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SECONDARY SCHOOL PUPILS' PERCEPTION OF THE PLANE SECTIONS OF SELECTED SOLID FIGURES.

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TWO METHODS OF RESPONSE DESCRIBED BY PIAGET AND INHELDER FOR DRAWING APPROPRIATE BOUNDARIES AND SELECTING APPROPRIATE PREDRAWN BOUNDARIES FOR HYPOTHETICALLY CUT, SOLID GEOMETRIC FIGURES WERE INVESTIGATED TO SEE WHETHER SECONDARY SCHOOL STUDENTS COULD RESPOND IN APPROPRIATELY TO THE MATHEMATICAL TASKS. THE RESPONSES OF 8TH-, 10TH-, AND 12TH-GRADE STUDENTS TO 16 SECTIONING TASKS WERE IDENTIFIED AND EVALUATED. NONE OF THE 72 SUBJECTS DEMONSTRATED THE ABILITY TO DRAW AND IDENTIFY GEOMETRIC SECTIONS WITH CONSISTENT ACCURACY. PREDICTED APPROPRIATE RESPONSES DESCRIBED BY PIAGET AND INHELDER WERE NOT VALIDATED. (GD)

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SECONDARY SCHOOL PUPILS' PERCEPTION OF THE PLANE SECTIONS OF SELECTED SOLID FIGURES

UNIVERSITATIS WISCONSINENSIS SIGILLUM

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Technical Report No. 13

SECONDARY SCHOOL PUPILS' PERCEPTION OF THE PLANE SECTIONS OF SELECTED SOLID FIGURES

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Based on a doctoral dissertation under the supervision of
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Research and Development Center
for Learning and Re-Education
The University of Wisconsin
Madison, Wisconsin

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PREFACE

This technical report is based on the doctoral dissertation of Barbara L. Boe. Members of the examining committee were Henry Van Engen, Chairman; Milton Beckman; Chester W. Harris; Herbert J. Klausmeier; and Milton O. Pella.

The R & D Center for Learning and Re-education has as its primary goal the improvement of cognitive learning in children and adults, commensurate with good personality development. Through synthesizing present knowledge and conducting research to generate new knowledge, we are extending the understanding of human learning and the variables associated with efficiency of school learning. Knowledge is being focused upon the three main problem areas of the Center: developing exemplary instructional systems, refining the science of human behavior and learning as well as the technology of instruction, and inventing new models for school experimentation, development activities, and so on.

In the development of the instructional system in mathematics at the R & D Center, research on mathematics learning is conducted and its findings incorporated into instructional procedures and materials. Mrs. Boe reports a study that was devised to ascertain more clearly students' ability to section solids. None of the students participating in the study demonstrated the ability to draw and identify with consistent accuracy sections hypothetically cut into solid figures. Differences between students in Grades 8, 10, and 12 were insignificant and, therefore, assumed to be chance occurrences. Relationships to the work of Piaget and Inhelder on which this study is based are discussed by Mrs. Boe. She also draws implications for curricular content in the area of industrial arts as well as in her field of mathematics.

Herbert J. Klausmeier
Co-Director for Research

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ABSTRACT

The seventy-two subjects in this study were randomly selected from five secondary schools in a large Midwest city and assigned to one of eighteen groups. The random selection was based upon the total population of each of Grades 8, 10, and 12 within the designated secondary schools. Two tests, differing only in their method of response, were designed for this study. Each test consisted of sixteen geometric sections that resulted from a hypothetical cut performed upon a solid figure. Test I, which was always given first, required the subject to draw his representation of the shape of the section, while Test II required the subject to select a pre-drawn representation of the boundary of the section. The choices for the response items for Test II were selected from the drawing responses of children in two previous pilot studies and from the comments recorded in Chapter 9 of The Child's Conception of Space (Piaget & Inhelder, 1963). A sixth choice was allowed, "None of these."

The subjects were tested individually. The order in which the problems appeared in each subject's tests was determined by a table of random numbers. No two subjects were presented with the same sequence of tasks, nor was any subject presented with the same sequence of tasks for both tests. The data recorded for each subject was his response to each section. Prior to the administration of the tests his response sheets were coded for sex, grade, ability level, and sequence of tasks.

With respect to the subjects a 2×3^2 factorial design constituting eighteen groups was used. Sex, three grade levels, and three ability levels made up the dimension for this design. The sixteen repeated measures constitute a 4×4 design given by four types of cuts and four solid figures.

The analysis of variance on the correct multiple-choice response scores, Test II, indicated five sources of variation significant at or beyond the .05 level. Three of these were main effects: (a) ability level, (b) solid figures, and (c) type of cut. The mean for the above average ability level was higher than the mean for the below average ability level. The mean for the sections performed on one nappe of the right circular cone was the lowest, with the rectangular prism next in difficulty followed by the right circular cylinder and cube. The oblique cut was the most difficult; the transverse cut was next in difficulty; the parallel and the longitudinal cuts were of equal difficulty. The other sources of variation were the interactions: (a) solid figure and type of cut, (b) ability level and solid figure. The analysis of variance on the appropriately drawn response scores for Test I showed the same five sources of variation significant at or beyond the .05 level. The order of difficulty of the solids was, from most difficult to easiest: one nappe of a right circular cone, right circular cylinder, rectangular prism, and cube. One significant main effect in addition to those mentioned was sex. The mean for the boys was higher than the mean for the girls. A Pearson product-moment correlation coefficient appeared to indicate that the two methods of response measure attributes other than just the subject's ability to appropriately respond to the sectioning tasks.

I

BACKGROUND TO THE PROBLEM

INTRODUCTION

Piaget and Inhelder (1963) contend that children do not have the same concept of the spatial world about them as do adults. Adults are more cognizant of the euclidean properties of size and distance while the child is more conscious of the continuity or separateness of an object. Until a child reaches a certain stage in his development, he fails to recognize that everyone does not see a physical situation as he does; he is not aware that other points of view exist. He is unaware of the projective relationships of space. Prior to the attainment of the euclidean concepts of Piaget and Inhelder's representational space, the child becomes aware of movement. Movement is lacking in a topologically oriented representational space. Measurement evolves from the idea of movement between objects and movement about an object. An awareness of movement is the precursor of euclidean concepts. The euclidean or geometric sections are a manifestation of this attainment level in the child's development of Piaget and Inhelder's (1963) representational space. Veblen and Young (1910), Coxeter (1964), and Meserve (1955) support Piaget and Inhelder's hypothesis that euclidean concepts emerge from the projective relationships with the attainment of the tasks of the euclidean sections.

Projective relationships do not consider parallel lines, angles, or distance. The connecting links between conservation of a set of neighboring elements to the conservation of distance mark the transition from topological and projective representational spaces to the euclidean representational space. In the topological space, because of the lack of configuration for a frame of reference, conservation of size and distance are inappropriate. In perspective space the different viewpoints give rise to variations in the size of objects. After the child becomes aware of the different viewpoints, the development of surfaces and the final passage from projective to euclidean representational space occurs as a "series of transition stages. . . consisting of affinities and similarities [Piaget & Inhelder, 1963, p. 301]."

Hence, according to Piaget and Inhelder, the child progresses in his understanding of representational space in the following order: topologic, projective, and euclidean. There is an intermingling of the projective and euclidean concepts as they evolve from the topologic, but the projective relationships appear to precede the euclidean concepts on actual attainment. The euclidean sections are important in the child's development of Piaget and Inhelder's representational space. It is within these tasks that the child encounters the problem of imagining a euclidean operation of sectioning a solid object and simultaneously imagining a projection of the solid object in order to mentally observe the section. The euclidean section then is of twofold importance. First, the child's achievement in sectioning solid figures including the conic sections indicates the emergence of euclidean concepts, and, secondly, the achievement of sectioning the cone is a link in the establishment of a correspondence between the euclidean and projective operations—Piaget's geometry of objects and geometry of viewpoints, respectively. The euclidean sections are therefore an "abstraction... based wholly upon the actions of the subject [Piaget & Inhelder, 1963, p. 270]." The child visually perceives a solid object, hears a verbal description concerning the intersecting plane, and then imagines this operation, mentally manipulating the solid, transforming it, and finally deciding the shape of the plane section. If a child can draw an appropriate boundary and select the appropriate pre-drawn boundary for the euclidean sections including those of the right circular cone, according to Piaget's hypothesis, the child has employed an intellectual operation upon his image of a solid figure (Piaget, 1966). With respect to the problem being investigated, successful sectioning implies that a subject will receive a perfect score. From Piaget's hypothesis such a perfect response score would imply that the child has begun his transition from a projective representational space to a euclidean representational space.

Piaget's work has been criticized on several points, notably his failure to indicate the number of subjects, the characteristics of the subjects beyond their age, lack of any statistical measurement on the responses, as well as a lack of information concerning the exact nature of the test instrument. Lovell (1959) questions Piaget's mathematical terminology. The results of Piaget's work as well as his techniques of research have stimulated both research efforts and methodology and content in the planning of elementary school mathematics. He has attempted to determine the natural evolution of concepts in children by analyzing responses to specific questions. In spite of the criticism leveled against his work, a considerable number of studies replicating part of his work have resulted in findings which confirm his hypothesis (Lunzer, 1960; Wallach, 1963).

The present problem will neither confirm nor deny Piaget's hypothesis of the development of the child's representational space. It will not confirm, nor will it deny, that the ability to appropriately respond to the tasks of sectioning solid figures through the right circular cone implies the emergence of the euclidean representational space. This study resulted from the unconfirmed, but often heard remarks of secondary school mathematics teachers that children have difficulty in sectioning solids; e.g., sectioning as applied to Cavalieri's principle and associating the curves of the quadratic equation with the boundaries of the conic sections. Whether these deficiencies result from teaching failure or from incomplete conceptual development can only be conjectured. The study will only show that children can or can not respond appropriately to the tasks using the two methods of response described by Piaget and Inhelder—drawing appropriate boundaries and selecting appropriate pre-drawn boundaries.

A basic assumption of this study is that secondary school mathematics teachers and curriculum planners are concerned about the responses of children to certain geometric situations. Current textbooks[1] and curriculum proposals[2] are increasing the geometry content throughout the entire school mathematics program. The study of volume—its cognitive mean-

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[1] These textbooks are merely a survey of the vast potential. It is beyond the scope of this study to delve into the problems related to textbook presentation with respect to the cognitive implications associated with geometric sections:

J. Houston Banks, Max A. Sobel, & William Walsh. Algebra: Its Elements and Structure: Book I. St. Louis: Webster Division/McGraw-Hill, 1965.

J. Houston Banks, Max A. Sobel, & William Walsh. Algebra: Its Elements and Structure: Book II. St. Louis: Webster Division/McGraw-Hill, 1965.

Ray C. Jurgensen, Alfred J. Donnelly, & Mary P. Dolciani. Modern Geometry-Structure and Method. Boston: Houghton Mifflin, 1963.

School Mathematics Study Group. Mathematics for High School, Intermediate Mathematics Part I. New Haven: Yale University Press, 1961.

Eugene D. Nichols & Wagner G. Collins. Modern Elementary Algebra. New York: Holt, Rinehart and Winston, 1961.

Arthur W. Weeks & Jackson B. Adkins. A Course in Geometry, Plane and Solid. Boston: Ginn, 1961.

[2] The committee reports listed below are presented in a chronological order. Their effects on mathematics content and teaching methodology are frequently controversial. Their recommendations are important to mathematics-education but the scope of the present study will not, at this time, permit a scrutiny of these various recommendations:

Provisional Report of the National Committee of Fifteen on Geometry Syllabus. School Science and Mathematics, 1911, 11, 329-355, 518-531.

Charles H. Butler. The Reorganization Report of 1923. The Mathematics Teacher, 1951, 44, 90-92.

Gordon D. Moch. The Perry Movement. The Mathematics Teacher, 1964, 56, 130-133.

Pre-Induction Course in Mathematics. The Mathematics Teacher, 1943, 36, 114-124.

The First Report of the Commission on Post-War Plans. The Mathematics Teacher, 1944, 37, 226-232.

The Second Report of the Commission on Post-War Plans. The Mathematics Teacher, 1945, 38, 195-221.

Program for College Preparatory Mathematics. Report of the Commission on Mathematics. New York: College Entrance Examination Board, 1959.

New Thinking in School Mathematics. O.E.E.C. Report. Paris: Office of Scientific and Technical Personnel, 1961.

ing and the derivation of formulas and their applications—makes use of geometric sectioning. The study of the conic sections, relating the euclidean sections to the algebraic equations, is found in some ninth grade algebra texts. Thus, the investigation of the responses of secondary-school age children to the sectioning of solid figures is a real situation stemming from the classroom. The implications which may evolve depend upon the children's spontaneous responses to the question, "What is the shape of the flat surface formed by cutting this figure?"

RELATED RESEARCH

There have been some replications of Piaget and Inhelder's space experiments. Beilin and his associates (Beilin, Kagan, & Rabinowitz, in press) have considered Piaget and Inhelder's water level tasks. Page (1959) has replicated the haptic perception experiments. Lovell (1959), Rivoire (1961), and Dodwell (1963) have done replications and modifications of several of Piaget and Inhelder's space experiments. Some of these researchers have investigated problems pertinent to the problem being considered. Their studies will be reviewed in some detail.

Lovell

Lovell (1959) replicated six experiments from The Child's Conception of Space. These six experiments were distributed among the first four chapters on topological space and the first of the five chapters on projective space: Haptic perception, elementary spatial relations in drawing, linear and circular order, knots, and projective straight line. Lovell attempted a controlled replication of Piaget and Inhelder's experiments. His evidence sometimes agreed with theirs. He did find a greater variability in performance within an age group than did Piaget and Inhelder. With respect to the projective straight line experiment, Lovell agreed that children, prior to six years of age, do not generally take aim, that is, sight along a line, but he disagreed with Piaget and Inhelder concerning children's ability to make straigh lines. Lovell's samples were able to make straight lines with slight irregularities.

Rivoire

Rivoire (1961) conducted "an investigation of the sequential development of representational space in children from 4 years 0 months to 14 years 11 months [p. 47]." Her test instrument consisted of twenty-eight items, seven per space; i.e., seven items in each of the four spaces she considered: topological, projective, affine, and euclidean. The spatial categorization of the items was conducted under the guidance of mathematicians. Because of the global nature of her study, the results of Revoire's investigation will be presented in more detail than those of Lovell's research.

> For projective space, the total number of items passed shows an increase through the eight year level, a plateau, and additional development between the ages of twelve and fourteen [p. 64].
>
> ...Euclidean space totals are low through the eight year olds, increasing for the ten and twelve year level, and increase noticeably at the fourteen year level.
>
> Projective items are hypothesized as beginning to form at approximately six years of age and as being almost complete by the age of eleven or twelve. The present data records the beginning of concepts of this space type by the age of four years and completion at fourteen years or later [p. 67].
>
> The Euclidean concepts, according to Piaget, should start to develop at six and be completed by the age of fourteen. [According to this study, euclidean concepts appear] above chance level for the first time at the ten year level. It would appear that these concepts were not completed by the end of the fourteenth year [p. 68].
>
> It is interesting to note that the concepts of sections of geometric solids, ...are not fully developed in all fourteen year old children in this study. This result does not correspond to the Piaget finding that by the age of thirteen or fourteen years these concepts are fully developed [p. 87].

Dodwell

Dodwell (1963) performed a series of replicative experiments from The Child's Conception

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History and Philosophy of S. M. S. G. Writing Teams: 1958-1960. The Board of Trustees of the Leland Stanford Junior University, 1962.

Goals for School Mathematics. The Report of the Cambridge Conference on School Mathematics. Boston: Houghton Mifflin, 1963. The Cambridge Report.

of Space. The fifth subgroup or experiment of this series was on the geometric sections and as such is of importance to this paper. His procedural questions were equivalent to the questions asked in the problem presently being studied. One difference in the procedure exists; Dodwell, like Piaget and Inhelder, actually performed the cut after the subject responded. After Dodwell cut the solid he asked, "Did you think it would look like that ?" Within the test instrument of the present study no cuts were performed; during the instructional period, however, a styrofoam ball was cut. Children 5 years 11 months to 11 years 3 months, were used in Dodwell's sample. This age range encompasses the period when drawing ability is developing. Drawing ability affects the subject's responses to representational space and drawing is affected by the subject's development of his representational space. In general, Dodwell does concur with Piaget and Inhelder's hypothesis on the development of representational space. There were 95 children in some of the three well-defined stages of development and 99 that Dodwell classified as mixed. He concluded "that the 'mental construction' of a section is not an 'all-or-nothing' affair, that is, constancy from one situation or object to another is comparatively exceptional [p. 152]." This conclusion is in agreement with Beilin and Franklin (1962). For the age range used in his study, Dodwell found a high correlation between achievement in the geometrical tasks and age "and an even more marked correlation with mental age [p. 55]."

In reading the replication studies it is obvious that further investigation into the responses of children to questions concerning representational space is needed. There are many facets of the individual which may affect his cognitive development and hence his responses. Piaget is frequently criticized for failing to elaborate upon his subjects' characteristics. The studies just mentioned attempt to rectify this deficiency. The present investigation will consider three variables for describing subject characteristics: sex, general intelligence, and age as measured by grade level in school.

In selecting the three stratifying variables used in the present study two points were considered, the criticism that Piaget does not define his subjects' ability levels and Piaget's reliance upon chronological age as his sole stratifying criterion. The inclusion of sex as the third variable was deemed necessary because research has shown that sex affects achievement level.

II
THE PROBLEM

The problem under investigation is a study of the responses of secondary school children to some of Piaget and Inhelder's geometric sections. An appropriate response to the task of sectioning requires a certain level of development in the child's representation of space. Representational space evolves from the child's imagery. The imagery the child uses is not a copy of reality but a product of "actual or potential action through signs and symbols [Piaget & Inhelder, 1963, p. 452]." Representational space and imagery may involve psychological premises that extend beyond the confines of the present study; they will be discussed within the context of Piaget's hypothesis only.

DEFINITIONS

Piaget's Representation Theory

In order to appreciate the significance of Piaget's representational space it is necessary to consider some of the characteristics of his psychological theory pertinent to representational space. Piaget approaches his theory of cognitive structure through the interaction of imagery and representation. For the purpose of the present investigation, Piaget's rebuttal to a critic will provide an adequate summation of this theory of representation.

> My whole conception of intellectual operations is based on the premise that to know or to understand is to transform reality and to assimilate it to schemes of transformations. . . . in my view, all concepts are derived first from the action and then from the operation, . . . One can, therefore, distinguish two components of cognitive functions:
>
> There is first of all a figural component that does not itself constitute a copy but rather a more or less approximate description of reality states and their configurations. This figurative component is derived from perception, imitation, and imagery (graphic or mental) or from interiorized imitation.
>
> There is secondly a cognitive component which takes account of transformations and which builds upon sensori-motor actions, interiorized actions, and finally thought operations which are derived from actions and not at all from imitation.
>
> In other words, when studying perception, mental imagery, etc. . . . I have tried to show that the figural aspects of cognitive functions are never sufficient to explain representative or conceptual knowledge.
>
> Imitation only plays the role of a symbolic instrument from the moment that sensorimotor play becomes symbolic.
>
> . . . symbolic signifiers are derived from imitation which, before becoming interiorized, is already a kind of symbolization in action. This act, however, in no way implies that symbolic instruments should be confused with figural aspects of thought. Perception is figurative, but not symbolic, while language is symbolic (in the broad sense), but not figurative. Interiorized imitations and images are, on the contrary, at the same time figurative and symbolic. Mental imagery in particular is the product of interiorized imitation and not the simple residue of perception. . . .
>
> If symbolic play uses imitation, it is exclusively as a symbolic instrument. This follows because there are only two ways that an absent situation can be represented; it can either be described by language or evoked by imitative gestures or images. This in no way means, however, that symbolic play can be reduced to imitation since play is exclusively an assimilation of reality to the self. Nonetheless, since it is symbolic it needs signifiers, and it borrows them either from language or from the only other source of symbols, . . . gestural or in-

teriorized imitation [Piaget, 1966, pp. 111-112].[3]

Representational Space

Piaget and Inhelder elaborate upon this theory in their description of representational space. They distinguish between that which is being signified and that which signifies it. Representational signifiers are the "signs (ordinary or mathematical language) and symbols (images, imitative gestures, sketches)" while the things they signify in the case of spatial representations, are the spatial transformations, spatial states. "The transition from perception to representation is a twofold problem, embracing...both image and thought [Piaget & Inhelder, 1963, p. 17]."

> Images, although seemingly similar to precepts in some ways are in reality very different, . . . [Lovell, 1958, p. 90].

> . . .our interpretation of the environment or perception is much more than the sensations received; it is sensation reinforced by ideas, images, and past experiences [Lovell, 1958, p. 90].

> The intellectual relationships which constitute the beginnings of representational space are at first linked to the image as a means of support . . . . as they attain to spatial transformations, as opposed to static forms, these relationships separate the figural from the motor elements of the image, and at the same time free themselves from the figural elements. . . .

> As a result of internalized imitation, the mental image benefits from the attainments of perceptual construction and is sooner or later...able to avail itself of ready-made forms, . . . Not that this stamps representational space as being euclidean or projective right from the start, for it is...more directly linked with symbolic imagery than with genuinely conceptual relationships. In the case of the latter, as opposed to the imitative images on which they depend, it may well be that spatial representation has to begin by once more establishing the topological relationships of which perception itself was part [Piaget & Inhelder, 1963, pp. 17-18].[4]

As in so much of Piaget's writing, the ideas may temporarily be lost in the words. Perhaps a non-mathematical example, a familiar one of Lovell, will simplify Piaget's specialized terminology. A young child learns that Spot is a dog. He associates "dog" with certain four legged furry animals that he sees (or hears). The child has achieved a representation of "dog" when he playfully pulls along behind him a piece of cloth saying, "Bow-wow!" "Dog!" or "Spot!" He does not need the actual presence of the animal to imagine "dog." He can take any object and assign to it the verbal equivalent of his imagined dog, his representation. He has a signifier for that which is signified, distinguishing between the animal and his symbol for it, the cloth and the word.

In a more elaborate, but analogous, form the child develops his representation of space. Since space is basically the result of abstracting from perceptual activities, the representation of space is based upon imagery (Gibson, 1963; Lovell, 1958; Piaget & Inhelder, 1963).

A child sees a railroad track, a fence, a ball, a triangular shaped pattern; he may recognize each when he encounters it in his environment. He may not, however, be able to construct any in the absence of the concrete stimulus or when seen in a more abstract form. Piaget's experiments reveal that a child draws parallel lines for the rails of the track, fence posts of equal height. The child has, however, seen the rails meet at a point on the horizon and the fence post diminish in size (Piaget & Inhelder, 1963). The child draws what he knows, not what he sees (Goodenough, 1926; Hachberg, 1962; Köhler, 1947; Lewis, 1962; Ronchi, 1957).

> The transition from the first stage to the second one is a gradual and a continuous process. [Children's early drawings are] graphic enumerations of items. Ideas...of the relative proportions of parts and spatial relationships are much later [Goodenough, 1926, p. 12].

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[3] Play may be considered as any actions of the child upon objects. The non-mathematical example of representation credited to Lovell is an example of Piaget's "play."

[4] The argumentative nature of the last sentence will not be pursued as the validity of the remark is inconsequential to the present investigation. It is included only to lay a foundation for the description of the subject's task and to show how the task belongs to Piaget's hypothesis of representational space.

The observations of Wertheimer (1959) concerning children's understanding the procedure for finding the area of a parallelogram in "standard position" and after a ninety degree rotation appears to be connected to the problem of picture inversion studied by Hunton (1955). Such observations raise questions pertaining to the role of representational space to the child's studies in school. The child's representation may not be fully developed although his perception may be adequate (Piaget & Inhelder, 1963). He sees concrete objects; through play he observes the results of his transformations upon these objects. His image, at this point, is a copy of reality. As he progresses in his development, the child can imagine objects in their absence, eventually performing mental transformations on his image.

Piaget and Inhelder (1963) summarize representational space when they write:

> . . . there appears a . . . type of image, one capable of anticipating the results of actions before they are carried out. This image is dynamic and mobile in character, . . . and entirely concerned with transformations of the objects. It is this plastic type of imagery which geometricians term the "concept of space" when it has become purely intellectual and transcends the bounds of sense perception, as opposed to elementary spatial notions . . . termed "pre-operational" [pp. 130, 138].[5]
>
> . . . this type of imagery only appears closely associated with fully developed operational systems.
>
> . . . the image is now no more than a symbol of an operation, an imitative symbol like its precursors, but one which is constantly out-paced by the dynamics of the transformations. Its sole function is not to express certain momentary states occurring in the course of such transformations by way of references or symbolic allusions [p. 296].

Thus, representational space is the child's response to his image of space. It is an outgrowth of his active manipulation of the spatial environment rather than a direct result of his visual perception. Spatial perception does precede the development of representational space. Representational space, however, is not based solely upon visual perception (Lovell, 1958, p. 85). The spatial studies of Drever (1955) and of Garry and Ascarelli (1960), employing blind subjects, support this.

Piaget and Inhelder hypothesize the importance of imaginal manipulation to the development of representational space. El Koussy's space factor, K, is often cited as a measure of the ability to manipulate shapes imaginatively (Lovell, 1958). The child's response to this imaginal manipulation is his signifier of space, his representational space.

Imagery

From the preceding discussion it is obvious that imagery is a vital component in the development of representational space. Throughout The Child's Conception of Space and Piaget's lectures at Cornell in 1964, the importance of imagery in cognitive development is emphasized. Imagination is stimulated by experience; a paucity of experience pre-supposes a limited or non-existent imagery, thus at least retarding conceptual development (Dienes, 1964). Piaget and Inhelder (1963) support this when they say that the child must construct "a system of intellectual operations if he is to be . . . able to form a mental image corresponding to his perception [p. 242]." The child who has not manipulated a sphere in the form of a ball or orange, who has not experienced the physical operation of cutting an orange, or of seeing an orange cut, will find it difficult, if not impossible, to comprehend, let alone anticipate, the sectioning of a sphere. This is not to imply that specific situations must be experienced before conceptualization can occur. It does imply that perceptual relationships will become conceptual when they are "coordinated with others in an overall grouping or group which combines the invariance of certain relations . . . with the variability of others . . . [Piaget & Inhelder, 1963, pp. 226-227]."

> Experience is always necessary for intellectual development. . . . [Just] being submitted to an experience (a demonstration) is not sufficient for a subject to disengage the structure involved. . . . The subject must be active, must transform things, and find the structure of his own actions on the objects [Duckworth, 1964, p. 4].

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[5] The pre-operational stage of development is characterized by the child's operation upon concrete objects; operating upon these things which are directly perceptible—visible and tangible.

It is from such perceptual experiences that the concepts develop, allowing for transfer for the representation of other sectionings.

Recently Piaget has been performing experiments in an attempt to delve into the imagery of children. He has repeatedly found that mere imagery alone will not give rise to the operations.

> ...imagery is not sufficient and anticipation is not possible as long as there is no operational transformation upon which the imagery can be based [Piaget, 1964, p. 30].[6]

Images which are not adequate to provide the operational level of representational development are "simply copies, more or less faithful, of what has already been seen [Piaget, 1964, p. 30]."

In spite of these seeming shortcomings of imagery, Dienes (1964) speaks of the importance of imagery manipulation and its basis in experience. He believes that no abstraction occurs unless there is an "accompanying development of the corresponding imagery," that understanding implies a successful carrying "out of a transformation of imagery corresponding to the symbols." Imagery, though not sufficient for thought, as Piaget (1964) has so aptly shown, is still necessary for it. "Whenever we think, we do not do so in vacuo, but in terms of images...[Dienes, 1964, p. 104]."

> Imagery is a kind of interiorized imitation which, like all imitations, supposes a certain amount of understanding and is thus directed by intelligence. So while intelligence still does not have a basis of operational structures, the imagery remains meager, remains static, remains unproductive. It does not attain the mobility of anticipatory imagery, but at the level where operations become possible and enable the child to think of transformations from one state to another, we find an action of the operation upon imagery. The imagery itself becomes more mobile, becomes anticipatory. Now it becomes an instrument of representation capable of serving the operations. It is a symbolic instrument and an auxiliary instrument which is not an element of thought itself, but is simply a tool, an aid to the process of thought—an aid that takes the form of figurative representations. These representations become much more helpful when they are supported by operations and so can deal with transformations themselves, not only fixed states [Dienes, 1964, p. 104].

Thus, imagery is the result of experiences—manipulative experiences with one's environment, a mental picture upon which the child can perform transformations without the use of a visual stimulus. Intellectual imagery is <u>not</u> merely a copy of reality, but an integral part of cognitive development and learning.

> Visual imagery is clearly connected with spatial ability in some way but the form of the relationship is unknown. Such imagery is likely to be of help in all cases where objects have to be manipulated imaginatively [Lovell, 1963, p. 92].

Sectioning Solids

Vinacke (1952) lists the hierarchy of concept development from the perceptual level to the abstraction or generalization level. The concepts of space follow the same hierarchical structure, from the concrete, manipulative play with objects, to the operational level, imagery (Piaget & Inhelder, 1963). The concepts of space also follow the same hierarchical structure as that of geometry: topological, projective, and euclidean (Meserve, 1955; Piaget & Inhelder, 1963). Within these three levels of spatial conceptualization, the sectioning of solid figures by a plane is of primary importance. It is at this stage, the anticipation of the boundary formed by the intersection of the plane and solid figure, that euclidean spatial concepts begin to emerge. The task, then, is the sectioning of geometric solid figures and needs to be further explained and defined.

There are two ways of describing sectioning: projectively, as the "geometry of viewpoints," and euclidean, a "geometry of objects." Piaget and Inhelder contend that from the point of view of the 4-12-year-old child the problem of sectioning is euclidean. This is because the operation of cutting embodies the "factor of measurement by passing from one object to another or around the periphery of an object. It thus brings movement or displacement to bear on the object... [p. 247]." They do state that this measurement and the object are regarded as being co-extensive.

They further state that either euclidean or projective methods could be used for a study

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[6] "Anticipatory images [are] images which imagine the result of a transformation as yet unknown, but which could be predicted on the basis of some reasoning...[Piaget, 1964, p. 31]."

of sectioning operations. The projective method is considered to be the more artificial of the two methods, "since in any case the euclidean method implies perspective or projective... [p. 248]." The sectioning of solids prior to the cognitive development of euclidean concepts "involves projecting the solid on a two-dimensional surface [p. 248]," thus providing a link in the conceptual development of space from one level of the mathematical and psychological space to the next. Another connection from the projective to the euclidean concepts is a series of transitional stages involving affinities and similarities.

Mathematically, Piaget and Inhelder develop the "continuous series of gradations" which maintain the topologic-projective-euclidean sequence. They endeavor to show that this same sequence occurs in the psychological development of spatial concepts.

> There is the extremely intimate connection between the development of euclidean and projective operations; the first dealing with the movement or displacement, and the second with their representation....
>
> In sectioning solids, the closest possible interaction is apparent at all levels of psychological development, between the euclidean operations which traverse a solid by an actual movement [distance]...and projective operations which represent the solid according to a given perspective cutting the three-dimensional figure along a plane [Piaget & Inhelder, 1963, p. 249].

The projective section of Piaget and Inhelder is the shadow formed by placing an object, such as a cone or pencil, between a light source and a screen. The objects are placed in various positions and the child is asked to draw or select the plane figure which represents the shape of the shadow prior to the child seeing the shadow. Coxeter (1964) considers these shadows as projective sections. He describes the "parallel projections" which are the basis of affine geometry, and the "central projections" which describe the properties of projective geometry. For example, a central projection casts shadows of rectangles that will be "distorted into quadrangles of various sizes, but their sides are still straight [Coxeter, 1964, p. 104]." The central projection is the projection Piaget and Inhelder employ in their sections. Piaget and Inhelder's geometry of view-points—projective geometry—is Young's (1911) geometry of position. Both the geometry of position and the geometry of viewpoints are appropriate nomenclature for Piaget and Inhelder's hypothesis concerning the child's development of perspectivity and projectivity.

In this study, the expression "sectioning solids" refers to the euclidean section, the boundary of the plane surface formed by the intersection of a plane with a solid figure. Sectioning a solid "consists simply of looking at objects made of [wood] ...and predicting the shape of the surface produced when the solid is cut along various planes with a large knife. ...It is...a question of causing an imaginary plane (the knifeblade, in imagination) to pass through the object which is so far intact [Piaget & Inhelder, 1963, p. 248]." The subject must predict, anticipate, the boundary or edge of the surface formed by the hypothetical cutting or sectioning. For example, an orange (sphere) cut in two by a large knife yields two hemispheres, each with an identical flat surface whose boundary constitutes a circle. A right cylinder cut longitudinally yields a surface whose boundary is a rectangle.

THE SECTIONING TASK

The task of sectioning solid figures is based upon Chapter 9 of The Child's Conception of Space (Piaget & Inhelder, 1963). The present investigation will study sixteen sections. These sections will be formed by four cuts—longitudinal, transverse, oblique, and parallel—which are to be imagined on each of four solid figures: rectangular prism, right circular cylinder, cube, and one nappe of a right circular cone. Unlike Piaget and Inhelder's tasks, the subjects of the present investigation will not see any of the sixteen sections of the test.

Piaget and Inhelder (1963) require two methods of response for each test item.

> To ensure that the child's responses are a genuine product of his spatial or geometric concepts, and not merely artifacts of the experimental technique, . . . [the child is asked to] (a), draw the expected surface . . .and (b), pick it out from a selection of comparison drawings...[p. 249].

PURPOSE

The purpose of this study is to investigate the responses of eighth, tenth, and twelfth grade secondary school children to sixteen sectioning tasks.

The design of this study permits the organization of the questions investigated into three groups. The first group of questions deals with the scores made by the subjects over the sixteen items: four different types of cuts performed on each of four solid figures. The second group of questions is concerned with the degree of difficulty of the sixteen items. The third group of questions is based upon the interactions of the variables in the first group with the variables in the second group.

Group I

1. Is there a significant difference between the mean of correct responses over the sixteen test items for boys and the mean of correct responses over the sixteen test items for girls?
2. Is there a significant difference among the means of correct responses over the sixteen test items for each of the three ability levels?
3. Is there a significant difference among the means of correct responses over the sixteen test items for each of the three grade levels?
4. Are there any significant interactions among the means of correct responses over the sixteen test items by sex with the three grade levels, by the three ability levels with the three grade levels, and by sex with the three ability levels and the three grade levels?

Group II

1. Is there a significant difference among the means of correct responses for each of the four solid figures?
2. Is there a significant difference among the means of correct responses for each of the four types of cuts?
3. Is there a significant interaction among the means of correct responses for the four solid figures by the four types of cuts?

Group III

1. Is there a significant difference between the means of correct responses for each of the four solid figures by boys and the means of correct responses for each of the four solid figures by girls?
2. Is there a significant difference between the means of correct responses for each of the four types of cuts by boys and the means of correct responses for each of the four types of cuts by girls?
3. Is there a significant interaction among the means of correct responses for each of the four solid figures by each of the three ability levels?
4. Is there a significant interaction among the means of correct responses for each of the four types of cuts by each of the three ability levels?
5. Is there a significant interaction among the means of correct responses for each of the four solid figures by each of the three grade levels?
6. Is there a significant interaction among the means of correct responses for each of the four types of cuts by each of the three grade levels?
7. Are there any significant higher-order interactions among the correct responses over the four solid figures or the four types of cuts by sex, ability level, or grade level?

III

DESIGN OF THE STUDY

THE SAMPLE

Seventy-two secondary school children were selected from five schools in a large Midwest city. These five schools are within a segment of the city that is representative of the other major areas excluding the inner core of the city. Within this wedge are found the semi-professionals and skilled workers. Hence the area from which this study's sample was drawn is typical of the city's major socio-economic population—middle class through professional people.

Table 1 lists the five schools, the number of children in each grade from each school, and the number of subjects selected from each grade in each school. There were 853 eighth grade children, 783 tenth grade children, and 684 twelfth grade children. Twenty-four children were randomly selected from the total population of each grade level. Each child was screened as to his sex and Lorge-Thorndike IQ score. If he met the required characteristics of one of the groups with respect to sex, grade, and ability level, he was designated as entry 1, 2, 3, or 4. If the four subjects for a cell had been selected, he was designated as an alternate. Each subject had an alternate. If a child did not qualify for a cell entry or if the cell sample and alternate cell sample were filled, he was by-passed and the next random number investigated. Neither an equal number of boys and girls nor an equal number of subjects was selected from each school. Schools were not a variable by this technique.

THE TESTS

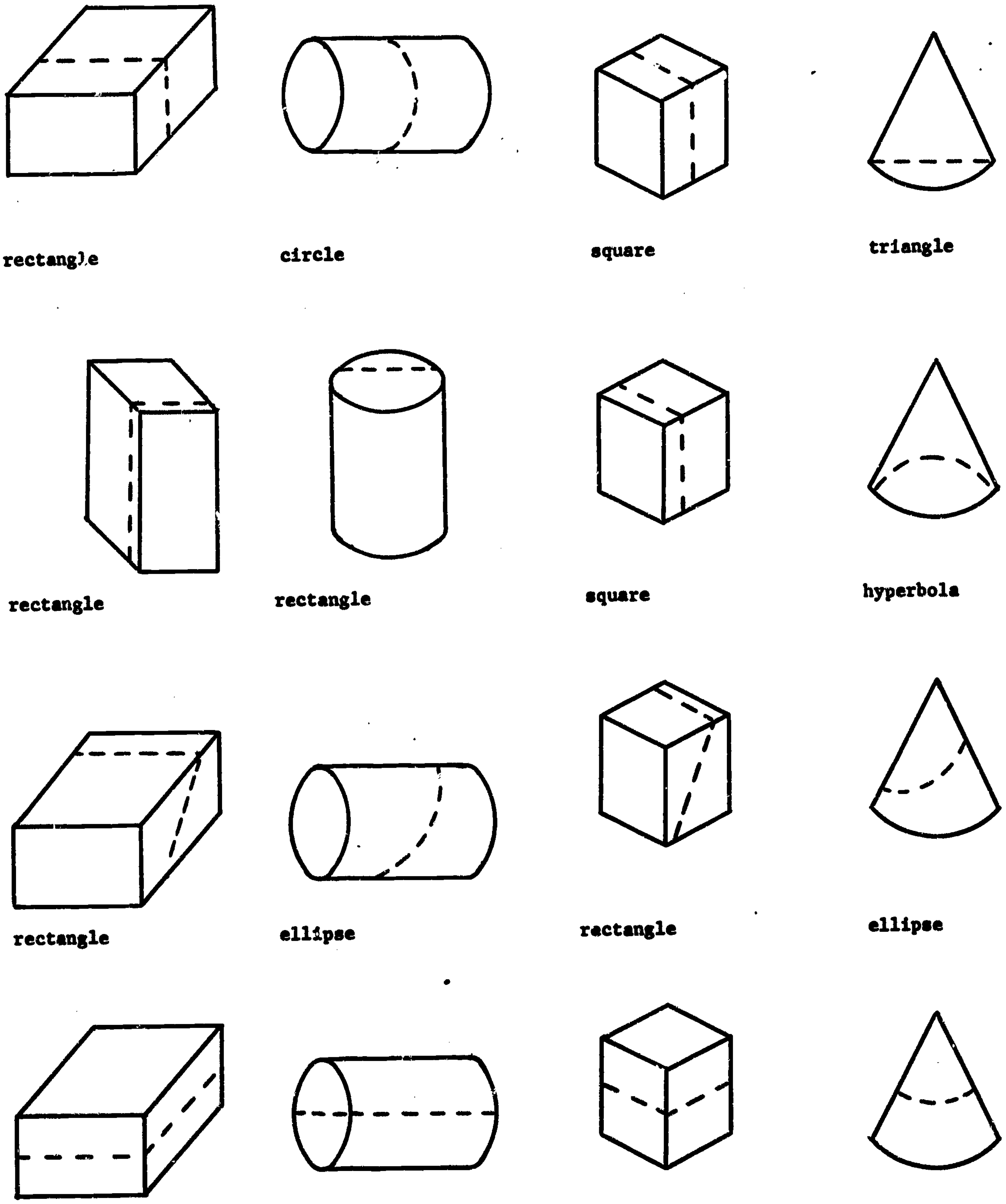
Two tests were designed for this study in accordance with Piaget and Inhelder (1963). The tests were identical; they differed only in the methods of response. The test instrument, depicted in Figure 1, consisted of sixteen items which are frequently referred to as cuts in this paper.

There were four types of cuts obtained from a rectangular prism, a right circular cylinder, a cube, and one nappe of a right circular cone: a longitudinal cut which was a perpendicular

Table 1

Schools from Which the Sample Was Selected

School	Grade	Number of Children in Each Grade	Number of Children Selected from Each Grade
Senior High School	10	353	17
	12	372	17
Junior-Senior High School	8	270	11
	10	430	7
	12	312	7
Junior High School A	8	189	5
Junior High School B	8	223	4
Junior High School C	8	171	4



rectangle	circle	square	triangle
rectangle	rectangle	square	hyperbola
rectangle	ellipse	ractangle	ellipse
rectangle	rectangle	square	circle
RIGHT RECTANGULAR PRISM	RIGHT CIRCULAR CYLINDER	CUBE	ONE NAPPE OF A RIGHT CIRCULAR CONE

Dashed lines indicate cuts hypothetically performed. Correct response below each figure.

Figure 1. Representation of the test.

bisector through the major axis; a transverse cut which was perpendicular, but not a bisector, through the major axis; an oblique cut which traversed the solid figure oblique to the surface upon which it rested, beginning and ending within the bounds of the solid figure; and a parallel cut which was parallel to the surface upon which the solid rested. The four cuts of the cone allowed the representation of two conic sections, a circle, and a triangle. The conic sections, according to Piaget and Inhelder, furnish important information in the child's development of representational space. The four cuts also provide representative sections often employed in the development and application of volume formulas.

Two pilot studies gave the examiner opportunity to: (1) determine the kinds of responses children of different ages and ability levels give to the problem of sectioning solid figures, (2) determine which three grade levels to study, (3) refine the dialogue for the preparation period prior to administering the test as well as for the test-interview period, (4) gain experience in working with the various ability and age groups, and (5) determine the average length of time a subject needed to complete both tests.

Test I, which was always given first, required the subject to respond by making a drawing of the boundary formed by hypothetically intersecting a plane and a solid figure. For Test II the subjects were required to choose a boundary from a set of pre-drawn boundaries. For each of the sixteen cuts there were five boundaries drawn in ink on an 18 inch by 22 inch white cardboard card. These boundaries were representative of the shapes and not proportional to the solid figures. The five drawings on the drawing selection cards either were selected from the children's responses in the two pilot studies or were responses given by Piaget and Inhelder's sample. The subject was given a sixth choice, "None of these."

Photographs of the solid figures and of the drawing selection cards are in the original report (Boe, 1966).

ADMINISTERING THE TESTS

The seventy-two subjects were tested individually by the researcher during the first three weeks of October, 1965. Each subject was first given the instruction period for the drawing task. Test I immediately followed. The sixteen items were randomly assigned for presentation to each subject. After an instruction period for the multiple choice responses, the sixteen items of Test II were presented in a unique random order. The order in which the subjects were given the problems was determined by a table of random numbers. No two subjects were presented with the same sequence of tasks; no subject was presented with the same sequence of tasks for both tests.

The examiner met the subject in the room where the interview was to be held. The day preceding the subject's interview an explanatory letter (See Boe, 1966.) was given to each subject along with his corridor pass. A few minutes prior to the instructional period were devoted to establishing rapport. An effort was made at this time to put the subject at ease; that this was an interview rather than a test was stressed. Only the subject and examiner were present during the entire interview.

The subject was told that he would be shown some solid figures and that he was to think about the surface formed when a solid figure was cut into two pieces, "such as the shape of the flat surface formed by cutting an orange." The basic test for the pre-test preparation and the test-interview is given in the Appendix to the original report (Boe, 1966). The time that the pre-test preparation required varied with each child as did the time needed to complete both sets of the sixteen items.

THE DATA

The data for each subject were recorded on response sheets: sixteen separate sheets for the items of Test I and one multiple choice response sheet for Test II. The response sheets were coded by letters and numbers indicating the subject's grade, ability level, sex, the cut being considered, and the random order of presentation. The type of response sheets may be found in the original report (Boe, 1966).

In Test I, if more than one drawing was made by a subject for any one cut, the subject was asked to indicate which was his final response. For Test II the examiner checked the subject's choice: A, B, C, D, E, or F, for each of the sixteen cuts.

To determine the IQ limits for the three ability levels a weighted mean IQ was computed. During the fall semester all children in Grades 6 and 8 were given the Lorge-Thorndike Intelligence Test by the school authorities. The three upper grade levels of this test employ both verbal and non-verbal items. School personnel appear to be more IQ oriented rather than being conscious of these two sub-scores, consequently the composite, or general IQ score, was considered for this study. The total scores from this test for the 2330 student population resulted in a weighted mean IQ of 111. The school district employs the standard deviation of 16 for this IQ test; consequently the researcher used the same standard deviation. The average ability level ranged from 103 to 119.

The average IQ for the below average group was 94; for the average ability group, 112; and for the above average ability group, 127. The average IQ for the eighth grade was 110; for the tenth grade, 112; and for the twelfth grade sample, 111.

THE STATISTICS

The Design

With respect to the subjects a 2×3^2 factorial design constituting eighteen groups was used. Sex, three grade levels, and three ability levels made up the dimensions of this design. The sixteen repeated measures form a 4×4 design given by four types of cuts and four solid figures. The level of significance was set at .05. A schematic representation of the design is given in Table 2.

Analysis of Data

Responses for both tests were scored either "1" or "0." In the case of Test I, a "1" indicated that the drawing was appropriate while a "0" indicated an inappropriate or incorrect response. An appropriate response implies, in the case of straight line drawings, not more than one error or two symmetrical errors, such as a rectangle for a square. In the case of curved drawings an appropriate response implied a figure clearly recognizable as a circle or an ellipse. For Test II a "1" indicated a correct choice, while a "0" was an incorrect choice.

A reliability check was made on the evaluation of the drawing responses. Two raters, the examiner and a graduate student familiar with the test instrument, independently evaluated the 1152 responses. There was 96.8 per cent agreement. Differences on the remaining items were resolved by a discussion of the drawings.

A Pearson product-moment correlation coefficient was used to determine whether the two methods of response yielded comparable data.

Table 2

A Schematic Representation of the Design of the Study

				F1				F2				F3				F4			
Sex	Grade	Ability	No. of Ss	C11	C12	C13	C14	C21	C22	C23	C24	C31	C32	C33	C34	C41	C42	C43	C44
Male	12	H	4																
		A	4																
		L	4																
	10	H	4																
		A	4																
		L	4																
	8	H	4																
		A	4																
		L	4																
Female	12	H	4																
		A	4																
		L	4																
	10	H	4																
		A	4																
		L	4																
	8	H	4																
		A	4																
		L	4																

IV

ANALYSIS OF THE DATA

As the data from Test II, the multiple-choice test, are more straightforward, they will be presented before the data from Test I, drawing.

RESULTS OF ANALYSIS OF VARIANCE ON CORRECT RESPONSE SCORES FOR TEST II

The analysis of variance for the five main effects and the significant interactions is presented in Table 3. The complete analysis of variance table is found in the original report (Boe, 1966).

Five sources of variation were found to be significant. Ability levels, solid figures, and the types of cut were significant beyond the .01 level. The interaction of the four solid figures and the four types of cuts was significant beyond the .01 level. The interaction of ability levels by solid figures was significant beyond the .05 level but not at the .01 level.

With the standard degrees of freedom three other interactions were found to be significant at the .05 level: sex by types of cut, ability levels by grade levels by solid figures, and ability levels by solid figures by types of cut. In a repeated measures design any main effects or interactions containing the repeated measure items which are significant when the standard degrees of freedom are used need to be tested for significance using a conservative statistic such as that developed by Greenhouse and Geisser (1959).

From the analysis, the main effects are not all significant; ability levels, solid figures, and types of cuts contain the major portion of the variability. When the critical values are determined for the three questionable interactions, it is found that their _F_ ratios are substantially smaller than the critical value. This would imply that these _F_ ratios are very near the lower bound of the questionable area between the critical values of acceptance. In view of this observation it is not deemed necessary to proceed with further calculation. A consideration of the tables of means for each of these questionable sources of variation may provide some clue to their possible significance.

Table 3

Analysis of Variance on Response Scores for Test II for the Five Main Effects and The Significant Interactions

Source of Variation	Degrees of Freedom	Mean Squares	_F_ ratio
Between Subjects			
A Sex	1	1.254	3.112
B Ability	2	9.815	24.355**
C Grade	2	0.311	<1
Within Subjects			
D Figures	3	1.168	9.898**
E Cuts	3	11.455	70.276**
D × E	9	1.977	17.342**
B × D	6	.546	4.627*

*p < .05
**p < .01

The mean of correct responses for boys was not significantly different from the mean of correct responses for girls, nor was there a significant difference attributable to differences in the three grade levels. None of the remaining F ratios, which are included in the complete analysis of variance table (Boe, 1966) was significant at the .05 level.

The total number of correct responses for Test II was 786; 319 were made by the above average ability group which represented 83.1 per cent of their 384 responses; 270 or 70.3 per cent by the average ability group; and 197 or 51.3 per cent by the below average ability group. The mean for the above average ability group was 13.292; for the average ability group, 11.250; and for the below average ability group, 8.208. The differences between the means for the three ability levels was significant beyond the .01 level.

It may be seen from Table 4 that the means of the correct response scores for each of the three grade levels: twelve, ten and eight, were 11.217, 11.167, and 10.375, respectively. The differences between these means were not significant. Table 4 also shows the mean total scores by grade in each of the three ability levels. This interaction was not significant.

The mean of the total correct responses under sex was 11.444 for the boys and 10.389 for the girls, as shown in Tables 5 and 6. The difference between the means under this variable was not significant. The means of the correct responses under sex by the three ability levels and by the three grade levels are shown in Tables 5 and 6. These interactions were not significant.

Table 4

Mean Correct Response Scores Test II, Ability Level and Grade Level

Grade	Above Average	Average	Below Average	Mean
Twelve	14.250	9.750	9.625	11.217
Ten	13.000	12.250	8.250	11.167
Eight	12.625	11.750	6.750	10.375
Mean	13.292	11.250	8.208	10.917

Table 5

Mean Correct Response Scores Test II, Sex and Ability

Sex	Above Average	Average	Below Average	Mean
Male	13.667	12.583	8.083	11.444
Female	12.917	9.917	8.333	10.389
Mean	13.292	11.250	8.208	10.917

Table 6

Mean Correct Response Scores Test II, Sex and Grade

Sex	Grade Twelve	Grade Ten	Grade Eight	Mean
Male	12.167	11.750	10.418	11.444
Female	10.250	10.583	10.333	10.389
Mean	11.217	11.167	10.375	10.917

Table 7

Mean Correct Response Scores for Test II for the Four Solid Figures, Three Ability Levels

Ability	Rectangular Prism	Right Circular Cylinder	Cube	Right Circular Cone	Mean
Above Average	3.375	3.625	3.250	3.042	3.323
Average	2.667	3.167	2.917	2.500	2.813
Below Average	2.250	1.792	2.625	1.542	2.052
Mean	2.764	2.861	2.931	2.361	2.729

Table 8

Mean Correct Response Scores on Test II for the Type of Cut, Sex

	Type of Cut				
Sex	Longitudinal	Transversal	Oblique	Parallel	Mean
Male	3.194	2.972	1.861	3.417	2.811
Female	3.194	3.000	1.222	2.972	2.597
Mean	3.194	2.986	1.542	3.194	2.729

There were 199 correct responses for selecting the appropriate boundaries formed by cutting the rectangular solid, which was 69.1 per cent of the 288 responses for this figure. For the right circular cylinder, the 206 correct responses represented 71.5 per cent of the 288 responses. Of the 288 responses on the boundaries formed by sectioning the cube, 211 responses were correct; this represented 73.3 per cent of the total responses. One hundred seventy correct responses were given by the subjects for selecting the appropriate boundaries for the four cuts described on the right circular cone; this represented 59.0 per cent of the 288 responses for this figure. The differences among the means were significant beyond the .01 level.

The means by ability level for the four solid figures are presented in Table 7. The interaction of ability level and solid figures was significant beyond the .05 level.

The total correct response score for the longitudinal cut was 230, 79.9 per cent of the 288 responses for this cut. The parallel cut was of an equivalent degree of difficulty as can be seen from the mean scores in Table 8. The total correct response score for the transverse cut was 215 or 74.7 per cent of the total possible score. The oblique cut was the most difficult, having a total correct response score of 111 or 38.5 per cent of the 288 responses. The differences in the means of correct responses for all the cuts were significant beyond the .01 level.

The interaction of the four cuts by the sexes is questionable, as was previously discussed. A survey of the means in Table 8 shows that the range of means is from 1.222 for females in the third or oblique cut to 3.194 which was received by both sexes in the longitudinal cut. The differences between the sexes on the longitudinal cut is 0.000; on the transverse cut, .028; on the oblique cut, .639; and on the parallel cut, .455. The third and fourth cuts have the largest difference between the means for males and females. This would appear to indicate that sex and these cuts do interact. The third cut is the most difficult cut as is indicated by its small mean score of 1.542; the fourth cut, however, is one of the easiest. With such unequal difficulties manifested in the four types of cuts, the significance of the interaction of sex and cuts remains questionable. A substantial difference exists between this interaction's *F* ratio and the critical values of the conservative test, which might imply that this interaction was not significant.

The mean scores for each of the four types of cuts by the four solid figures are shown in Table 9. From these means it appears that the

Table 9

Mean Correct Response Scores on Test II, Figures, Cuts

Type of Cut	Rectangular Prism	Right Circular Cylinder	Cube	Right Circular Cone	Mean
Longitudinal	.611	.944	.944	.694	.799
Transverse	.931	.722	.931	.403	.747
Oblique	.417	.431	.319	.375	.386
Parallel	.806	.764	.736	.889	.799
Mean	.691	.717	.733	.590	.682

Table 10

Mean Correct Response Scores on Test II
Three Ability Levels, Three Grade Levels, Four Solid Figures

Ability Level	Grade Level	Rectangular Prism	Right Circular Cylinder	Cube	Right Circular Cone	Mean
Above Average	Twelve	3.750	3.450	3.625	3.125	3.563
	Ten	3.250	3.750	3.000	3.000	3.250
	Eight	3.125	3.375	3.125	3.000	3.156
Average	Twelve	2.250	3.000	2.125	2.365	2.438
	Ten	3.125	3.250	3.125	2.750	3.063
	Eight	2.625	3.250	3.500	2.365	2.938
Below Average	Twelve	2.365	2.625	2.875	1.750	2.406
	Ten	2.000	1.750	2.750	1.750	2.063
	Eight	2.365	1.000	2.250	1.125	1.688
	Mean	2.764	2.861	2.931	2.361	2.729

Table 11

Mean Correct Response Scores on Test II
Three Ability Levels, Four Solid Figures, Four Types of Cuts

Ability Level	Figure	Cut				Mean
		Longitudinal	Transverse	Oblique	Parallel	
Above Average	Rectangular Prism	.875	1.000	.542	.975	.844
	Right Circular Cylinder	.975	1.000	.667	1.000	.906
	Cube	1.000	1.000	.500	.750	.813
	Right Circular Cone	.933	.500	.625	1.000	.760
Average	Rectangular Prism	.500	.875	.542	.750	.667
	Right Circular Cylinder	1.000	.833	.500	.833	.792
	Cube	.933	.875	.375	.750	.729
	Right Circular Cone	.833	.417	.417	.833	.625
Below Average	Rectangular Prism	.458	.933	.167	.708	.563
	Right Circular Cylinder	.875	.333	.125	.458	.448
	Cube	.933	.933	.083	.708	.656
	Right Circular Cone	.333	.292	.083	.833	.385
	Mean	.799	.747	.386	.799	.682

longitudinal cuts on the right circular cylinder and on the cube were the least difficult while the oblique cut on the cube was the most difficult. The oblique cut received the lowest mean for correct responses. The right circular cone was the most difficult figure on which the cuts were described. The interaction of figures by cuts was found to be significant beyond the .01 level.

Table 10 shows that the means of correct responses for the twelfth grade were higher than the means for the other grade levels in all three ability levels with one exception; for the average ability group, the tenth grade subjects had the highest mean score for the sectioning of the four solid figures. The interaction of ability level, grade level, and solid figures was significant at the .05 level with the standard degrees of freedom. It was not significant when the conservative test was used. There was a substantial difference between the F ratio and the critical value.

Table 11 shows that the means of correct responses for the three ability levels were highest for the above average ability level when each of the four cuts on each of the four solid figures is considered. In each ability level the right circular cone had the smallest mean correct response scores. The transverse cut on the right circular cone and the oblique cut on the cube were the most difficult for the above average ability group. The latter section was also the most difficult for the average ability group. The oblique cut on all four solid figures was the most difficult for all groups, but especially so for the below average ability group, as can be seen by the means in Table 11. The right circular cone was their most difficult. The interaction of ability level, solid figure, and type of cut was significant at the .05 level using the standard degrees of freedom; it was not significant when the conservative test was employed. Once again the difference between the F ratio and the critical value was substantial.

RESULTS OF THE ANALYSIS OF VARIANCE ON THE APPROPRIATE RESPONSE SCORES FOR TEST I

The analysis of variance on the scores for Test I, a drawn representation of the boundary formed by sectioning a solid figure, for the five main effects and the significant interactions is presented in Table 12. The complete analysis of variance table is found in the original report (Boe, 1966).

The scores referred to as the appropriate-response-by-drawing scores were derived as described in Chapter III.

It may be noted in a comparison of the results of the analyses of variance for Test I and Test II that the means of the same main effects were significantly different. The main effect of sex, in addition to ability levels, solid figures, and type of cut, was significant in Test I beyond the .05 level. Eleven different inter-

Table 12

Analysis of Variance of the Scores for Test I

Source of Variation	Degrees of Freedom	Mean Squares	F ratio
Rows			
A Sex	1	1.605	4.126*
B Ability	2	10.011	25.735**
C Grade	2	1.191	3.062
Columns			
D Figures	3	5.378	86.742**
E Cuts	3	11.117	91.125**
D × E	9	1.622	28.964**
Rows × Columns			
B × D	6	.738	11.903**

*p < .05
**p < .01

Table 13

Mean Correct Response Scores on Test I, Sex, Ability Level

	Ability Level			
Sex	Above Average	Average	Below Average	Mean
Male	14.833	13.667	9.667	12.722
Female	13.417	12.417	8.750	11.528
Mean	13.709	13.042	9.209	12.125

Table 14

Mean Scores for Appropriate Drawing Responses, Solid Figures by Type of Cut

	Type of Cut				
Solid Figure	Longitudinal	Transverse	Oblique	Parallel	Mean
Rectangular Prism	.986	.972	.569	.931	.865
Right Circular Cylinder	.944	.722	.431	.736	.708
Cube	.972	.958	.583	.972	.872
Right Circular Cone	.792	.250	.389	.917	.587
Mean	.924	.726	.493	.889	.758

Table 15

Mean Scores for Appropriate Drawing Responses, Ability Level by Solid Figures

	Ability Level			
Figure	Above Average	Average	Below Average	Mean
Rectangular Prism	3.750	3.667	2.958	3.458
Right Circular Cylinder	3.583	3.333	1.583	2.833
Cube	3.750	3.625	3.083	3.486
Right Circular Cone	3.042	2.417	1.583	2.347
Mean	3.531	3.260	2.302	3.031

Table 16

Mean Scores for Appropriate Drawing Responses, Sex, Type of Cuts

Sex	Longitudinal	Transverse	Oblique	Parallel	Mean
Male	3.750	2.944	2.306	3.722	3.181
Female	3.638	2.861	1.639	3.389	2.882
Mean	3.694	2.903	1.972	3.556	3.031

actions reach the point of significance when the standard degrees of freedom are used. When the conservative test is used, the interactions solid figure by type of cut and ability level by solid figure are the only significant interactions in this set of data, as was true in the set of data for Test II. Ability by grades by solid figures was not significant with the standard degrees of freedom as it was in the analysis of the data for Test II.

Six sources of variation were found to be significant. Of the main effects, ability level, solid figures, and type of cut were significant beyond the .01 level; sex was significant beyond the .05 level. The interactions of solid figures by types of cuts and ability level with solid figures were significant beyond the .01 level. Nine additional interactions were significant using the standard degrees of freedom but not significant when the conservative test was used.

The total number of boundaries recorded as appropriately drawn was 873, 458 by the boys and 415 by the girls. The mean total scores of appropriate drawings of the boundaries was 3.181 for boys and 2.882 for girls. The difference between the means was significant beyond the .05 level.

There were 339 appropriate drawings done by the above average ability group, 313 by the average ability group, and 221 by the below average ability group. The mean total scores of appropriate drawings of the boundaries was 3.531 for the above average ability group, 3.260 for the average ability group, and 2.302 for the below average ability group. The differences among the means were significant beyond the .01 level.

The means for the drawing of the boundaries found on each of the four solid figures, shown in Table 14, are significantly different beyond the .01 level. The means for the drawing of the boundaries formed by each of the four types of cuts are also presented in Table 14. The differences among the means were significant beyond the .01 level. The interaction of solid figures with types of cuts was significant beyond the .01 level.

From the table of means given in Table 14, it appears that the longitudinal cut on the rectangular prism was the least difficult for the subject to draw. The lowest mean for the sixteen tasks was for the transverse cut on the right circular cone. The oblique cut was the most difficult cut to draw in all cases except on the cones. The means Tables 9 and 14 for the interactions of figures and cuts for both correct responses and appropriate drawings of the boundaries, show that some cuts were not equated with low scores or with high scores for all four solid figures, thus the significant interaction.

Table 15 shows the means for the interaction of ability levels by solid figures. A comparison of Tables 7 and 15 reveals that the same solid figures were not equated with low scores or with high scores for all three ability levels, thus the significant interaction.

Tables 16-24 illustrate the mean scores for the appropriate drawing responses for those interactions which may or may not be significant. When considering the mean scores of the other significant sources of variation, these mean scores are relatively small. The interpretation of the higher-order interactions is also questionable. Whatever patterns of responses that can be seen from these potential sources can best be determined from the means tables.

The differences between the means of the boys' scores on the four cuts and the girls' scores on the four cuts range from .083 for the transverse cut to .665 for the oblique cut. The difference between the boys' mean on the longitudinal cut and the girls' mean is .112. For the parallel cut, the difference between the boys' mean and the girls' mean is .333. The variation between the *F* ratio and the critical value was not too large. This would imply that the *F* ratio is not too far into the area of questionability. It appears to be nearer the critical value associated with the conservative test.

Except for the oblique cut drawn by the below average ability group, the mean scores of the three ability groups for each of the four cuts lie within a range of 1.750 from highest to next lowest mean score. The differences between the above average ability level group and the average ability group on the parallel cut is .042; the difference between the average ability group and the below average ability group is 1.000. It is questionable whether such differences between the mean scores represent a significant interaction. This *F* ratio is nearer the critical value associated with the conservative test which indicates a potential nonrejection if a more sensitive test was applied.

In comparing Table 18 with Table 17, the same basic pattern is revealed with respect to the degree of difficulty in drawing the boundary of the section formed by these four cuts. Except for the longitudinal cuts and the transverse cuts, the higher grade level implies a higher degree of appropriate representation of the various sections' boundaries. The significance of the interaction of grade level by type of cut remains questionable. The *F* ratio is much further from the critical value associated with the critical test, which would imply a rejection of the significance of this interaction.

As can be seen from Tables 19 and 20, the same basic relationships are in evidence as were revealed in Tables 13, 14, and 15. In all but one section, the transverse cut on the cube, the boys received a higher mean score than the girls. The differences between the corresponding means range from 0.000 in two instances to one instance of .222, the section formed by the oblique cut on the cube. The average difference between the means is .072. Whether

Table 17

Mean Scores for Appropriate Drawing Responses, Ability Level, Type of Cut

	Ability			
	Above Average	Average	Below Average	Mean
Longitudinal	3.958	3.875	3.250	3.694
Transverse	3.458	3.042	2.208	2.902
Oblique	2.792	2.250	.875	1.972
Parallel	3.917	3.875	2.875	3.556
Mean	3.531	3.260	2.302	3.031

Table 18

Mean Scores for Appropriate Drawing Responses, Grade Level, Type of Cut

	Cut				
Grade	Longitudinal	Transverse	Oblique	Parallel	Mean
Twelve	3.959	3.042	2.333	3.792	3.281
Ten	3.459	2.792	2.125	3.459	2.958
Eight	3.667	2.875	1.458	3.417	2.854
Mean	3.694	2.903	1.972	3.556	3.031

Table 19

Mean Scores for Appropriate Drawing Responses, Sex, Solid Figures, Type of Cut

		Figure				
Sex	Cut	Rectangular Prism	Right Circular Cylinder	Cube	Right Circular Cone	Mean
Male	Longitudinal	1.000	.944	.979	.833	.938
	Transverse	.979	.778	.944	.250	.736
	Oblique	.694	.500	.722	.389	.576
	Parallel	1.000	.778	1.000	.944	.931
Female	Longitudinal	.972	.944	.972	.750	.812
	Transverse	.972	.667	.972	.250	.667
	Oblique	.500	.361	.500	.388	.416
	Parallel	.861	.694	.944	.889	.847
	Mean	.865	.708	.872	.587	.757

the interaction of sex, solid figures, and type of cut is a significant interaction or whether its significance with the standard degrees of freedom is due to the large mean square of the interaction of solid figures by cut remains questionable at this time. The critical value for the conservative test, however, is not too much larger than the *F* ratio.

The means for appropriate drawing responses for the sixteen test items by ability level are presented in Table 20. The means are higher for the longitudinal cut in all but four cases; for these four situations the parallel cuts had the higher means. Three of these four means were for the parallel cut on the right circular cone while the fourth mean was for the parallel cut on the cube. Two of these means were in the below average ability group; there was one each in the other two ability groups.

The interaction of ability level by solid fig-

Table 20

Mean Scores for Appropriate Drawing Responses
Ability Level, Solid Figures, Type of Cut

		Figure				
Ability	Cut	Rectangular Prism	Right Circular Cylinder	Cube	Right Circular Cone	Mean
Above Average	Longitudinal	1.000	1.000	1.000	.958	.990
	Transverse	1.000	.958	1.000	.500	.865
	Oblique	.750	.708	.750	.583	.698
	Parallel	1.000	.917	1.000	1.000	.979
Average	Longitudinal	1.000	1.000	1.000	.875	.969
	Transverse	1.000	.875	1.000	.167	.760
	Oblique	.667	.500	.667	.417	.563
	Parallel	1.000	.958	.958	.958	.969
Below Average	Longitudinal	.958	.833	.917	.542	.813
	Transverse	.542	.333	.875	.083	.552
	Oblique	.292	.083	.333	.167	.219
	Parallel	.792	.333	.958	.792	.719
	Mean	.865	.708	.872	.587	.757

Table 21

Mean Scores for Appropriate Drawing Responses,
Grade Level, Solid Figure, Type of Cut

Grade	Cut	Rectangular Prism	Right Circular Cylinder	Cube	Right Circular Cone	Mean
Twelve	Longitudinal	1.000	1.000	1.000	.958	.990
	Transverse	1.000	.875	.958	.208	.760
	Oblique	.625	.542	.708	.458	.583
	Parallel	.958	.875	1.000	.958	.948
Ten	Longitudinal	.958	.917	.958	.682	.865
	Transverse	.958	.682	.958	.250	.698
	Oblique	.583	.542	.542	.458	.531
	Parallel	.958	.682	.917	.958	.865
Eight	Longitudinal	1.000	.917	.958	.792	.917
	Transverse	.958	.667	.958	.292	.719
	Oblique	.500	.208	.500	.250	.365
	Parallel	.875	.708	1.000	.833	.854
	Mean	.865	.708	.872	.587	.758

ure by type of cut was significant at the .01 level when the standard degrees of freedom were used; it failed to be significant when the conservative test was used. The critical value for the conservative test was rather large, but not large enough to lead the investigator to hypothesize about the apparent trend toward rejection or acceptance of the significance of this interaction.

The means for appropriate drawing responses for the test items by grade level are presented in Table 21. The means within the table follow the pattern indicated in the other tables; the oblique cuts, right circular cone, and the transverse cut on the right circular cone received the lowest mean scores. It is interesting to note the relatively low mean for the twelfth grade response to the transverse cut on the right cir-

Table 22

Mean Scores for Appropriate Drawing Responses,
Sex, Ability Level, Grade Level, Type of Cut

Sex	Grade Level	Ability Level	Cut				Mean
			Longitudinal	Transverse	Oblique	Parallel	
Male	Twelve	Above Average	1.000	.875	.750	1.000	.906
		Average	1.000	.750	.688	1.000	.859
		Below Average	1.000	.688	.625	.938	.812
	Ten	Above Average	1.000	.938	.938	1.000	.969
		Average	1.000	.750	.875	1.000	.906
		Below Average	.688	.500	0.000	.813	.500
	Eight	Above Average	1.000	.875	.750	1.000	.906
		Average	1.000	.813	.438	.938	.797
		Below Average	.750	.438	.125	.688	.500
Female	Twelve	Above Average	1.000	.750	.875	1.000	.906
		Average	1.000	.750	.438	1.000	.797
		Below Average	.938	.750	.125	.750	.641
	Ten	Above Average	.938	.813	.500	.875	.781
		Average	.875	.750	.563	.875	.766
		Below Average	.688	.438	.313	.625	.516
	Eight	Above Average	1.000	.938	.375	1.000	.828
		Average	.938	.750	.375	1.000	.766
		Below Average	.813	.500	.125	.500	.488
Mean			.924	.726	.493	.889	.758

cular cone. It is the lowest mean score recorded for this cut by the three grades; it is equal to the score received by the eighth grade subjects on the oblique cut performed on the right circular cylinder.

The interaction of these four variables, sex, grade level, solid figures and type of cut was significant at the .01 level using the standard degrees of freedom. It was not significant when the conservative test was applied. In considering the value of the F ratio with respect to the critical values, it appears to indicate that the critical value (4.21) is of sufficient size to warrant rejection of the F ratio. With respect to the statistical procedure employed, the question of significance remains open.

Table 22 contains the mean scores for the appropriate drawing responses on the four types of cuts, stratified with respect to sex, grade level, and ability level. The mean scores indicate nearly perfect responding to the longitudinal cut. Of the mean scores for the longitudinal cut, the lowest mean score was received by both below average tenth grade groups. The male tenth grade below average group had the lowest mean score of all the groups for the oblique cut, with the female below average ability twelfth and eighth grade groups next in receiving low mean scores. This interaction was significant beyond the .01 level when the standard degrees of freedom were used. It was not significant when the conservative test was used; in fact the critical value was rather substantial in comparison to the F ratio.

The standard degrees of freedom indicated that the interaction of sex, ability level, solid figures, and type of cut was significant at the .05 level. Using the conservative test, the critical value was found to be 4.21, which appears to imply that this interaction might be very near the area of rejection. Whether or not this interaction is significant cannot be determined. The range between the critical value and the actual value of F would tend to imply rejection.

There are five extremely low mean scores, only one of which appears above the low ability group. Table 23 seems to indicate a homogeneity of scores comparable to previous tables of mean scores.

Table 23

Mean Scores for Appropriate Drawing Responses,
Sex, Ability Level, Solid Figure, Type of Cut

Sex	Ability Level	Type of Cut	Figure				Mean
			Rectangular Prism	Right Circular Cylinder	Cube	Right Circular Cone	
Male	Above Average	Longitudinal	1.000	1.000	1.000	1.000	1.000
		Transverse	1.000	1.000	1.000	.583	.896
		Oblique	1.000	.750	.917	.583	.813
		Parallel	1.000	1.000	1.000	1.000	1.000
	Average	Longitudinal	1.000	1.000	1.000	1.000	1.000
		Transverse	1.000	1.000	1.000	.083	.771
		Oblique	.750	.667	.833	.417	.667
		Parallel	1.000	1.000	1.000	.917	.979
	Below Average	Longitudinal	1.000	.833	.917	.500	.813
		Transverse	.917	.333	.833	.083	.542
		Oblique	.333	.083	.417	.167	.250
		Parallel	1.000	.333	1.000	.917	.813
Female	Above Average	Longitudinal	1.000	1.000	1.000	.917	.979
		Transverse	1.000	.917	1.000	.417	.833
		Oblique	.500	.667	.583	.583	.583
		Parallel	1.000	.833	1.000	1.000	.958
	Average	Longitudinal	1.000	1.000	1.000	.750	.938
		Transverse	1.000	.750	1.000	.250	.750
		Oblique	.583	.333	.500	.417	.458
		Parallel	1.000	.917	.917	1.000	.958
	Below Average	Longitudinal	.917	.833	.917	.583	.813
		Transverse	.917	.333	.917	.083	.563
		Oblique	.250	.083	.250	.167	.188
		Parallel	.583	.333	.917	.667	.625
Mean			.865	.708	.872	.587	.758

Table 24 contains the means of the response scores per cut entry. The standard degrees of freedom resulted in this interaction being significant beyond the .01 level. The critical value for significance when the conservative test was used was 4.63. This implies that this interaction could be significant.

The mean scores of 0.000 were predominantly received by the below average ability groups; two noticeable exceptions are the female twelfth-grade, above average group score for the transverse cut on the right circular cone, and the male twelfth-grade average group score for the same section. The largest set of zero mean scores was attained by the male tenth-grade below average ability group. Four sections involving the more difficult cuts, oblique and transverse on the cylinder and cone, resulted in zero mean scores for the female eighth-grade below average ability group.

There are a few inconsistencies with the consolidated data, for example, the female tenth-grade above average ability group's mean score of .250 on the section formed by the oblique cut on the rectangular prism; only the below average eighth-grade group received such a mean score on this section. Basically the mean scores per cut are comparable to the consolidated data with respect to the five main effects.

Table 24

Mean Scores for Appropriate Drawing Responses, Sex, Ability Level, Grade Level, Solid Figures, Type of Cut

Sex	Grade Level	Ability Level	Type of Cut	Figure				
				Rectangular Prism	Right Circular Cylinder	Cube	Right Circular Cone	Mean
MALE	Twelve	Above Average	Longitudinal	1.000	1.000	1.000	1.000	1.000
			Transverse	1.000	1.000	1.000	.500	.875
			Oblique	1.000	.750	1.000	.250	.750
			Parallel	1.000	1.000	1.000	1.000	1.000
		Average	Longitudinal	1.000	1.000	1.000	1.000	1.000
			Transverse	1.000	1.000	1.000	0.000	.750
			Oblique	.750	.750	.750	.500	.688
			Parallel	1.000	1.000	1.000	1.000	1.000
		Below Average	Longitudinal	1.000	1.000	1.000	1.000	1.000
			Transverse	1.000	.750	.750	.250	.688
			Oblique	.750	.250	1.000	.500	.625
			Parallel	1.000	.750	1.000	1.000	.938
	Ten	Above Average	Longitudinal	1.000	1.000	1.000	1.000	1.000
			Transverse	1.000	1.000	1.000	1.000	.938
			Oblique	1.000	1.000	.750	1.000	.938
			Parallel	1.000	1.000	1.000	1.000	1.000
		Average	Longitudinal	1.000	1.000	1.000	1.000	1.000
			Transverse	1.000	1.000	1.000	0.000	.750
			Oblique	1.000	1.000	1.000	.500	.875
			Parallel	1.000	1.000	1.000	1.000	1.000
		Below Average	Longitudinal	1.000	.750	1.000	0.000	.688
			Transverse	1.000	0.000	1.000	0.000	.500
			Oblique	0.000	0.000	0.000	0.000	0.000
			Parallel	1.000	.250	1.000	1.000	.813
	Eight	Above Average	Longitudinal	1.000	1.000	1.000	1.000	1.000
			Transverse	1.000	1.000	1.000	.500	.875
			Oblique	1.000	.500	1.000	.500	.750
			Parallel	1.000	1.000	1.000	1.000	1.000
		Average	Longitudinal	1.000	1.000	1.000	1.000	1.000
			Transverse	1.000	1.000	1.000	.250	.813
			Oblique	.500	.250	.750	.250	.438
			Parallel	1.000	1.000	.750	.750	.875
		Below Average	Longitudinal	1.000	.750	.750	.500	.750
			Transverse	.750	.250	.750	0.000	.438
			Oblique	.250	0.000	.250	0.000	.125
			Parallel	1.000	0.000	1.000	.750	.688
		Above Average	Longitudinal	1.000	1.000	1.000	1.000	1.000
			Transverse	1.000	1.000	1.000	0.000	.750
			Oblique	.750	1.000	.750	1.000	.875
			Parallel	1.000	1.000	1.000	1.000	1.000

Table 24 (continued)

Sex	Grade Level	Ability Level	Type of Cut	Figure Rectangular Prism	 Right Circular Cylinder	 Cube	 Right Circular Cone	Mean
FEMALE	Twelve	Average	Longitudinal	1.000	1.000	1.000	1.000	1.000
			Transverse	1.000	.750	1.000	.250	.750
			Oblique	.500	.500	.500	.250	.438
			Parallel	1.000	1.000	1.000	1.000	1.000
		Below Average	Longitudinal	1.000	1.000	1.000	.750	.938
			Transverse	1.000	.750	1.000	.500	.813
			Oblique	.250	.750	.500	.500	.500
			Parallel	1.000	.500	1.000	1.000	.875
	Ten	Above Average	Longitudinal	1.000	1.000	1.000	.750	.938
			Transverse	1.000	.750	1.000	.500	.813
			Oblique	.250	.750	.500	.500	.500
			Parallel	1.000	.500	1.000	1.000	.875
		Average	Longitudinal	1.000	1.000	1.000	.500	.875
			Transverse	1.000	.750	1.000	.250	.750
			Oblique	.750	.250	.750	.500	.563
			Parallel	1.000	.750	.750	1.000	.875
		Below Average	Longitudinal	.750	.750	.750	.500	.688
			Transverse	.750	.250	.750	0.000	.438
			Oblique	.500	.250	.250	.250	.313
			Parallel	.750	.250	.750	.750	.625
	Eight	Above Average	Longitudinal	1.000	1.000	1.000	1.000	1.000
			Transverse	1.000	1.000	1.000	.750	.938
			Oblique	.500	.250	.500	.250	.375
			Parallel	1.000	1.000	1.000	1.000	1.000
		Average	Longitudinal	1.000	1.000	1.000	.750	.438
			Transverse	1.000	.750	1.000	.250	.750
			Oblique	.500	.250	.250	.500	.375
			Parallel	1.000	1.000	1.000	1.000	1.000
		Below Average	Longitudinal	1.000	.750	1.000	.500	.813
			Transverse	1.000	0.000	1.000	0.000	.500
			Oblique	.250	0.000	.250	0.000	.125
			Parallel	.250	.250	1.000	.500	.500
MEAN				.865	.708	.872	.587	.758

RESULTS OF PEARSON PRODUCT-MOMENT CORRELATION COEFFICIENT

The Pearson product-moment correlation was applied to the data in order to learn whether the two methods of response were equivalent forms as implied by Piaget and Inhelder. The correlation coefficient would be the reliability coefficient. One thousand one hundred fifty-two paired observations—the correct responses to each of the test items of Test II and the appropriate drawing response of Test I—were available. The correlation coefficient of .55 was derived from the raw data.

SUMMARY

The results of the analysis of variance performed on the scores for correct responses to the pre-drawn boundary selection cards and on the appropriate drawing responses indicated six sources of variation significant at or beyond the .05 level. Four of these were: (a) sex, (b) ability level, (c) solid figures, and (d) type of cuts. The means for boys were higher than the means for girls on the appropriate drawing response, but were not significant for the selection from the pre-drawn boundary response cards. The means for the above average ability level were higher than the means for the average ability level, which in turn were higher than the means for the below average ability level. The degree of difficulty presented by the four solid figures and the four types of cuts was not equivalent; the right circular cone and the oblique cut were the most difficult. The significant interactions were: (a) solid figures and type of cuts, and (b) ability level and solid figures.

There were three questionable interactions in the analysis of the data for the responses by selection: (a) sex and type of cut, (b) ability level, grade level, and solid figure, and (c) ability level, solid figure, and type of cut. Nine questionable sources of variation were found in the analysis of the appropriate drawing responses: (a) sex and type of cut, (b) ability level and type of cut, (c) grade level and type of cut, (d) sex, solid figure, and type of cut, (e) ability level, solid figure, and type of cut, (f) grade level, solid figure, and type of cut, (g) sex, ability level, grade level, and type of cut, (h) sex, ability level, solid figure, and type of cut, and (i) sex, ability level, grade level, solid figure, and type of cut.

The results of the Pearson product-moment correlation coefficient to check the reliability of the two methods of response was .55.

V

CONCLUSIONS AND IMPLICATIONS

CONCLUSIONS

The general purpose of this study was to determine whether the secondary school pupils in the sample could successfully respond to the sectioning tasks as hypothesized by Piaget and Inhelder. It was shown that there were significant differences in the responses made by the pupils of different ability levels. That significant differences existed among the tasks was also shown. The subject's responses were recorded as two separate tests: (I) the drawing of the sections hypothetically cut through the solid figure and (II) the selection of representative boundaries for these same sections.

The first major result of this study comes from the raw data. Piaget and Inhelder (1963) say that by the age of 12 all the geometric sections they presented to their sample had been mastered. The 16 sections of the present study, which are included in Piaget and Inhelder's sections, should therefore be appropriately drawn and correctly selected by all the subjects. Only ten subjects, however, achieved such success; seven of these subjects had appropriately drawn all 16 sections; three had correctly selected all 16 sections. No one received a total response score of 16 on both Test I and Test II. Considering that 72 subjects had responded by both drawing the boundaries of the sections and selecting the appropriate representation, the degree of success does not concur with Piaget and Inhelder's conclusion. Although Rivoire's (1961) sample did not include subjects beyond the age of fourteen years, she too could not verify Piaget and Inhelder's statement.

Piaget and Inhelder maintain that the two methods of response are equivalent measures of the pupil's responses to his spatial representations of the geometric sections. This would imply that the two methods of response are highly correlated. A correlation coefficient of .55 is not considered an indication of reliability. The research of Goodenough (1926) on the effects of ability on drawing tasks may provide a partial explanation for this relatively low correlation coefficient. Dodwell (1963) attributed his sample's inappropriate drawing responses to the fact that the age group he studied was at a critical stage in their drawing development. The sample of the present study may be mature enough to subdue these effects. The fact that ability and sex differences are manifested in tasks involving drawing and that sex is a significant main effect only in the drawing responses may account for part of the low correlation between the two methods of response. Hence, it appears that the two methods of response need not measure the same thing, namely the child's ability to appropriately respond to the sectioning tasks.

The analyses of variance show that there are significant differences among the solid figures, the types of cuts, and the 16 sections that result. These significant differences will not permit corroboration of Piaget and Inhelder's statement (1963) that, "In the course of Stage III (7-8 to 10-11 years) [p. 266]" the child has no greater difficulty with "the cylinder, the prism, the parallelepiped [p. 251]," and "the conic sections [p. 259]." Not only had the sample of the present study not mastered these tasks, but they exhibited a pattern of difficulty too great to be attributed to chance. The cuts hypothetically performed on the cone were consistently the most difficult.

Another discrepancy between the conclusions of Piaget and Inhelder and the results obtained from the sample of the present study concerns the age variable. Piaget and Inhelder repeatedly base their levels of development upon an age criterion. They rarely provide data on the intellectual ability of their samples. In the problem under investigation, age, as measured by grade in school, was not a significant variable for either method of response. Ability level, however, was significantly different for both methods of response. This may be illustrated by the ten subjects who appropriately responded to all sixteen sections by one method of response. Of the ten, three were twelfth graders, four were tenth graders, and three were eighth graders; all three grade levels were represented. When these same ten subjects were grouped according to ability level, there were six above average ability, four average ability, and no below average ability subjects. Thus, ability levels appear to play an important role in the child's responses to the sectioning tasks while age level, for this sample, is inconsequential.

It appears that the sample of the present study did not respond to the sectioning tasks in a manner comparable to that stated by Piaget and Inhelder. None of the 72 subjects of the present study drew an appropriate boundary and selected a correct representative boundary for each of the sixteen sections. Ninety-one per cent of the sample did not appropriately draw the boundary for some of the sixteen sections; 96 per cent did not select the correct representative boundary for some of the 16 sections. These percentages of subjects' inappropriate responses do not substantiate the findings reported by Piaget and Inhelder.

As mentioned earlier in this paper, Piaget and Inhelder rely upon the child's development of his mental imagery in his successful attainment of representational space. Most surely there is imagery involved in hypothetically sectioning a solid figure and reproducing a concrete representation of it. Inhelder's article on imagery (1965) supports such a conjecture. If the mental imagery is developed, it seems realistic to suppose that the multiple choice response test would result in a higher response score than the drawing response test. This was not the case. Thus, it appears that mental imagery may not be as highly developed for this age group as Piaget and Inhelder's observations would imply.

It is interesting that the main effect of grade level was not significant. This seems to imply that although performance was less than 100 per cent, no additional improvement occurred through all three grade levels. The mean scores for the three grade levels differed only by chance. This fact may be in accord with some of Piaget's thinking concerning the performance of youngsters when they lack true operational thought.

IMPLICATIONS FOR EDUCATION

There were two motivating factors for this study: the observations of secondary school mathematics teachers concerning the apparent lack of understanding by their pupils regarding the relationship of the quadratic equations in two variables and the conic sections, and a firm belief that mathematics teachers and curriculum planners are concerned about the spontaneous responses of children to geometrical (spatial) representations. From the data secured there appear to be two areas of concern to educators: the mathematics content areas where sectioning is employed, and sectioning tasks found in industrial arts programs. The latter content area lies outside the investigator's field of study.

The current introduction to the study of volume in the elementary school is implicitly based upon the geometric sections, particularly the sections formed by the parallel cut (Hartung, Van Engen, & Knowles, 1963; Peters, 1962). In the secondary school a further extension, explicitly based upon the parallel cut of the geometric sections, is found in the development and application of Cavalieri's principle (Kelley & Ladd, 1965). It is conceivable that some of the 72 subjects in this study had encountered these concepts in their public school instruction. Of the 16 subjects in the two top ability groups of the twelfth grade, only 3 demonstrated mastery of the parallel cut. Perhaps teachers of mathematics should consider the remarks by Willerding (1955) that corroborate the above observation. She was concerned about college students who have difficulty with integration problems. The teaching of volume as layers of cubic units is analogous to the development of integration as area under a curve (Courant & Robbins, 1960), and is similar to the use of integration in finding volume (Johnson & Kiokemeister, 1959). Willerding advocates a readiness program for the secondary school to alleviate the latter problem. She suggested that the pupils be provided with "informal and concrete" experiences. Children need to "see the cross sections which will later be used in the integration process." Those pupils who had such opportunities "had less difficulty with volume, particularly those involving a known parallel cross section [Willerding, 1955, p. 411]." Her suggestions support the finding of this study that secondary school children have not mastered the sectioning tasks. If this is a valid observation then such experiences need to be incorporated into the secondary school mathematics program. Perhaps the new approach to area and volume in the elementary school will provide some of the needed experiences.

In surveying the contemporary textbooks for the secondary school there appears to be a trend to introduce at an earlier grade the conic sections along with the quadratic equations

(Banks, Sobel, & Walsh, 1965). Grade level for the sample of the present study was not a significant variable, so the earlier introduction from this point of view is feasible. The fact that mastery of the conic sectioning tasks had not been achieved by the majority of the subjects does not support an earlier introduction to the conic sections. The advent of more unfamiliar material may result in additional confusion; however, presenting a concrete example of the abstract equations may provide needed experience to give meaning to the mathematics. Herein lies a possible research study. Select two groups of children, equated in as many of the following variables as possible: ability level, grade level, sex, interest or motivation, teacher qualification. The ability level of the children may be a general intelligence level, a non-verbal ability level, or mathematics achievement level. One group would be presented with quadratic equations as a contained content area, similar to that found in many first year algebra texts. The other group would receive the same algebraic content plus the introduction of the conic sections. If experiences in the visualization of the concrete situation relating to quadratic equations is an educational advantage, such a study would provide the needed evidence.

There is another area of consideration related to the industrial arts program. Many below average ability males pursue this program. The below average ability subjects in the present study appropriately responded to 51 per cent of the sections presented in Test I and 60.04 per cent of the sections presented in Test II. If we could increase these percentages of appropriate responses by providing concrete sectioning experiences, perhaps these young men would be more successful in their programs. This is merely a conjecture, but worthy of consideration, especially since these pupils are frequently high school drop-outs. Approximately 64 per cent of the responses of the below average twelfth-grade males were appropriate, as compared to 86 per cent for the above average twelfth-grade males. The below average eighth-grade males appropriately responded to less than 41 per cent of the sections; the above average eighth-grade males had over 76 per cent of their responses appropriate. These young men need additional concrete experiences. These are observations based upon the results of this study's sample, but there are implications for potential research problems within this topic for those qualified in industrial arts curriculum.

If Piaget and Inhelder's theory concerning the hierarchical structure of the evolvement of representational space is true, then it seems possible to question the development of representational space in this sample of secondary school pupils. If these children have not developed to the point where they are capable of appropriately representing the initial stages of euclidean space, are they truly capable of comprehending the euclidean relationships currently being presented to them? This question cannot be answered by this research. Its solution lies in future research that may answer the questions raised by Lovell and others concerning the validity of Piaget and Inhelder's hypothesis on the child's conception of space.

Further research is needed to determine whether specific activities can be used to make sectioning tasks more meaningful, not only to increase the child's mastery in the sectioning of solids, but also to make Willerding's report of success in later mathematics a reality for more children. It is possible that the "new" approach to area and volume in the elementary school will be beneficial in providing the necessary background for future course work in these areas by the pupils. A study concerning the effect of some or all of the space factors with the geometric sections may prove to be not only interesting but helpful to the child and his teachers. An analysis of achievement scores in topics using geometric sections with the response scores on sectioning tasks as a stratifying variable may provide information concerning success in these topics as measured by this variable. These appear to be important questions arising from the obtained data. Only through research will effective teaching methods, appropriate content, and successful school achievers be found.

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