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RULE GENERALITY AND CONSISTENCY IN MATHEMATICS LEARNING.
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PSYCHOLOGICAL PRINCIPLES INVOLVED WITH RULE GENERALITY (DEGREE OF NONSPECIFICITY) AND PERFORMANCE CONSISTENCY IN MATHEMATICAL PRESENTATIONS WERE STUDIED. SPECIFICALLY, THE PURPOSES WERE (1) TO DETERMINE IF TEST BEHAVIOR CONFORMS TO THE SCOPE OF A VERBALLY ADMINISTERED TEST RULE, (2) TO EXPLORE THE INTERPRETABILITY OF VERBAL TEST RULES, AND (3) TO DETERMINE WHETHER "WITHIN SCOPE" USE OF A RULE IMPLIES "BEYOND SCOPE" USE WHEN NO INFORMATION IS GIVEN AS TO WHEN A RULE IS AND IS NOT APPROPRIATE. TWO EXPERIMENTS WERE CONDUCTED. IN EXPERIMENT 1, 85 COLLEGE STUDENTS PARTICIPATED IN A GAME OF NUMBERS, USING 1 OF 3 RULES (OF VARYING GENERALITY) FOR WINNING THE GAME. IN EXPERIMENT 2, THE VARIABLES WERE RULE GENERALITY (3 LEVELS) AND EXAMPLE (GIVEN-NOT GIVEN). THE MATERIALS, BASED ON ARITHMETIC SERIES, WERE PRESENTED TO 114 JUNIOR HIGH SCHOOL STUDENTS. EACH STUDENT WAS TESTED ON THREE PROBLEMS (1) WITHIN THE SCOPE OF THREE SEPARATE RULES, (2) WITHIN THE SCOPE OF THE TWO MORE GENERAL RULES, AND (3) ONLY WITHIN THE SCOPE OF THE MOST GENERAL RULE. FINDINGS WERE THAT (1) EACH GROUP'S PERFORMANCE WAS AT ESSENTIALLY THE SAME LEVEL ON THE "WITHIN SCOPE" PROBLEMS, (2) THE RULE TAUGHT FOR ONE PROBLEM TENDED TO BE USED BY THE SUBJECTS ON SUCCEEDING PROBLEMS WHETHER APPROPRIATE OR NOT, AND (3) THE "MOST SPECIFIC" RULE WAS BETTER LEARNED THAN OTHERS. PRACTICAL IMPLICATIONS FOR TESTING AND THEORETICAL QUESTIONS WERE DISCUSSED. (RS)

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ABSTRACT

Rule Generality and Consistency in Mathematics Learning

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Two experiments were conducted. In experiment one, 51 college *Ss* were taught one of three rules, of varying generality, for winning the game of NIM. Two additional groups of 17 *Ss* each served as controls. In experiment two, the variables were rule generality (3 levels) and example (given-not given). The materials, based on arithmetic series, were presented to 114 junior high school *Ss*. All *Ss* were tested on three problems, the first within the scope of each rule, the second within the scope of the two more general rules, and the third only within the scope of the most general rule.

The results generally justify the categorization of verbally presented rules as to generality. There was positive transfer to an outside scope problem in only one case and each group's performance was at essentially the same level on the within scope problems. In experiment one, the most specific rule was better learned than the others; a similar, but weaker, effect was noted in experiment two.

A third facet of the study dealt with response consistency. Except for one case where the effect was rather directly attributable to prior learning, those Ss who used the rule taught on one problem tended also to use it on succeeding problems whether or not the rule was appropriate.

Both practical implications for testing and theoretical questions were discussed.

Rule Generality and Consistency in Mathematics Learning[1]

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In many instructional situations, the question often arises as to how general the presentation of material ought to be. Some proponents emphasize that the more general a principle the more useful it will be; others, that the more specific the principle, the better the learning. There is a real need to better understand the psychological principles involved but previous studies dealing with rule (or principle) learning (e.g., Craig, 1956; Gagne and Brown, 1961; Haselrud and Meyers, 1958; Kersh, 1958, 1962; Kittle, 1957; (Scandura, 1964, 1966;) Wittrock, 1963), have dealt only indirectly with this question.

This study represents a first attempt to provide a rigorous definition of principle (or rule) generality and to contrast the logically determined behavioral implications of this definition with the results actually obtained. The definition of generality used is a natural extension of a definition introduced earlier (Scandura,

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1 This research was conducted at the Florida State University where the senior author was formerly located and was supported by the U.S. Office of Education.

Although a legitimate distinction may be made between rules and principles, the distinction is fine and was not recognized until after the study was completed. The terms have been used synonymously throughout the paper except in the concluding section where the distinction is outlined.

1966a) as part of a new scientific Set-Function Language (SFL) for formulating research questions on meaningful learning. The denotation (i.e., observable aspect) of the principle, which is the basic behavior unit (as opposed to the association) in the SFL, is defined as a function--that is, as a set of ordered stimulus-response pairs, $f = \{(S_i, R_i) | i = 1, \ldots, n, \ldots\}$, in which each stimulus is paired with a single response. When viewed in terms of sets, principles are naturally orderable as to generality; one set (principle) may be said to be more general than another if it includes all instances (S-R pairs) of the latter plus some of its own.

More particularly, concern here was with principle statements of the form, "If A, then B" (see Gagne, 1965; Scandura, 1966a, 1966d), --e.g., given some numerical series, the sum may be determined by squaring the number of terms in the series. The associated test stimuli (e.g., number series) and responses (e.g., sums) were used to test for the acquisition of stated principles.

In accordance with the above definition, the statement of a highly general principle was expected to induce appropriate performance on a wide variety of tasks. At the same time, it was felt that ease of applying a presented principle might vary directly with its specificity. In short, the generality of a principle is a structure variable, one whose effects on interpretability and transfer we wanted to determine.

Principle generality was not viewed simply as an empirical variable. To the contrary, we feel, as do Wittrock and Keislar*, that principle generality may be a fundamental variable underlying the results of some of the rule related studies cited above. For illustrative purposes, consider the Wittrock (1963) study. It is recent and well designed, and so provides an excellent case in point. The independent variables in Wittrock's study were (1) rule (given-not given), (2) answer (given-not given). The treatments involved presenting (or not presenting) a rule along with a problem to which the rule applied with (or without) the answer. For our purposes, it will suffice to consider the resultant performance on the original problem(s) and on new problems to which the rules applied. In view of the proposed definition of principle generality, the answer given groups were effectively presented with a very restrictive principle--a principle applicable to exactly one stimulus. The rule given groups were shown a more widely applicable rule statement. When looked at in this way it is not at all surprising that (1) the answer given groups did as well as the rule given groups on the learning test, (2) the rule given groups did better on new problems to

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*Personal communication.

which the rule was applicable. Furthermore, the relatively poor performance of the rule and answer given group, as compared to the rule given group, on the latter measure, is suggestive of the relative difficulty of learning principles of different generality. The obtained results could be explained by postulating that the rule and answer given <u>S</u>s took the path of least resistance and simply remembered the answer while ignoring the rule.

Of course, the results can be interpreted equally well in terms of the variables explicitly manipulated in the Wittrock (1963) study. In what sense is rule generality more basic than the rule and answer variables? The answer lies in what both we and Wittrock and Keislar* believe to be the greater explanatory power of what might be called generality laws (hypotheses)--e.g., a rule of lesser generality is easier to learn and apply than one of greater generality--as compared with those laws (results) obtained earlier by Wittrock--e.g., giving rules is better than not giving rules.... The generality laws can be used to explain these results, but the relationships found (i.e., the results) cannot easily be used to explain the proposed generality laws. It is to Wittrock's credit that he <u>emphasized</u> the phenotypic nature of the rule and answer variables and later helped to explicate one form of the generality hypothesis (based on an S-R mediation argument).

Another facet of this research concerns the

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*Personal communication.

*Personal communication. In reacting to the manuscript on which this article is based, Wittrock called our attention to another form of the generalization hypothesis. In two recent studies (Wittrock, Keislar, & Stern, Verbal Cues in Concept Identification, Journal of Educational Psychology, 1964, 55, 195-200; Wittrock & Keislar, Verbal Cues in the Transfer of Concepts, Journal of Educational Psychology, 1965, 56, 16-21.), Ss were pretrained on a hierarchically arranged associative structure, composed of word or pictorial stimuli, and then, during learning, were given cues at one level in the hierarchy. These cues were presumed to facilitate discovery of the concepts and/or rules underlying appropriate behavior on the learning tasks. These authors,

> "hypothesized that in a problem-solving test where there was a low probability of success without cues, transfer would be related to the type of cue used earlier during instruction. The class cue should produce the greatest transfer to new instances of the same class; the specific cue should produce the greatest initial learning; and the general cue should produce the greatest remote transfer or transfer to new concepts (Wittrock & Keislar, 1965, 16)."

Support was obtained for the first two hypotheses, but not the last. Since the prior training was common to all Ss, the cues presumably acted as much to induce an appropriate responding set as to modify "what was learned." To the extent, then, that word and pictorial cues, pretraining on hierarchically arranged associative structures, and einstellung effects or learning by discovery, can be equated with rule or principle statements, the prior learning had by the college Ss used in this study, and what is learned by exposition, respectively, the hypotheses proposed by Wittrock and Keislar and those proposed here are very similar.

Nonetheless, the present hypotheses were derived in a manner quite distinct from the way in which Wittrock and Keislar derived theirs. For one thing, transferability was equated with the logically determined scope of principles--the inclusiveness of its denotative set of instances (ordered S-R pairs). The interpretability hypothesis, however, was originally based simply on intuition. It was only after a post hoc analysis of the experimental results led to a more complete formulation of the SFL that a formal rationale was proposed. Wittrock and Keislar based their hypotheses on S-R mediation theory. In addition to being based on the principle, rather than the association, the SFL seems to be leading to quite a different set of theoretical assumptions than those underlying S-R theories. In particular, (continuous) generalization gradients, based on response strength (see footnote in Discussion Section), do not appear necessary in SFL formulations.

Further discussion is beyond the scope of this paper. Some related issues have been dealt with more fully elsewhere (Scandura, 1966c, 1966d) and theoretical work is still in progress.

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consistency with which a presented principle (or concept) is applied. In an earlier study[2], Greeno and Scandura (1966) found, in a verbal concept learning situation, that after learning a common response to one or more stimulus exemplars of a concept, S either gave the correct response the first time he saw a transfer stimulus (i.e., a new exemplar of the concept) or the transfer item was learned at the same rate as its paired control in a (transfer) paired-associate list. Scandura (1966a, 1966c, 1966d) later reasoned that if transfer obtains on trial one, if at all, then responses to additional transfer items, under appropriate conditions, should be contingent on the response given to the first transfer stimulus. The results of a pilot study were revealing (Scandura, 1966c). In 47 of 52 cases, in which the first test response indicated that a concept had been acquired, the concept also provided the basis for responding to a second test (i.e., transfer) stimulus. Similar pilot results obtained when the test responses were based on principles of the form, "If the stimulus object is large, then the response is the name of its color" (Scandura, 1966a, 1966d).

The primary purpose of this research was to determine whether test behavior conforms to the logically

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2 Conducted during the summer of 1962.

determined scope of a verbally stated principle. Assuming no *a priori* knowledge of related principles, appropriate responding is to be expected only within the scope of the principle. However, there sould be no systematic within scope differences--all stimuli within the scope of a principle should be of approximately the same difficulty. A secondary purpose of the study was to explore the question of interpretability. When and why do verbally stated principles differ in ease of learning as determined by performance on within scope items? Originally, it was thought that generality itself might be the sole crucial factor. The final purpose of this research was to obtain further data on Scandura's (1966a, 1966c, 1966d) response consistency hypothesis in a more complex situation. Does within scope use of a rule imply beyond scope use when no information is given as to when a rule is and is not appropriate?

To obtain evidence on these points two experiments were conducted concurrently, one with college *Ss* and the other with junior high school *Ss*. Mathematical materials were used in both experiments.

EXPERIMENT ONE

Method

Material and Subjects. The material consisted of a variant of the number game, "NIM" (Banks, 1964, 55-58). In the game, two players alternately select numbers from a specified set of consecutive integers (including 1) and keep a running sum. The winner is the one who picks the last number in a series having a predetermined sum. If this sum is 31 and the set consists of the integers 1-6, the players select numbers from 1-6 until the cumulative sum is either 31 or above (in which case no one wins).

Each game of NIM can be characterized by two integers, an ordered pair (n, m) where n is the largest integer in the selection set and m is the predetermined sum.

Rules are available by which the person making the first selection can always win. Some of these rules are particular to specific games whereas others are more general. The most specific (S) rule referred only to (6, 31) games. It was stated:

"There is a pattern to the game which will enable you to win whenever you are allowed to make the first selection. You must, however, make an appropriate first selection and then proceed in a precise manner. In order to win the game you should make 3 your first selection. Then you should make selections so that the sums corresponding to your selections differ by 7."

The rule of intermediate (SG) generality referred to all games of the form (6, m) and was stated:

> "...In order to win the game, the appropriate first selection is determined by dividing the desired sum by 7. The remainder of this division is precisely the selection which should be made first...."

The most general (G) rule referred to games of the form (n, m) and was stated:

> "...In order to win the game the appropriate first selection is determined by adding one to the largest number in the set from which the selections must come and dividing the desired sum by this result. The remainder of this division is precisely the selection that should be made first. Then you should make selections so that the sums corresponding to your selections differ by one greater than the largest number in the set from which the selections must come."

All of the materials were reproduced by mimeograph and were combined into four 8 1/2" x 11" booklets; introduction, treatment (i.e., rule), test material, and answer sheet. The introductory booklet contained six pages. Page 1 simply indicated that the experimental results would be made available to the Ss and asked that they not divulge information about the experiment to others who might be participating. The nature of the (6, 31) game was explained and an example given on page 2. Pages 3 and 5 consisted of two different (6, 31) games and required S to compute the running sum in each in accordance with a specified sequence of selections. This was done to ensure that the Ss knew the objective of and how to play the game. Knowledge of results on these practice games was given on pages 4 and 6. Nothing was said about the game winning procedures, but it was mentioned that there are many variations of NIM.

There were five different treatment booklets. Three included one of the game winning rules and a common (6, 31) game to which the rule was applied. The other two booklets served as controls. The C treatment booklet consisted of a short topic on divisibility with questions about divisibility on the second page. Booklet E consisted of the divisibility topic and the (6, 31) example. In this common game, the running sums were 3, 5, 10, 13, 17, 23, 24, 25, 31. Those numbers in italics resulted directly from the hypothetical winner. By remembering these sums it would be possible to win any new (6, 31) game.

The test booklet consisted of seven pages, the first of which explained how to use the booklet. S was told to make a selection on the answer sheet and then turn to one of the remaining six pages in the materials booklet for the opponent's selection. This order was scrambled for each of the three problems presented. Problem one consisted of a (6, 31) game, problem two a (6, m) game with m = 25, and problem three an (n, m) game with n = 4 and m = 22.

The Ss were 85 Florida State University undergraduates enrolled in a mathematics education course for elementary teachers. Participation was a class requirement.

Design and Procedure. The Ss were assigned randomly to three treatment groups and two controls so that each group contained 17 Ss. Group S was given the (6, 31) rule and example, group SG the (6, m) rule and example, group G the (n, m) rule and example, group E only the example, and group C nothing relevant. The experiment was run in groups of 17 or fewer Ss with all but group E represented in approximately equal numbers. Group E was run at one sitting shortly after the other Ss were run to determine whether the example itself had a significant effect on learning.

At the beginning of each experimental session, the Ss were presented with the common instruction booklet and one of the five treatment booklets as indicated above. They were told to read the material carefully. After these booklets were completed, they were collected and S was given the test and answer booklets. The experiment was self-paced. Time for completion of the entire experiment varied between 15 and 40 minutes.

Two binary criterion measures were used, use of appropriate pattern (AP) and use of the rule (UR) taught.

S was given credit for using the AP if he won the game and employed an appropriate game winning strategy. Credit was given for UR regardless of whether or not the strategy used was appropriate for winning the game in question (e.g., UR credit was given to the S group Ss for applying rule S on test game 2 and/or 3).[3]

All of the tests conducted were applied to 2 x 2 contingency tables (with no pooling over treatment groups or test problems). When the measures were independent, the exact Fisher-Yates test was used; when correlated, a different nonparametric test, based on χ^2, was used (McNemar, 1954, 358-359). Alpha levels, for the former test, may be obtained directly from tables prepared for this purpose (Finney, 1948; Latscha, 1953). One-tailed tests were used in conjunction with the stated hypotheses with an alpha level of .05.

Results

Table 1 shows the number of Ss in each group who used the AP on problems one, two, and three.

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[3] There were two exceptions. One S summed incorrectly once but otherwise established an appropriate pattern and was given credit for a successful execution. Another S made the first four selections appropriately, deviated on the fifth, but still won the game; this person was also credited with a success.

Rule use was determined from the first and second test selections only. This was done because it was impossible for the Ss in groups S and SG to continue using their rule beyond that point in test problem three without choosing a number outside the selection set.

Table 1

Number of Appropriate Patterns

	N	Problem One (6, 31)	Problem Two (6, 25)	Problem Three (4, 22)
Group C	17	0	0	0
Group E	17	3	0	1
Group S	17	13	0	1
Group SG	17	5	4	0
Group G	17	5	5	4

The three treatment groups performed according to prediction. Thirteen of 17 Ss in group S were able to apply the (6, 31) rule to within scope problem one, but none discovered the more general (6, m) pattern and only one discovered pattern (n, m). The differences between problem one and problems two and three were both highly reliable ($p < .001$). As hypothesized, the SG Ss also used an AP only on those problems within the scope of the rule taught. Significantly more SG Ss solved problem one than problem three ($p < .02$); the corresponding problem two-problem three difference attained the .05 level. There were five successes on problem one and four on problem two but none on problem three. In addition, three of the E Ss did use the game winning (6, 31) pattern on problem one after having seen it used on the (6, 31) example but none of the small differences in performance on the three test problems was significant in either of the control groups (i.e., C and E).

Equally important, there was essentially no difference in within scope performance. Of the five SG Ss who used an AP on problem one, four did so on problem two. The same five G Ss used an AP on within scope problems one and two; only one of these Ss failed to use an AP on problem three. Even the two minor deviations noted could conceivably have been classified as AP.

Table 1 also provides a basis for comparing different treatment groups as to ease of learning. Again, 2 x 2 contingency tables were used with groups and success-failure being the dimensions. Cell entries were the number of Ss in each category on a given problem.

Prior to conducting the experiment, it was conjectured that performance may be enhanced most by stating a rule in as specific a form as possible so long as the criterion is within the scope of the rule. This general hypothesis leads to the following predictions: (1) problem one, $S > SG > G$, (2) problem two, $S < SG > G$, and (3) problem three, $S = SG < G$.

The results only partially confirmed these expectations. On problem one, group S performed better than groups SG and G ($p < .005$ in both cases), but groups SG and G performed at essentially the same level. On problem two, group SG performed better than group S ($p < .05$); but again, groups SG and G did not differ appreciably. On problem three, there was essentially no S-SG difference but the G-SG difference was in the expected direction ($p < .05$). Representing nonsignificant differences by equal signs, these results may be summarized: (1) problem one, S > SG = G, (2) problem two, S < SG = G, and (3) problem three, S = SG < G.

Consistency was based on the UR measure. In each case, rule users and non-rule users on a given problem were compared as to rule use on succeeding problems.

Table 2 shows that the rules taught were used on all problems to about the same degree. A more intensive

Table 2

Use of Rule Taught

	N	Problem One (6, 31)	Problem Two (6, 25)	Problem Three (4, 22)
Group S	17	13	9	8
Group SG	17	7	7	5
Group G	17	6	6	6

individual analysis indicated that, in general, rule users on problem one were rule users on problems two and three. Non-rule users on problem one tended to be non-rule users on the remaining problems. There were four Ss in group S who used the rule taught on problem one, but not on problem two; there was one who used the rule on problem two, but not on problem three. In group SG, the corresponding numbers were zero and two; in group G, they were zero and zero. In all, there was a total of five cases in which non-rule users later used the rule taught. Of these, three were Ss in group S who, for some unknown reason, failed to use the rule on problem two but did so on problem three.

In group S, significantly more rule users on problem one were rule users on problem two than was the case with non-rule users ($p < .02$). The same relationship held for problems two and three ($p < .04$). The corresponding significance levels, in group SG, were beyond .001 and .01 and, in group G, were beyond .001 and .001, respectively.

Discussion

These results certainly provide strong support for our original hypotheses: (1) performance on within-scope problems did not differ appreciably, even though the common example was more similar to problem one than the others, and successful problem solving was limited almost exclusively to within-scope problems, (2) rule S proved easier to apply than rules SG and G, and (3) the rules taught tended to be used consistently on all problems whether they were appropriate or not.[4]

About the only major unanticipated result in experiment one was that Rule G proved as easy to interpret as

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[4] The first mentioned result has particular relevance for the psychologist since it tends to cast doubt on the typically made assumption that there is a generalization gradient (Based on response strength) associated with S-R generalization. According to this assumption, performance on the first test problem, which was similar to the common example, (both were (6, 31) games), should have been superior to that on the other problems. Even S-R associationists (e.g., Berlyne, 1965, 171-174) are generally agreed that the lack of such an effect provides indirect support for a rule or principle interpretation.

Even if a generalization gradient is eventually demonstrated, S-R theorists will need to consider the possibility that such a result is simply an artifact of averaging individual differences in perceptual discrimination (and temporal) over continuous dimensions. To the extent that the variables involved in meaningful learning are discrete, a rule interpretation may prove more useful.

The consistency results also deserve comment. The observed consistency probably was due in no small part to the lack of information indicating when a presented rule was and was not appropriate. The unfamiliarity of the material and the lack of negative feedback also may have been important contributing factors. These boundary conditions were similar to those obtaining in the pilot studies cited in the introduction. It would appear that S will continue to respond as instructed unless confronted with an inappropriate stimulus or feedback otherwise indicates that the rules have changed.

The generalization of the pilot results to a more complex setting lends further credence to our belief that consistency of responding is a basic rule of behavior which has far-reaching practical as well as theoretical implications. These and other related issues have been discussed more fully elsewhere (Scandura, 1966a, 1966c, 1966d) and need not be repeated here.

Rule SG. Since these groups were not performing significantly better than control group L on problem one, we were tempted to attribute the lack of an anticipated SG-G difference to scale insensitivity and thereby not be forced to accept a "no difference" hypothesis--especially since the power of our tests were unknown.

EXPERIMENT TWO

Method

Materials. The materials were based on arithmetic series--i.e., number series, of the form a + (a + d) + (a + 2d) + ... + (a + nd) where a, d, and n are integers. Three types of arithmetic series were considered: (1) series (S) beginning with 1, ending with 99, and having a common difference of two, (2) series (SG) beginning with 1 and having a common difference of two, and (3) arbitrary arithmetic series (G). These categories were ordered in the sense that all S series were also SG series and all SG series were also G series.

Rules are available for finding the sums of S, SG, and G series. These rules were stated as follows:

S - The sums of some arithmetic series may be obtained by multiplying 50 x 50;

SG- ... by multiplying the number of terms in the series by itself;

G - ... by adding the first number in the series to the last number in the series, dividing the resulting sum by 2, and then multiplying the number you get by the number of terms in the series.

All of the materials were reproduced by mimeograph and were combined into a single 5 1/2" x 8 1/2" booklet. On page 1, S was introduced to the symbols used (e.g., " • " for multiplication) and was then asked to compute: 89 • 74, (57 + 95)/4 • 27, and(X + Y)/T • W where X = 3, Y = 5, T = 4, and W = 6. Arithmetic series were defined on page 2, with examples, and the use of dots, as a labor saving device in writing number series, was introduced and illustrated. Finally, S was asked to fill in the missing terms in the series 7 + 10 + __ + 16 + __ + __ + 25. Page 3 differed according to the treatment but always included one of the rule statements and sometimes the series 1 + 3 + 5 + ... + 99 illustrating its use. The first test series was identical to the example and was presented on page 4. Test series two, 1 + 3 + 5+...+ 79, was presented on page 5, and test series three, 2 + 4 + 6 + ... + 48, on page 6.

Subjects and Design. The Ss were students at the Florida State University Campus School. There were 29 Ss in grade six, 76 in grade seven, and 79 in grade eight.

Two variables were independently manipulated, rule generality (S, SG, G) and example (given, not given). The Ss were assigned at random to the six treatment conditions so that each treatment was as nearly equally represented in each of the seven classes (one sixth, three seventh, and three eighth) as possible.

Since a large number of Ss did poorly on both the pretest (pages 1 and 2) and the three post-test series, data were presented only for those 114 Ss who got at least three of the four pretest problems correct. The post-test results of the poorer students were in the same direction as the others but so few were successful on the test series that the overall power of the statistical tests used would have been reduced.

Procedure. The experiment was conducted in classrooms during periods scheduled for mathematics instruction. The regular teachers administered the materials in their respective classes under the direct supervision of Frank Lee. Five minutes was allowed on each of the first two pages in the booklet, two minutes on page 3, and three minutes on each test problem, 21 minutes in all. To insure uniformity, the teachers consulted with Lee and were given a page of explicit directions to follow. The teacher read a prepared note urging the Ss to do their best and telling them that they would be informed of the experimental results. The teachers were given a report of the results, for dissemination, as promised.

The dependent variables were correctness of sum and use of the rule taught. Fischer's exact test was supplemented with χ^2-tests when marginals were greater than 20 and beyond the scope of the available tables (Finney, 1948; Latscha, 1953).

Results

The results shown in Table 3 for the no-example Ss closely parallel those in experiment one. Only three of 59 Ss solved an extra scope problem; that S S who was successful on test series two used the appropriate rule, 40 x 40. Scrutinizing the test papers suggested that the two S Ss who correctly determined the series three sum did so by brute force methods. A fast adder could probably have succeeded in the time allowed (series three was considerably shorter than the others--46 was the largest term as opposed to 99 and 79). In group S, the performance differences between problem one and problems two and three were significant ($p < .01$ and $.02$, respectively). Also as in experiment one, significantly more SG Ss solved problem one than problem three ($p < .02$); the corresponding problem two-problem three difference was also significant ($p < .05$). Performance on within scope problems, again did not differ appreciably.

On problem one, the no example groups were ordered S > SG > G, but only the S-G difference was significant ($p < .05$). On problem two, groups SG and G (proportionately) did not differ appreciably, but groups SG and G were more successful than was group S ($p < .05$ and $.01$, respectively). None of the groups differed appreciably in their ability to solve problem three due to the poor performance of the G Ss on this problem.

The availability of an illustrative series, however, not only improved overall test series one performance ($p < .005$), but contrary to expectation, improved group S performance significantly on problem two ($p < .01$).

Table 3

Number of Correct Answers

	Rule - No Example				Rule - Example			
	N	1	2	3	N	1	2	3
S	20	8	1	2	21	20	9	3
SG	20	5	4	0	15	11	8	1
G	19	3	5	2	19	18	14	3

Another result was also unanticipated. Whereas 18 of 19 Ss in group G-with-example were successful on test series one and 14 were successful on series two, only three found the correct sum of series three. There were proportionately more successes on series two than on series three ($p < .003$).

In experiment two, consistency was more difficult to determine since many of the Ss apparently did some or all of the calculations in their heads. Nonetheless, what *modi operandi* could be identified were relatively consistent between problems in groups SG and G. It was also noted that inappropriate attacks also tended to be used consistently in these groups.

Table 4

Use of Rule Taught

	Rule - No Example				Rule - Example			
	N	1	2	3	N	1	2	3
Group S	20	8	1	1	21	20	0	0
Group SG	20	5	5	5	15	12	9	6
Group G	19	5	7	7	19	18	16	15

The major exception was group S. Only one of those 28 Ss in the two S groups who used the rule taught on series one also used it on the other problems. Apparently, the same reluctance to respond to test series one, with the answer shown, which characterized those Ss shown only rule S, was magnified on problems two and three.

Discussion

Although the results of experiment two paralleled those of experiment one in most respects, there were several important differences. First, the presence of the example (problem one) along with Rule S resulted in significantly better performance on problem two than when Rule S was shown alone, the only case in either experiment where non-negligible success was noted on an extra scope problem. This effect may have been due to the form of the combining operation, "50 x 50," in the Rule S statement. "50 x 50" is clearly an instance of the most general SG combining rule, "$N \times N = N^2$." Presumably, the statement of Rule S, together with the common example, 1 + 3 + 5 +...+97 + 99, provided the successful S Ss with enough cues to generalize. In particular, they may have discovered that this series had 50 terms. An analysis of the test papers tended to substantiate this interpretation. At least three of the nine successful S Ss squared 40 to get the answer. Two more Ss multiplied 30 x 30.[5] Hindsight suggests that this difficulty might have been overcome by simply stating the sum, 2500, of the illustrative series rather than "50 x 50."

Second, only three of the nineteen G-with-example Ss solved problem three whereas 18 solved problem one and 14 solved problem two. The reason for this difference was not immediately apparent especially since 15 of these Ss applied Rule G to the third problem. A more intensive *post hoc* analysis of the situation,

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[5] Note that there are 10 odd integers between 99 (last term in series one) and 79 (last term in series two) and that 50 - 10 is 40. 50 - 20 (the difference between 99 and 79) is 30. Three other Ss apparently found the correct series two sum by subtracting the sum of the odd integers between 79 and 99 from 2500 (sum of series one).

however, suggested that the result may have been due to a difference in ease of determining N, the number of terms, for use in the G combining rule, [(A + L)/2]N. N could be determined from problem series one and two by taking the average of the first and last terms (i.e., A and L). A careful examination of the test papers suggested that this led to an incorrect value (25 rather than 24) for N in the third series, 2 + 4 + 6 +...+ 46 + 48. In short, the difficulty was not in the rule itself but in finding the correct value of N. Such difficulties may be circumvented in future experimentation by controlling for such unwanted differences.[6]

Third, although the results of experiment two were in the hypothesized direction, only the overall effect of scope on interpretability was significant. This led us to wonder whether the interpretability of a rule statement depends <u>solely</u> on its generality. Could the rule statements have also differed as to the difficulty of interpreting the actual terms or symbols

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[6] At least 7 of the 15 Ss who made an attempt to apply Rule G determined the number of terms in series three to be 25. This was a reasonable selection in view of our results concerning consistency, <u>Einstellung</u> if you will, (A + L)/2 = (2 + 48)/2 = 25.

It may be desirable to think of properties, such as N, as being derived from lower order (i.e., more easily discernible) stimulus properties. Thus, the rule, (A + L)/2, worked for problems one and two whereas L/2 was required for problem three.

used?[7] After consideration of this possibility, it was rejected as an important factor in experiment two since a recheck convinced us that we had succeeded reasonably well in stating each principle as clearly as possible. Perhaps a more likely interpretation is that the Ss' were sufficiently familiar with the terms used to compose rules SG and G to reduce the effects of statement generality.

Fourth, only one of the Ss who was shown the rule, 50 x 50, applied it to problems two and three. This result can probably also be attributed to an interfering effect due to familiarity with addition problems. The

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[7] Whereas "N," for example, might suffice for one S, another might require, "the number of terms in a series." Both have the same referent, but the former at once symbolizes the referent more succinctly and requires more specialized knowledge for interpretation. Similarily, one S may be able to interpret "compute [(A + L)/2]N" whereas another could not, requiring instead a statement like, "add A to L, then, divide the resulting sum by 2, and finally, multiply the quotient so determined by N." The latter rule statement simply makes clear the sequence of steps and binary operations implied by the algebraic statement.

Why one way of symbolizing a statement is more interpretable than another, rather than vice versa, is a difficult question to answer, but it probably relates to the order in which symbolizations are learned. Ordinarily, shorter statements are substituted for longer ones as their use becomes more frequent. Perhaps this is a natural process resulting from man's tendency to recode information into a manageable number of "chunks" (e.g., Miller, 1956; Scandura & Roughead, 1966). At any rate, the senior author has recently completed a study which demonstrates that shorter symbolic representations are more easily learned and remembered, whether or not they are familiar, but that the ability to apply them depends critically on the ability to operationally use the constituent symbols and the grammatical schemas relating them in the principle statement (Scandura, 1966d).

Ss may simply have mistrusted Rule S. How could a rule, like 50 x 50, having only one answer, be the sum of all three problem series? Most junior high school Ss would find it unreasonable that the series 1 + 3 +...+ 99 (problem one) and 1 + 3 +...+ 79 (problem two) have the same sum (50 x 50). Some such reluctance may also have obtained on problem one with group S-without-example. Nonetheless, we were surprised that only 8 of those 20 Ss, not presented with the example, gave the correct sum (2500) for problem one.[8]

Implications and Theoretical Comment

The results of these experiments demonstrate, in a rather conclusive fashion, the behavioral relevance of principle generality. For the most part, successful performance was noted only on tasks within the scope of verbally stated principles. When principles are presented in an expository fashion, it is normally too much to expect generalization to problems to which the principle does not immediately apply.

Of perhaps even greater practical significance were the lack (there was one exception) of performance differences on within scope problems and the consistency results.

[8] It should be emphasized that Rule S, in experiment two, was conceptually different from the others used in experiments one and two. Rule S applied to only one stimulus (series) whereas each of the others applied to a set of stimuli. The fundamental nature of this difference has been discussed in detail elsewhere (Scandura, 1966a, 1966d).

The former result demonstrates that (almost) any stimulus within the scope of a principle is equally as difficult to respond to correctly as any other. Furthermore, coupled with the consistency data cited in the introduction, the obtained consistency results suggest that only one (new) test stimulus is needed to determine whether, in fact, a stated principle has been learned (i.e., correctly interpreted). No more information is gained by using additional test instances. These results could have far-reaching implications for the development of highly efficient measuring instruments.

In addition, the pronounced tendency of the *Ss* to attack all of the test problems in the same way, irrespective of whether the procedure used was appropriate, suggests that the ability (i.e., knowing how) to solve problems and knowing *when* to solve them are quite distinct. Testing for the latter ability necessarily must involve the presentation of extra-scope problems.

More important than the results of these exploratory experiments were the *post hoc* analyses they made both necessary and possible. In particular, the preceding discussion strongly suggests that the roles played by various aspects of a principle statement need to be more clearly specified. The form, "If $\bar{A}$, then $\bar{B}$" does not detail all that appears relevant. For one thing, it was not possible in this study to distinguish between the roles played by A, L, and N (the stimulus variables entering into the rule $[(A + L)/2]N$) and the algebraic expression, $[(X + Y)/2]Z$ (the form of the combining operation of the rule by which the appropriate response sums are determined). The variables relate to properties of arithmetic series stimuli, while the algebraic expression represents a ternary operation by which another property (e.g., sums) may be derived. $\bar{X}$, of course, while it played no role in this study is also critical. It tells when a rule can and can not be applied. Thus, the rule, N^2, is appropriate whenever an arithmetic series consists of the odd integers beginning with 1 while $[(A + L)/2]N$ works whenever there is a common difference between adjacent terms.

These observations suggest that a principle statement may be represented more appropriately by the form, "If I', then O' (D') ≡ R'," where I' refers to the set of stimulus properties which indicate when the rule, denoted O' (D'), should be applied, D' refers to the set of those properties which determine the responses, and O', to the operation from which the responses, denoted by R', may be derived from the properties referred to by D'.[9] That part of a principle statement represented by O' (D') corresponds to what is typically called a rule.

9 Similarly, a principle, that internalized representation which determines a learner's responses to stimuli, may be characterized by an ordered four-tuple (I, D, O, R). Primes, of course, have been used to distinguish between the referents (e.g., O) and the symbols used to represent them (e.g., O', O", etc.). These definitions, along with that given in the introduction, form the basic elements of the Set-Function Language (Scandura, 1966a, 1966d).

Although the actual symbols used in a statement may be an important factor, as suggested above, the hypothesis advanced in this study to the effect that rule generality and interpretability are inversely related finds a formal rationale in the nature of the characterizing elements. Making operational use, for example, of the arithmetic series property (i.e., dimension), "the difference between adjacent terms is some common value," necessarily presumes that, "the difference between adjacent terms is 2," "...3," "etc.," can all be correctly interpreted. The converse does not necessarily follow. A similar relationship exists with respect to the rules, 50 x 50 and N x N. To correctly apply the latter, more general, rule to any particular series requires the ability to determine any value of the dimension N, including 50. Being able to apply 50 x 50 does not.

It would appear that the more general the principle the more is expected of the learner. Whether such differences will be reflected in behavior, however, may depend on not only rule generality but the population involved, particularly on whether the Ss have the necessary requisite abilities (Gagne, 1962; Scandura, 1966b).

In effect, differences in generality appear, on analysis, to be equivalent to differences in abstraction level. Thus, the number _2_ is more abstract than the property _two oranges_ because the former applies to a collection of sets only one of which has the latter property. For the same reason, the property represented by the placeholder _X_ is more abstract than the number _2_ since it refers to a still higher order collection. Unfortunately, we have not yet conducted a study designed to provide definitive information on these points. For the present, this analysis remains hypothetical.

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RULE GENERALITY AND CONSISTENCY IN MATHEMATICS LEARNING

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BACKGROUND

In many instructional situations, the question often arises as to how general the presentation of material ought to be. Some proponents emphasize that the more general the presentation the more useful it will be; others, that the more specific the presentation the better the learning. There is a real need to better understand the psychological principles involved but previous studies dealing with rule or principle learning have dealt only indirectly with this question.

A related problem concerns the consistency with which a learned principle is applied. In an earlier study, Greeno and Scandura found that after learning a common response to one or more stimulus exemplars of a concept, S either gave the correct response the first time he saw a transfer stimulus (i.e., a new exemplar of the concept) or the transfer item was learned at the same rate as its paired control. Scandura later reasoned that if transfer obtains on trial one, if at all, then responses to additional transfer items, under appropriate conditions, should be contingent on the response given to the first transfer stimulus. In short, having learned a concept (or principle), S should respond in a consistent manner.

OBJECTIVES

This study represents a first attempt to provide a rigorous definition of principle (or rule) generality and to contrast the logically determined behavioral implications of this definition with the results actually obtained. The definition of generality used is a natural extension of a definition introduced earlier as part of a new scientific Set-Function Language (SFL) for formulating research questions on meaningful learning.

More particularly, concern here was with principle statements of the form,"If A, then B"--e.g., given some numerical series, the sum may be determined by squaring the number of terms in the series. The associated test stimuli (e.g., number series) and responses (e.g., sums) were used to test for the acquisition of stated rules. The statement of a highly general principle was expected to induce appropriate performance on a wide variety of tasks. At the same time, it was felt that ease of applying a presented principle might vary directly with its specificity. Principle generality was not, however, viewed simply as an empirical variable. To the contrary, principle generality may be a fundamental variable underlying the results of some of the rule related studies cited above.

The primary purpose of this research was to determine whether test behavior conforms to the logically determined scope of a verbally stated principle.

Assuming no a priori knowledge of related principles, appropriate responding is to be expected only within the scope of the principle. However, there should be no systematic within scope differences--all stimuli within the scope of a principle should be of approximately the same difficulty. A secondary purpose of the study was to explore the question of interpretability. When and why do verbally stated principles differ in ease of learning as determined by performance on within scope items? Originally, it was thought that generality itself might be the sole crucial factor. The final purpose of this research was to obtain further data on Scandura's response consistency hypothesis in a more complex situation. Does within scope use of a rule imply beyond scope use when no information is given as to when a rule is and is not appropriate?

PROCEDURE

To obtain evidence on these points two experiments were conducted concurrently, one with college Ss and the other with junior high school Ss. Mathematical materials were used in both experiments.

In experiment one, the material consisted of a variant of the number game, NIM. In the game, two players alternately select numbers from a specified set of consecutive integers (including 1) and keep a running sum. The winner is the one who picks the last number in a series having a predetermined sum. If this sum is 31 and the set consists of the integers 1 - 6, the players select numbers from 1 - 6 until the cumulative sum is either 31 or above (in which case no one wins). Each game of NIM can be characterized by two integers, an ordered pair (n, m) where n is the largest integer in the selection set and m is the predetermined sum.

Rules are available by which the person making the first selection can always win. Three rules of varying generality were considered. The most specific (S) rule referred only to (6, 31) games. The rule of intermediate (SG) generality referred to all games of the form (6, m). The most general (G) rule referred to games of the form (n, m).

All of the materials were combined into four booklets; introduction, treatment (i.e., rule), test material, and answer sheet. The introductory booklet introduced the Ss to the experiment, explained and illustrated a sample (6, 31) game, and provided two practice (6, 31) games to make sure that the Ss could play the game (but not necessarily win). There were five different treatment booklets. Three included one of the game winning rules and a common (6, 31) game to which the rule was applied. The other two booklets served as controls. Three problems were included in the test booklet. Problem one was a (6, 31) game; problem two a (6, m) game with m = 25, and problem three an (n, m) game with n = 4 and m = 22.

The 85 experimental Ss (college undergraduates) were assigned randomly to three treatment groups and two controls so that each group contained 17 Ss. Group S was given the (6, 31) rule and a (6, 31) example, group SG the (6, m) rule and the example, group G the (n, m) rule and the example, group E only the example, and group C nothing relevant. The experiment was run in groups of 17 or fewer Ss.

At the beginning of each experimental session, the Ss were presented with the common instruction booklet and one of the five treatment booklets. After these booklets were completed, they were collected and S was given the test and answer booklets. The experiment was self-paced.

Two binary criterion measures were used, use of appropriate pattern (AP) and use of the rule (UR) taught. S was given credit for using the AP if he won the game and employed an appropriate game winning strategy. Credit was given for UR regardless of whether or not the strategy used was appropriate for winning the game in question (e.g., UR credit was given to the S group Ss for applying rule S on test games 2 and/or 3).

In experiment two, the materials were based on arithmetic (number) series. Analogous to experiment one, three types of arithmetic series were considered. These categories were ordered in the sense that all S series were also SG series and all SG series were also G series. Rules are available for finding the sums of S, SG, and G series.

All of the materials were combined into a single booklet. The first page consisted of a short pretest. Arithmetic series were defined and illustrated on page 2. Page 3 differed according to the treatment but always included one of the rule statements and sometimes the common series illustrating its use.

Twenty-nine sixth grade, 76 seventh grade, and 79 eighth grade pupils participated in this experiment. Two variables were independently manipulated, rule generality (S, SG, G) and example (given, not given). The Ss were assigned at random to the six treatment conditions so that each treatment was as nearly equally represented in each of the classes as possible. Since a large number of Ss did poorly on both the pretest and the three post-test series, data were presented only for those 114 Ss who got at least three of the four pre-test problems correct.

The experiment was conducted in classrooms during periods scheduled for mathematics instruction. The regular teachers administered the materials in their respective classes under the direct supervision of

Frank Lee. Each class took twenty minutes to complete the experiment. To insure uniformity, the teachers were given a page of explicit directions to follow and were closely supervised.

The dependent variables were correctness of sum and use of the rule taught.

RESULTS

The results of experiment one provided strong support for the original hypotheses: (1) performance on within-scope problems did not differ appreciably, even though the common example was more similar to problem one than the others, and successful problem solving was limited almost exclusively to within-scope problems, (2) rule S proved easier to apply than rules SG and G, and (3) the rules taught tended to be used consistently on all problems whether they were appropriate or not.

About the only major unanticipated result in experiment one was that Rule G proved as easy to interpret as Rule SG. Since these groups were not performing significantly better than control group E on problem one, we were tempted to attribute the lack of an anticipated SG-G difference to scale insensitivity and thereby not be forced to accept a "no difference" hypothesis--especially since the power of our tests were unknown.

Although the results of experiment two paralleled those of experiment one in most respects, there were several important differences. First, the presence of the example (problem one) along with Rule S resulted in significantly better performance on problem two than when Rule S was shown alone, the only case in either experiment where non-negligible success was noted on an extra scope problem. This effect may have been due to the form of the combining operation, "50 x 50," in the Rule S statement. "50 x 50" is clearly an instance of the more general SG combining rule, "$n \times n = n^2$." Presumably, the statement of Rule S, together with the common example, provided the successful S Ss with enough cues to generalize. An analysis of the test papers tended to substantiate this interpretation. Second, only three of the nineteen G-with-example Ss solved problem three whereas 18 solved problem one and 14 solved problem two. The reason for this difference was not immediately apparent especially since 15 of these Ss applied Rule G to the third problem. A more intensive post hoc analysis suggested that while it was no more difficult to apply the rule itself to the third problem series, it was more difficult to determine the appropriate value of one of the variables (i.e., n) entering into the combining operation (i.e., $[(a + 1)/2]n$). Third, although the results of experiment two were in the hypothesized direction, only the overall effect of scope on interpretability was significant. Fourth, only one of the group S *S*s applied rule S to problems two and three. This result can probably also be attributed to an interfering effect due to familiarity with addition problems. The *S*s may simply have mistrusted Rule S. How could a rule, like 50 x 50, having only one answer, be the sum of all three obviously different problem series?

CONCLUSIONS AND IMPLICATIONS

The results of these experiments demonstrate, in a rather conclusive fashion, the behavioral relevance of principle generality. For the most part, successful performance was noted only on tasks within the scope of verbally stated principles. When principles are presented in an expository fashion, it is normally too much to expect generalization to problems to which the principle does not immediately apply.

Of perhaps even greater practical significance were the lack (there was one exception) of performance differences on within scope problems and the consistency results. The former result demonstrates that, under certain specifiable conditions, any stimulus within the scope of a principle is equally as difficult to respond to correctly as any other. Furthermore, coupled with the consistency data cited in the introduction, the obtained consistency results suggest that only one (new) test stimulus is needed to determine whether, in fact, a given principle has been learned. No more information is gained by using additional test instances. These results could have far-reaching implications for the development of highly efficient measuring instruments.

In addition, the pronounced tendency of the Ss to attack all of the test problems in the same way, irrespective of whether the procedure used was appropriate, suggests that knowing how to solve problems and knowing when to use this knowledge are quite distinct. Testing for the latter ability necessarily must involve the presentation of extra-scope problems.

More important than the results of these exploratory experiments were the post hoc analyses they made both necessary and possible. In particular, the preceding discussion strongly suggests that the roles played by various aspects of a principle statement need to be more clearly specified. The form "If A, then B" does not detail all that appears relevant. For one thing, it was not possible in this study to distinguish between the stimulus variables entering into the rule and the combining operation of the rule by which the appropriate responses are determined. With the arithmetic series, for example, the variable referred to such things as the number of terms in the series while the G combining operation was of the form, $[(x + y)/2]z$. The variables relate to properties of arithmetic series stimuli, while the algebraic expression represents a ternary operation by which another property (e.g., sums) may be derived. The A, in "If A, then B," while it played no role in this study is, of course, also critical. It tells when a rule can and can not be applied. Thus, the rule, n^2, is appropriate whenever an arithmetic series consists of the odd integers beginning with 1 while $[(a + 1)/2]n$ works whenever there is a common difference between adajacent terms.

These observations suggest that a principle statement may be represented more appropriately by the form, "If I', then O' (D') = R'," where I' refers to the set of stimulus properties which indicate when the rule, denoted O' (D'), should be applied, D' refers to the set of those stimulus properties which determine the responses, and O', to the operation from which the responses, denoted by R', may be derived from the properties referred to by D'. Notice that that part of a principle statement represented by O' (D') corresponds to what is typically called a rule.

Although the actual symbols used in a statement may be an important factor, as suggested above, the hypothesis advanced in this study to the effect that rule generality and interpretability are inversely related finds a formal rationale in the nature of the characterizing elements. Making operational use, for example, of the arithmetic series property (i.e., dimension), "the difference between adjacent terms is some common value," necessarily presumes that, "the differences between adjacent terms is 2," "...3," "etc.," can all be correctly interpreted. The converse does not necessarily follow. A similar relationship exists with respect to the rules, 50 x 50 and n x n. To correctly apply the latter, more general, rule to any particular series requires the ability to determine any value of the dimension n, including 50. Being able to apply 50 x 50 does not.

It would appear that the more general the principle the more is expected of the learner. Whether such differences will be reflected in behavior, however, may depend on not only rule generality but the population involved and, particularly, on whether the Ss have the necessary requisite abilities.

In effect, differences in generality appear, on analysis, to be equivalent to differences in abstraction level. Thus, the number 2 is more abstract than the property two oranges because the former applies to a collection of sets only one of which has the latter property. For the same reason, the property represented by the placeholder X is more abstract than the number 2 since it refers to a still higher order collection. Future studies should be designed to provide definitive information on these points.

BIBLIOGRAPHY

There are 23 different references listed in the final report.

PUBLICATIONS

A draft of the final report has been accepted for publication in the American Educational Research Journal subject to certain editorial revisions. It should appear in that journal sometime during the year 1967.

Experiment #1

Common to All Subjects

NAME ______________________

GENERAL DIRECTIONS

This experiment is sponsored by the Mathematics Education Department of Florida State University. It is designed to determine how well you can relate and generalize certain mathematical patterns. You will be given some material to learn and then be tested on this material. The results of this test will be made known to your instructor and he may pass this information on to you.

Other people may be participating in this experiment at a later date, so please do not spoil the experiment by talking to anyone about it. Your cooperation is appreciated.

As a participant in this experiment, you are going to be asked to learn to play a game. The game is a number game played between two people. The game has many variations, but at present we will examine only one of the possibilities. In order to learn to play, you must read very carefully.

The game is initiated by one person making a selection of a number from the set $\{1, 2, 3, 4, 5, 6\}$. Participants then make alternating selections from this set and a running sum is kept. A number may be selected more than once. The object of the game is to make the selection which makes the sum exactly 31.

EXAMPLE 1

John is playing against Mary

	running sum	
John selects 4	4	
Mary selects 2	6	(from 4 + 2)
John selects 5	11	(from 6 + 5)
Mary selects 4	15	(from 4 + 11)
John selects 6	21	(from 6 + 15)
Mary selects 6	27	(from 6 + 21)
John selects 4	31	(from 4 + 27)

John wins since his last selection made the sum 31.

Note that John and Mary make alternating selections.

EXAMPLE 2

You fill in the blanks

	running sum
John selects 3	3
Mary selects 3	6
John selects 6	___
Mary selects 4	16
John selects 5	___
Mary selects 2	23
John selects 4	___
Mary selects 4	31

__________wins the game.

Turn to the next page to check your answers.

SOLUTION FOR EXAMPLE 2

	running sum	
John selects 3	3	
Mary selects 3	6	
John selects 6	12	(from 6 + 6)
Mary selects 4	16	
John selects 5	21	(from 5 + 16)
Mary selects 2	23	
John selects 4	27	(from 4 + 23)
Mary selects 4	31	

________ wins the game. (Because she made the selection which resulted in the sum of 31).

After you have completed checking your answers, go on to the next page.

EXAMPLE 3

Fill in the blanks

running sum

Mary selects 6 ___

John selects 6 ___

Mary selects 5 ___

John selects 3 ___

Mary selects 2 ___

John selects 4 ___

Mary selects 5 ___

__________ wins the game.

Turn to the next page to check your answers.

SOLUTION FOR EXAMPLE 3

	running sum
Mary selects 6	6
John selects 6	12
Mary selects 5	17
John selects 3	20
Mary selects 2	22
John selects 4	26
Mary selects 5	31

__Mary__ wins the game.

So far we have discussed only the $\{1,2,3,4,5,6\}$ and sum 31 game. As suggested on page 1, the game has man, iations. These variations come from varying the allowable selections and also the desired sum. Thus, if we allow selections from the set $\{1, 2, 3, 4, 5\}$ and allow the desired sum to be 28, we get a game similar to the one described.

LEVEL 1

One of the important mathematical operations is division. Let us restrict division here to division of whole numbers. What does division mean? In elementary school you were probably taught that division indicated that you were to find out how many of one number was in another number. For example, 12 ÷ 4 probably meant that you were to find how many 4's there were in 12. A more sophisticated approach can be developed, however.

How did you check long division? By multiplication? Probably! Then why not define division in terms of multiplication? That is exactly what mathematicians do. They say that:

(1) $12 \div 4 = 3$ because $4 \times 3 = 12$ or

(2) $36 \div 9 = 4$ because $9 \times 4 = 36$.

Thus, in order to divide effectively, you must be able to multiply.

Let us examine division involving 0. In order to do this, we must first examine multiplication involving 0. What is 9 x 0?, 12 x 0?, 4 x 0? How about 0 x 9?, 0 x 12?, 0 x 4? The answer to all these questions is 0. In general then, if you are multiplying two numbers and one of the numbers is 0, the product will be 0. What about division involving 0? Remember what we mean by division. Look back to (1) and (2). Now consider:

$0 \div 4$

Is $0 \div 4$ equal to 1? No, because $4 \times 1 = 4$, not 0. Well, what will work?

$0 \div 4 = 0$ because $4 \times 0 = 0$.

Level 1

That wasn't bad. Now, try this one:

$4 \div 0$

Is $4 \div 0$ equal to 0? No, because $0 \times 0 = 0$. Well, how about 4? Is $4 \div 0$ equal to 4? No, because $0 \times 4 = 0$. What will work? There isn't a number which will work. Hence $4 \div 0$ is not defined.

Now try: $0 \div 0$

Is $0 \div 0$ equal to 0? Well, certainly $0 \times 0 = 0$. But what about 1? Is $0 \div 0$ equal to 1? Well, $0 \times 1 = 0$. How about 2?, 3?, 4? . . . . Yes, each number will work. Hence mathematicians consider $0 \div 0$ to be undefined.

In general then, whenever the divisor is 0, mathematicians say the division is not defined.

You should now be ready for the test. Close your booklet and raise your hand. A proctor will bring your test to you. You must turn in this booklet when you receive your test.

EXAMPLE 4

	Sum
You select 3	3
Your opponent selects something, say 2	5
You select something to make the sum 10, in this case, 5	10
Your opponent selects something, say 3	13
You select something to make the sum 17, in this case, 4	17
Your opponent selects something, say 6	23
You select something to make the sum 24, in this case, 1	24
Your opponent selects something, say 1	25
You select something to make the sum 31, in this case, 6	31

You win!

When you think you understand how to win the game, close the booklet and raise your hand, and you will be given the test. You must turn in this booklet when you get your test.

LEVEL 2

There is a pattern to the game which will enable you to win whenever you are allowed to make the first selection. You must, however, make an appropriate first selection and then proceed in a precise manner. In order to win the game (see example 4 - next page), you should make 3 your first selection. Then you should make selections so that the sums corresponding to your selections differ by 7.

EXAMPLE 4

	Sum
You select 3	3
Your opponent selects something, say 2	5
You select something to make the sum 10, in this case, 5	10
Your opponent selects something, say 3	13
You select something to make the sum 17, in this case, 4	17
Your opponent selects something, say 6	23
You select something to make the sum 24, in this case, 1	24
Your opponent selects something, say 1	25
You select something to make the sum 31, in this case, 6	31

You win!

When you think you understand how to win the game, close the booklet and raise your hand, and you will be given the test. You must turn in this booklet when you get your test.

LEVEL 3

There is a pattern to the game which will enable you to win whenever you are allowed to make the first selection. You must, however, make an appropriate first selection and then proceed in a precise manner. In order to win the game (see example 4 - next page), the appropriate first selection is determined by dividing the desired sum by 7. The remainder of this division is precisely the selection which should be made first. Then you should make selections so that the sums corresponding to your selections differ by 7.

EXAMPLE 4

	Sum
You select 3	3
Your opponent selects something, say 2	5
You select something to make the sum 10, in this case, 5	10
Your opponent selects something, say 3	13
You select something to make the sum 17, in this case, 4	17
Your opponent selects something, say 6	23
You select something to make the sum 24, in this case, 1	24
Your opponent selects something, say 1	25
You select something to make the sum 31, in this case, 6	31

You win!

When you think you understand how to win the game, close the booklet and raise your hand, and you will be given the test. You must turn in this booklet when you get your test.

LEVEL 4

There is a pattern to the game which will enable you to win whenever you are allowed to make the first selection. You must, however, make an appropriate first selection and then proceed in a precise manner. In order to win the game (see example 4 - next page), the appropriate first selection is determined by adding one to the largest number in the set from which the selections must come, and dividing the desired sum by this result. The remainder of this division is precisely the selection that should be made first. Then you should make selections so that the sums corresponding to your selections differ by one greater than the largest number in the set from which the selections must come.

EXAMPLE 4

	Sum
You select 3	3
Your opponent selects something, say 2	5
You select something to make the sum 10, in this case, 5	10
Your opponent selects something, say 3	13
You select something to make the sum 17, in this case, 4	17
Your opponent selects something, say 6	23
You select something to make the sum 24, in this case, 1	24
Your opponent selects something, say 1	25
You select something to make the sum 31, in this case, 6	31

You win!

When you think you understand how to win the game, close the booklet and raise your hand, and you will be given the test. You must turn in this booklet when you get your test.

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TEST DIRECTIONS

You are now going to be tested on your ability to play and win the type of game previously described. You will be playing against a person merely described as "your opponent." You will be allowed to make the first selection in each game, and you will be given instructions on how to determine your opponent's selection. Look at the accompanying Answer Sheet, Page 1. Do not write anything on that sheet until you have read and understand the directions. Blanks are left to indicate your selections. Also, blanks are left to indicate the sum. After your selection has been made and the sum computed and entered, you are told to turn to a certain page to determine your opponent's selection. That page is part of this Test Directions booklet. His selection should be indicated by you in the prescribed blank and the sum should be calculated and entered. The process is then continued until you win the game or your opponent wins or makes a selection which will make the sum larger than 31. You must fill in all the blanks until the game is completed, but there will probably be some extra blanks at the bottom of the page. Play the game in order, and do not determine your opponent's selection until you have made your selection which proceeds his. If you don't understand, raise your hand and a proctor will help you.

No erasures or mark outs are allowed, so be sure of your entry before you mark it down. Now direct your attention to the answer sheet.

Print your name in the blank provided for it and begin the test.

Your opponent selects 1.

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Your opponent selects 2.

Your opponent selects 3.

Your opponent selects 4.

Your opponent selects 5.

Your opponent selects 6.

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NAME ______________________________

ANSWER SHEET

Read the Test Direction Sheet before you begin. No erasures or mark outs are allowed.

TEST 1

The game is the one which allows selections from the set {1, 2, 3, 4, 5, 6}

and where the desired sum is 31.

I select ____. The sum is then ____.

Turn to page 4 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 6 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 5 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 2 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 3 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 2 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 3 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Go to the next page.

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TEST 2

The game is the one which allows selections from the set {1, 2, 3, 4, 5, 6} and where the desired sum is 25. Remember no erasures or mark outs are allowed.

I select ____. The sum is then ____.

Turn to page 6 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 5 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 7 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 2 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 3 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 2 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 4 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Go to the next page.

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TEST 3

The game is the one which allows selections from the set {1, 2, 3, 4} and where the desired sum is 22. Remember no erasures or mark outs are allowed.

I select ____. The sum is then ____.

Turn to page 5 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 3 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 7 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 3 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 2 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 2 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

Turn to page 7 for your opponent's selection.

His selection is ____. The sum is then ____.

I select ____. The sum is then ____.

When you have completed this page, hand in your test booklet and you will be excused.

Experiment #2 **Experimenter only**

INSTRUCTIONS

(Please study before handing out the experimental materials.)

Your assistance in conducting this experiment is deeply appreciated.

From the experimental point of view, it is extremely important that the students work solely with the materials and instructions given them in the test booklet. DO NOT ANSWER ANY QUESTIONS concerning these materials, as this might seriously alter the effects of the experiment. After the experiment has been concluded, we will inform you of the purpose and results.

First, please read the attached general instruction sheet to the students. If there are questions, you may repeat, but not add to, any part of the attached sheet.

Second, distribute the booklets face down. Tell the students not to turn the booklets over until told to do so.

When all have received booklets, tell them to turn over their booklets and start work.

Before directing them to turn to the next page, allow them exactly:

5 minutes on page 1,

5 minutes on page 2,

2 minutes on page 3,

3 minutes on page 4,

3 minutes on page 5,

3 minutes on page 6--then direct them to stop work and close their booklets.

Collect all the booklets and place them in the envelope in which they were received.

Thank you.

NAME________________________

GENERAL DIRECTIONS

This experiment is sponsored by the Mathematics Education Department of Florida State University. It is designed to determine how well you can relate and generalize certain mathematical patterns. You will be given some material to learn and then be tested on this material. The results of this test will be made known to your instructor and he may pass this information on to you.

Other people may be participating in this experiment at a later date, so please do not spoil the experiment by talking to anyone about it. Your cooperation is appreciated.

This page common to all Subjects

NAME ____________________

Print your name in the space provided at the upper right corner of this page.

In this booklet you will be given some problems to solve. You may have to add, multiply, or divide numbers to get the solutions. The signs for adding, multiplying, and dividing are shown in these examples:

$1 + 3$ means to add 1 and 3. $2 \cdot 4$ means multiply 2 by 4.

$\frac{6}{3}$ means divide 6 by 3.

Would you fill in the blanks with answers for the following problems: (Use this page for any written work you have to do to obtain the answers.)

a) $89 \cdot 74 =$ __________(Answer).

b) $\frac{(57 + 95)}{4} \cdot 27 =$ __________(Answer).

c) $\frac{(X + Y)}{T} \cdot W =$ __________(Answer), if $X = 3$, $Y = 5$, $T = 4$, and $W = 6$.

You will be asked to give the sum of number series in which each number is larger than the number it follows by the same amount. The series $3 + 5 + 7 + 9 =$ _____, and $4 + 8 + 12 \ldots + 40 + 44 =$ _____, are examples of such series. Notice that when there are many terms (that is, numbers) in the series, we use dots to save us from writing all the terms. We can do this because the first three terms show the amount by which each term is larger than the term it follows. In the example given above ($4 + 8 + 12 + \ldots + 40 + 44$) we use the dots to save us from writing $16 + 20 + 24$ etc. since we are somewhat, but not too, lazy. These number series are called arithmetic series. Before we go on, would you fill in the blanks to complete the following arithmetic series:

$7 + 10 +$ ____ $+ 16 +$ ____ $+$ ____ $+ 25$.

DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

Problem 1

Write the sum of the following arithmetic series in the space provided for it. Use this page for any written work you have to do to obtain this sum.

$$1 + 3 + 5 + 7 + \dots + 97 + 99 = \_\_\_\_\_\_\_\_\_\_ .$$

DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

Problem 2

Write the sum of the following arithmetic series in the space provided for it. Use this page for any written work you have to do to obtain this sum.

$$1 + 3 + 5 + \ldots + 77 + 79 = \underline{\hspace{3cm}}.$$

DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

Problem 3

Write the sum of the following arithmetic series in the space provided for it. Use this page for any written work you have to do to obtain this sum.

$$2 + 4 + 6 + \ldots + 46 + 48 = \underline{\qquad\qquad}.$$

WHEN YOU FINISH, CLOSE THIS BOOKLET AND WAIT UNTIL TOLD TO TURN IT IN. DO NOT LOOK BACK AT OTHER PAGES.

Rule 8

The sum of some arithmetic series may be obtained by multiplying 50 by 50.

DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

Rule S and Example

The sum of some arithmetic series may be obtained by multiplying 50 by 50. One such arithmetic series is:

$$1 + 3 + 5 + 7 + \ldots + 97 + 99 = \underline{50 \cdot 50 = 2500}$$

DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

Rule 56

The sum of some arithmetic series may be obtained by multiplying the number of terms in the series by itself.

DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

Rule SG and Example

The sum of some arithmetic series may be obtained by multiplying the number of terms in the series by itself.

One such arithmetic series is:

$$1 + 3 + 5 + 7 + \ldots + 97 + 99 = \underline{50 \cdot 50 = 2500}$$

DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

Rule G

The sum of some arithmetic series may be obtained by adding the first number in the series to the last number in the series, dividing the resulting sum by 2, and then multiplying the number you get by the number of terms in the series.

DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

Rule [illegible] and Example

The sum of some arithmetic series may be obtained by adding the first number in the series to the last number in the series, dividing the resulting sum by 2, and then multiplying the number you get by the number of terms in the series.

One such arithmetic series is:

$$1 + 3 + 5 + 7 + \ldots + 97 + 99 = \frac{(1 + 99)}{2} \cdot 50 = 2500$$

DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.