boston, Mass.: Allyn and Bacon, Inc., 1966.

2. Brumfiel, Eicholz, Shanks, O'Daffer, Principles of Arithmetic. Reading, Mass.: Addison-Wesley Press, 1963.

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APPENDIX E

Instructional Observation Data Sheet

ession ______________ Observer ______________________________

e ______________ Group ______________________________

me in nutes	Instructor's Questions	Instructor's Responses	Instructor Lecture and Directions	Student Quest.	Student Resp.	Nothing	Comment

Observation Sheet-2

Time in minutes	Instructor's Questions	Instructor's Responses	Instructor Lecture and Directions	Student Quest.	Student Resp.	Nothing	Comment
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